Advanced Algorithms 2012A
Lecture 7 – Cheeger's inequality (cont'd)*

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1 Continuing with Cheeger's inequality

Last week we stated the following Theorem and proved only its first inequality.

Theorem [Alon, Alon-Milman, Sinclair-Jerrum, Mihail, after Cheeger]: Let λ_2 be the second smallest eigenvalue of the normalized Laplacian $\hat{L}_G$. Then

$$\tfrac{1}{2}\lambda_2 \le \phi(G) \le \sqrt{2\lambda_2}.$$

Exer: Let G be a graph of maximum degree $d_{\max}$. Show a connection between λ_2 (or all the eigenvalues) of L and $\hat{L}$, and derive from it an analogue of Cheeger's inequalities that relates the isoperimetric number/edge-expansion $h(G)$ to $\lambda_2(L)$.

1.1 Interesting consequence for planar graphs

Theorem [Spielman-Teng]: Every (unweighted) planar graph of bounded degree has $\lambda_2(L) \le O(1/n)$.

An immediate corollary (using the exercise) is that such graphs always have a cut of edge-expansion $O(1/\sqrt{n})$. Moreover, such a cut can be derived from an (approximate) eigenvector of λ_2, giving formal justification for spectral partitioning algorithms. As we discuss later today, this bound implies that such graphs have a 2/3 balanced-cut of with $O(\sqrt{n})$ edges.

1.2 Tightness of these inequalities

The cycle graph. Let G be a cycle on n vertices with unit-weight edges. To compute $\phi(G)$, we use a variant of our homework exercise which shows that $\phi(G)$ is attained by a connected set S of

*These notes summarize the material covered in class, usually skipping proofs, details, examples and so forth, and possibly adding some remarks, or pointers. The exercises are for self-practice and need not be handed in. In the interest of brevity, most references and credits were omitted.

size $s \le n/2$:

$$\phi(G) = \min_{S\subset V} \tfrac{w(S,\bar{S})}{\min\{d(S),d(\bar{S})\}} = \min_{1\le s\le n/2} \tfrac{2}{2s} = \tfrac{2}{n}.$$

We can show an upper bound on $\lambda_2(\hat{L})$ by "guessing" a vector $\vec{0} \ne x \perp \vec{1}$ and computing its Rayleigh quotient. We can set $x_j = n/4 - j$ and $x_{n/2+j} = -n/4 + j$ for $j = 0, \ldots, n/2$, then $x \perp \vec{1}$ by its symmetry and thus

$$\lambda_2 \le \frac{x^T \hat{L} x}{x^T x} = \frac{x^T L x}{r x^T x} = \frac{\sum_{ij\in E}(x_i - x_j)^2}{2\sum_i x_i^2} = \frac{n \cdot 1^2}{8\sum_{j=1}^{n/4} j^2} \le O(1/n^2).$$

We thus obtain that for the difficult direction is tight on the cycle:

$$\Omega(1/n) \le \phi(G) \le \sqrt{2\lambda_2} \le O(1/n).$$

The hypercube graph. Let G be the hypercube graph $\{0,1\}^k$ of dimension $k = \log_2 n$, with unit edge-weights. This graph is k-regular. We now denote a vertex as v instead of i. By considering dimension cuts $S_p = \{v \in \{0,1\}^k : \ v_p = 0\}$ for any $p \in \{1, \ldots, k\}$ (by symmetry, it does not matter which p), we see that

$$\phi(G) \le \tfrac{w(S_p,\bar{S}_p)}{\min\{d(S_p),d(\bar{S}_p)\}} = \tfrac{n/2}{k\cdot n/2} = 1/k.$$

The eigenvalues of $\hat{L}$ can be computed exactly. We will not do it here, but only exhibit one corresponding eigenvector x: The coordinate x_v corresponding to vertex $v \in V(G) = \{0,1\}^k$ is the bit v_p transformed into $+1/-1$ namely $(-1)^{v_p}$ (for arbitrary p, again it does not matter which one). A simple calculation shows this is indeed an eigenvector with eigenvalue $2/k$, and we will not prove here that this is actually $\lambda_2(\hat{L})$. We conclude that the easy direction is tight:

$$1/k = \tfrac{1}{2}\lambda_2 \le \phi(G) \le 1/k.$$

1.3 The difficult direction

Overview. We will prove something stronger (and useful algorithmically). Given any vector $x \perp \vec{d}^{1/2}$ with "small" Rayleigh quotient, say $\lambda' > 0$, we will find a cut $S \subset V$ with small conductance $\phi_G(S) \le \sqrt{2\lambda'}$. The idea is to partition V using a random threshold t on the x_i values. Notice that $x^T L x$ involves terms of the form $(x_i - x_j)^2$, and our earlier technique for going from a tree metric (for which the line metric is a special case) into a cut works when we have $|x_i - x_j|$. The trick will be to use Cauchy-Schwarz inequality (plus other stuff).

First attempt. Consider nonzero $x \perp \vec{d}^{1/2}$. Using our earlier observation, define $y = D^{-1/2}x \ne \vec{0}$ and then $\lambda' = \frac{x^T\hat{L}x}{x^Tx} = \frac{\sum_{ij\in E} w_{ij}(y_i-y_j)^2}{\sum_{i\in V} d_i y_i^2}$, and $0 = x^T d^{1/2} = y^T D^{1/2} d^{1/2} = y^T d$.

