Randomized Algorithms 2021A – Lecture 2 (second part)
Undirected Connectivity and Cover Time*

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1 Undirected Connectivity

We finished the discussion from last time, including Markov's inequality.

Markov's inequality: Let X be a nonnegative random variable with finite expectation. Then

$$\forall t > 1, \quad \Pr[X \geq t \cdot \mathbb{E}[X]] \leq 1/t.$$

Or equivalently,

$$\forall s > 1, \quad \Pr[X \geq s] \leq \mathbb{E}[X]/s.$$

2 Cover Time

The *cover time* from vertex u, denoted $\mathrm{cov}_u(G)$, is the expected number of steps until a random walk that starts at u has visited all vertices of G. Formally, let $T' = \min\{t \geq 0 : \{X_0, X_1, \ldots, X_t\} = V\}$ and let $\mathrm{cov}_u(G) = \mathbb{E}[T']$.

The cover time of a graph G is defined as $\mathrm{cov}(G) = \max_u \mathrm{cov}_u(G)$, i.e., according to the "worst-case" starting vertex.

Example: In the n-clique, the cover time is $O(n \log n)$. (It is just the coupon collector problem, as we will soon see).

Exer: Prove it.

Theorem 6: $\mathrm{cov}(G) \leq 2(n-1)|E|$.

Proof: Was seen in class, using a spanning tree T of the graph.

Theorem 7 (Matthews' bound): Let G be a connected graph on n vertices, and let $H_{\max} = \max\{H_{uv} : u, v \in V\}$. Then

$$H_{\max} \leq \mathrm{cov}(G) \leq O(\log n) H_{\max}.$$

*These notes summarize the material covered in class, usually skipping proofs, details, examples and so forth, and possibly adding some remarks, or pointers. The exercises are for self-practice and need not be handed in. In the interest of brevity, most references and credits were omitted.

Proof: Was seen in class.

Exer: Show that each of these inequalities is tight (up to constants) for some graph G.