Advanced Algorithms 2012A – Problem Set 4

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1. Prove that the number of connected components in a graph G is equal to the multiplicity of 0 among the eigenvalues of L_G (the Laplacian matrix of G).

   Remark: The goal is to obtain a stronger statement than what was done in class (about whether G is connected or not). You can use things proved in class.

2. Design a polynomial-time algorithm with the following guarantee: when given a graph G and a number $\phi^* > 0$ as input, if G has a 1/2-balanced cut of conductance $\leq \phi^*$, then the output is a 2/3-balanced cut of conductance $O(\sqrt{\phi^*})$. (Otherwise, the output can be anything like an arbitrary cut or "failure".)

   You can use what we said in class (without going through all details) that the Cheeger inequalities are constructive and can be implemented in polynomial time.

3. Consider the setting where experts' either make a mistake or not (i.e., binary losses), and consider deterministic algorithms that follow one of the experts (not a convex combination of experts). Let m^* be the number of mistakes of the best expert in hindsight and recall that Weighted Majority never makes more than roughly $2(1 + \varepsilon)m^*$ mistakes.

   Prove that no algorithm can guarantee fewer than $2m^*$ mistakes, namely that for every algorithm and every integer m^*, there is an input sequence where the algorithm makes at least $2m^*$ mistakes.

   Hint: Construct the sequence on the fly by peeking into the algorithm's next step. How many experts are needed for this proof?

Extra credit:

4. Let G be a graph of maximum degree $d_{\max}$. Show a connection between $\lambda_2(L_G)$ and $\lambda_2(\hat{L}_G)$ (i.e., bound one in terms of the other). Show also a connection between the conductance $\phi(G)$ and the isoperimetric number $h(G)$. Finally, use the above and Cheeger's inequalities seen in class, to derive an analogue that relates the isoperimetric number $h(G)$ to $\lambda_2(L_G)$.

   Note: These inequalities may involve (i.e., depend on) $d_{\max}$.