By scaling y, we may assume WLOG that all $y_i \in [-1,1]$. Choose $t \in (0,1)$ uniformly at random, and let $S_t = \{i \in V : y_i^2 \geq t\}$. Some calculations show that

$$\min_t \frac{w(S_t, \bar{S}_t)}{d(S_t)} \leq \frac{\mathbb{E}_t[w(S_t, \bar{S}_t)]}{\mathbb{E}_t[d(S_t)]} \leq \sqrt{\frac{2\sum_{ij\in E} w_{ij}|y_i - y_j|^2}{\sum_{i\in V} d_i y_i^2}} = \sqrt{2\lambda'},$$

We can ensure $S_t \neq \emptyset$ by scaling so that some $|y_i| = 1$, but the problem is that we might have $S_t = V$ (in fact, even $d(S_t) > d(V)/2$ would be problematic for us).

Remark: So far we did not really use the fact that $y \perp d$.

Second attempt. Let m be a median of the y_i's i.e.

$$0 < \sum_{i:y_i<m} d_i \leq d(V)/2, \quad \text{and} \quad 0 < \sum_{i:y_i>m} d_i \leq d(V)/2.$$

Define $z^+ \in \mathbb{R}^V$ by increasing values that are smaller than m, i.e. $z_i^+ = \max\{y_i, m\}$, and similarly define $z_i^- = \min\{y_i, m\}$. As seen in class, at least one of them, say z^+, can be used with some manipulations to find $z \in \mathbb{R}^n$ such that

$$\lambda' \geq \frac{\sum_{ij\in E} w_{ij}(z_i^+ - z_j^+)^2}{\sum_{i\in V} d_i(z_i^+ - m)^2} \geq \frac{\sum_{ij\in E} w_{ij}(z_i - z_j)^2}{\sum_{i\in V} d_i z_i^2}.$$

Now we can apply the analysis of our first attempt (choosing random t and defining S_t using z_i^2 instead of y_i^2), to conclude that

$$\min_t \frac{w(S_t, \bar{S}_t)}{d(S_t)} \leq \frac{\mathbb{E}_t[w(S_t, \bar{S}_t)]}{\mathbb{E}_t[d(S_t)]} \leq \sqrt{\frac{2\sum_{ij\in E} w_{ij}|z_i - z_j|^2}{\sum_{i\in V} d_i z_i^2}} \leq \sqrt{2\lambda'}.$$

It was important here that for all t we have $0 < d(S_t) \leq d(V)/2$, and thus the LHS is indeed $\min_t \phi_G(S_t)$. Applying the above to $\lambda' = \lambda_2$ we get $\phi(G) \leq \sqrt{2\lambda_2}$.

Exer: Prove a statement similar to the Theorem that relates $\lambda_2(L)$ to the isoperimetric number/edge-expansion $h(G) = \min_{S\subset V} \frac{w(S,\bar{S})}{\min\{|S|,|\bar{S}|\}}$. Note that now the inequalities might involve the maximum degree $d_{\max} = \max_{i\in V} d_i$.

2 Applications of sparse-cut

2.1 From sparse-cut to edge-expansion

Consider a graph $G(V,E)$ with edge-capacities $c(e) \geq 0$. The edge-expansion or isoperimetric number (also the Cheeger constant) of G is defined as:

$$h(G) = \min_{S\subset V} \frac{c(S,\bar{S})}{\min\{|S|, |V \setminus S|\}}.$$

Observation: the edge-expansion objective is approximated within factor 2 by uniform-demands sparse-cut, i.e., when every pair of vertices forms a demand-pair and thus the objective is $\frac{c(S,\bar{S})}{|S|\cdot|V\setminus S|}$.

Proof: Asumming WLOG $|S| \leq |V|/2$ then $|V|/2 \leq |V \setminus S| \leq |V|$.

We remark that both problems are NP-hard. But recall that our theorem about flow/cut gap actually yields a polynmial-time algorithm with approximation $O(\log k)$ for sparse-cut (and we have $k = \binom{n}{2}$ in our case of uniform-demands).

Corollary: The problem of finding S that minimizes edge-expansion can be approximated within factor $O(\log n)$ in polynomial time.

2.2 From edge-expansion to balanced-cut

Let $b \in [\frac{1}{2}, 1)$. In b-balanced cut, the input is a graph $G(V, E)$ with edge-capacities and the goal is to find a minimum capacity cut $(S, \bar{S})$ under the restriction that both $|S|, |V \setminus S| \leq b|V|$. The case $b = 1/2$ is called Minimum Bisection.

The following algorithm computes a 2/3-balanced cut, whose capacity (cost) can be "compared" to the optimal 1/2-balanced cut. (This type of guarantee is called bicriteria approximation algorithm.)

Bicriteria algorithm for Minimum Bisection

Input: graph $G = (V, E)$ with edge capacities

Output: a cut $(V', V \setminus V')$

1. Initialize $V' \leftarrow V$.

2. Repeat while $|V'| \geq \frac{2}{3}|V|$

2a. Find in $G[V']$ a cut S that approximately minimizes edge-expansion

2b. Remove S (the smaller side) i.e. $V' \leftarrow V' \setminus S$.

3 Output V'.

Theorem [bicriteria approximation for Minimum Bisection]: For every graph G, the above algorithm reports a cut $(V', V \setminus V')$ that is 2/3-balanced and its capacity is at most $O(\text{OPT}_{1/2} \log n)$ where $\text{OPT}_{1/2}$ is the minimum bisection of G.

Exer (similar bound based on spectral arguments): Design a polynomial time algorithm whose input is a graph G and $\phi^* > 0$, if G has 1/2-balanced cut of conductance $\leq \phi^*$, then the algorithm finds a 2/3-balanced cut of conductance $O(\sqrt{\phi^*})$.