Seminar on Algorithms and Geometry 2014B – Problem Set 2

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We discussed in class the random projection method for dimension reduction and an algorithm for approximating Max-Cut via semidefinite programming.

1. Recall the random linear map $L : \mathbb{R}^m \to \mathbb{R}^d$ seen in class for dimension reduction of n points in $\mathbb{R}^m$.

   Prove for every two unit-length vectors $u, v \in \mathbb{R}^m$, with high probability

   $$\langle Lu, Lv \rangle = \langle u, v \rangle \pm O(\varepsilon).$$

   This shows that the map L approximately preserves inner products additively.

   Hint: Just rely on what we already know about L preserving distances.

2. Recall the following notation from class. OPT is the optimum Max-Cut value for a given graph G (we saw it is equal to the value of the quadratic program Z_{QP}), and z_{VP} is the value of the vector-program (relaxation of Max-Cut on G).

   Consider adding to that vector program the following constraints

   $$\begin{aligned} \|v_i - v_j\|^2 &\le \|v_i - v_k\|^2 + \|v_k - v_j\|^2, && \forall i, j, k \in V, \\ \|v_i - v_j\|^2 &\le \|v_i - (-v_k)\|^2 + \|(-v_k) - v_j\|^2, && \forall i, j, k \in V, \end{aligned}$$

   and denote the value of the resulting vector program by z_{NEW}.

   Prove that for every graph G,

   $$OPT \le z_{NEW} \le z_{VP}.$$

   Extra credit: Show a graph G for which $z_{NEW} < z_{VP}$. (Hint: Consider a triangle (3-cycle), bound z_{VP} by exhibiting a vector solution, and bound z_{NEW} by plugging into its objective the second constraint above.)