Randomized Algorithms 2017A – Problem Set 1

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1. Recall we defined parallel composition as the graph $\bar{G}$ obtained by a disjoint union of two graphs G_1 and G_2, where we identify vertices $s_1 \in G_1$ and $s_2 \in G_2$ (calling it $\bar{s}$) and identify $t_1 \in G_1$ with $t_2 \in G_2$ (calling it $\bar{t}$). Prove that

$$\frac{1}{\mathrm{R_{eff}}^{\bar{G}}(\bar{s},\bar{t})} = \frac{1}{\mathrm{R_{eff}}^{G_1}(s_1,t_1)} + \frac{1}{\mathrm{R_{eff}}^{G_2}(s_2,t_2)},$$

and use it to drive an explicit expression for all hitting times in a cycle graph on n vertices (wlog, let the vertices be $\{0,\ldots,n-1\}$ and express $H_{0,v}$).

Hint: Use Ohm's Law.

2. A random walk in a *directed* graph is defined by picking, at every step, a uniformly random outgoing edge. The hitting time is defined analogously to undirected graphs. Show that for every n, there exists a directed graph on n vertices that is strongly connected and has two vertices u, v for which the hitting time is $H_{uv} = 2^{\Omega(n)}$.

Extra credit:

3. For two vertices s, t in a graph G, define P_{st} as the probability that a random walk started at s, hits t before returning to s (it is called the *escape probability*). Prove that

$$P_{st} = \frac{1}{\deg(s)\,\mathrm{R_{eff}}(s,t)}.$$

Hint: Define a function $q : V \to \mathbb{R}$, where $q(v)$ is the probability that a random walk started at v, hits t before hitting s. Let $\phi : V \to \mathbb{R}$ be the potential function induced on G when we create a potential difference of 1 between s and t, i.e., $\phi(s) = 1$ and $\phi(t)$. Establish a connection between the functions q and ϕ (hint: each of them is harmonic on $V \setminus \{s,t\}$). Then derive an expression for P_{st} using $\{q(v) : v \in N(s)\}$, and similarly for $1/\mathrm{R_{eff}}(s,t)$.