Randomized Algorithms 2019A – Problem Set 1

Robert Krauthgamer and Moni Naor

Due: Dec. 6, 2018

1. Let $G = (V, E)$ be an undirected graph. Use direct arguments about electrical flows to prove the triangle inequality

   $$\forall u, v, w \in V, \qquad R_{\text{eff}}(u, w) \le R_{\text{eff}}(u, v) + R_{\text{eff}}(v, w).$$

   Remark: It also follows from the triangle inequality for commute times, which we saw in class, but I ask here for a direct proof.

2. Prove that for every n-vertex graph $G = (V, E)$,

   $$\text{cov}(G) \le 2(n-1)|E|.$$

   Hint: Follow a spanning tree T of the graph.

3. A random walk in a *directed* graph is defined by picking, at every step, a uniformly random *outgoing* edge, and following that edge. The hitting time H_{uv} is defined analogously to undirected graphs. Show that for every n, there exists a directed n-vertex graph that is strongly connected and has two vertices u, v for which the hitting time is $H_{uv} = 2^{\Omega(n)}$.

   Extra credit:

4. Prove that for every $G = (V, E)$ and every $w \in V$,

   $$\text{cov}_w(G) \ge \Omega(\log n) H_{min},$$

   where $H_{min} = \min_{u \ne v \in V} H_{uv}$ is the minimum (non-trivial) hitting time in the graph.

   Hint: Let $\{X_t\}_{t \ge 0}$ be a random walk that starts at $X_0 = w$, and consider for sake of analysis a *random* permutation $\pi_1, \ldots, \pi_{n-1}$ of $V \setminus \{w\}$. Define the random variable Z_k, for $k = 1, \ldots, n-1$, to be the first time by which all of $\pi_1, \ldots, \pi_k$ were visited. Show that for all $k \ge 2$,

   $$\begin{aligned}&\mathbb{E}[Z_k - Z_{k-1} \mid Z_{k-1}, X_0, \ldots, X_{Z_{k-1}}] \\ &\quad = \Pr\left[\pi_k \notin \{X_0, \ldots, X_{Z_{k-1}}\} \mid Z_{k-1}, X_0, \ldots, X_{Z_{k-1}}\right] \cdot H_{X_{Z_{k-1}}, \pi_k}.\end{aligned}$$

   Notice that we consider the randomness of both the walk and the permutation, and condition on the walk up to time Z_{k-1}.