Seminar on Algorithms and Geometry 2014B
Lecture 12 – Planar Separators*

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1 Vertex Separators

Definition: Let $G = (V, E)$ be a graph with vertex weights $w : V \to \mathbb{R}_{\geq 0}$. An *vertex-separator* of G is a partition $V = A \cup B \cup S$ such that

- both $w(A), w(B) \leq \alpha \frac{2}{3} w(V)$ (sometimes a different constant $\alpha \in (0, 1)$ is more convenient)
- there are no edge connecting between A and B (i.e., removing S disconnected A from B)

Remark: It is instructive to think about unit weights, but sometimes the weighted version is needed.

Exer: A different variant is that every connected component of $G \setminus S$ has weight at most $\frac{1}{2} w(V)$. Show that this variant implies an (A, B, S). (Note that A, B in the above definition need not be connected.)

2 The Planar Separator Theorem

Planar Separator Theorem [Lipton-Tarjan'79]: Every n-vertex planar G with vertex weights w, admits a vertex-separator (A, B, S) of cardinality $|S| \leq 4\sqrt{n}$. Moreover, such a vertex-separator can be computed in linear time (given a planar drawing of G).

Remark: Better constants and also multiple proofs are known.

Examples:

Every tree has a vertex-separator of size 1.

Every grid has a separator of size $O(\sqrt{n})$.

The theorem does not extend to all graphs (e.g., a complete graph) or to all sparse graphs (e.g., an expander).

*These notes summarize the material covered in class, usually skipping proofs, details, examples and so forth, and possibly adding some remarks, or pointers. The exercises are for self-practice and need not be handed in. In the interest of brevity, most references and credits were omitted.

Fundamental Cycle Lemma: Let G be a plane graph (a planar graph with a specific drawning in the plane). Let T be a spanning tree of G rooted at $r \in V$, and let d be the tree's depth. Then G has a vertex-separator of size $|S| \leq 2d+1$. Moreover, this vertex-separator consists of two paths to the root in T.

Proof (Sketch): First, triangulate G, i.e., add edges so that every face has 3 edges. Now every non-tree edge e defines a cycle C_e in $T+e$ (which consists of e and two paths to the root). Such a cycle contains $|C_e| \leq 2d+1$ vertices, and removing it separates the plane, and thus the remaining vertices, into the interior and exterior, with no edge connecting between the two. Define

$$\text{balance}(C_e) = \max\{w(\text{interior}(C_e)), w(\text{exterior}(C_e))\}.$$

Consider a non-tree edge e that minimizes $\text{bal}(C_e)$. Assume this value is $> \frac{2}{3}w(V)$, as otherwise we're done. Without loss of generality, the balance is attained by the interior, i.e., $w(\text{interior}(C_e)) \geq \frac{2}{3}w(V)$, and moreover break ties by taking e for which the interior has minimum number of faces. Now consider the face that contains e and is in the interior of C_e, and consider the two other edges on that face, with some case-analysis on whether they belong to T or not, to get a contradiction to e being a minimizer.

Proof of Theorem: The idea is to Reduce the diameter to $O(\sqrt{n})$ (if it is larger) by removing a few vertices (adding them into S) to chop the graph into pieces, and then applying in each piece the fundamental cycle lemma (which adds more vertices into S).

The proof was shown in class.

3 Recursive Separator

Recursive Separator Theorem: For every n-vertex planar graph $G = (V, E)$ and $r \leq n$, there is a subset $S \subset V$ of cardinality $|S| \leq O(n/\sqrt{r})$ such that every connected component of $G \setminus S$ has cardinality at most r.

Proof: Repeatedly apply the Planar Separator Theorem on every connected component whose size $> r$ (always using unit weights). By definition, all final CCs (at end of the recursion) have size $\leq r$.

The bound on the total number of vertices removed throughout the recursion was shown in class.

4 Application to Maximum Independent Set

Theorem (example application): There is a PTAS for maximum independent set in planar graphs.

Proof: Given G, the algorithm computes a recursive separator S for $r = \log\log n$. In every connected component of $G \setminus S$, compute a maximum independent set by exhaustive search in time $2^r \leq O(logn)$, and output their union. It is indeed an independent set, because the different components have no edges between them.

The analysis was shown in class, and uses the fact that G is 4-colorable, hence $OPT \geq n/4$.

5 Course wrap-up: Open Problems

Some problems mentioned throughout the course that are still open:

- Do planar metrics embed into ℓ_1 with $O(1)$ distortion?

  The currently known bound is $O(\sqrt{\log n})$, and holds even when embedding into ℓ_2.

- Can one achieve dimension reduction for doubling subsets of ℓ_2 (better than JL)? Namely, we want both the dimension and distortion to depend on the doubling dimension but not on the number of points n.

- Use geometric tools in conjunction with SDPs (or stronger programs) to improve the approximation ratio for multicut or for sparsest-cut (which is essentially a problem of embedding into ℓ_1)?

- Better algorithms for NNS, either for (high-dimensional) Euclidean metrics, or for specific metrics or under other restrictions (like the doubling dimension seen in class).