Sublinear Time and Space Algorithms 2018B – Problem Set 3

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General instructions: Please keep your answers short and easy to read. You can use results, calculations or notation seen in class without repeating them, unless asked explicitly to redo them.

1. Let $h', h'' \in \mathbb{R}^n$ be supported on disjoint sets $T', T'' \subset [n]$ respectively, and let the matrix $A \in \mathbb{R}^{m \times n}$ be $(|T'| + |T''|, \varepsilon_0)$-RIP. Show that

   $$|\langle Ah', Ah'' \rangle| \leq \varepsilon_0 \|h'\|_2 \|h''\|_2.$$

   Hint: Consider first h', h'' that have unit length, and apply the formula $\|u+v\|_2^2 - \|u-v\|_2^2 = 4\langle u, v\rangle$ to $u = Ah'$ and $v = Ah''$.

2. Show how to reconstruct an unknown k-sparse $x \in \mathbb{R}^n$ from a $(2k, 0.1)$-RIP matrix $A \in \mathbb{R}^{m \times n}$ and $y = Ax$, in time $n^{O(k)}$ and with no errors.

   Hint: start by showing that there is a unique suitable x.

3. Let $u \sim \mathrm{Exp}(1)$ be a random variable with exponential distribution. Show that

   $$\mathbb{E}[\tfrac{1}{u} \mid \tfrac{1}{u} \leq n^3] = O(\log n).$$

Extra credit:

4. Show how to improve the multiplicative error in the Precision Sampling Lemma to $1+\varepsilon$. How would it increase the total cost of the estimator?

   Hint: Use independent repetitions.

5. Show that for an arbitrary $x \in \mathbb{R}^n$, if some $\tilde{x}$ satisfies the ℓ_1/ℓ_2 guarantee

   $$\|x - \tilde{x}\|_2 \leq \tfrac{O(1)}{\sqrt{k}} \; \|x_{tail(k)}\|_1$$

   (e.g., as seen in class using an RIP measurement matrix), then $x^* = \tilde{x}_{top(k)}$ satisfies the ℓ_1/ℓ_1 guarantee

   $$\|x - x^*\|_1 \leq O(1) \|x_{tail(k)}\|_1.$$

   Hint: Let T be the top k coordinates of x, and $\tilde{T}$ the top k coordinates of $\tilde{x}$. Split the coordinates into $\tilde{T}$, $T \setminus \tilde{T}$, and the rest.