# Randomized Algorithms 2021A – Lecture 3 (second part)
# Electrical Networks*

Robert Krauthgamer

## 1 Electrical Networks

It turns out that random walks are "equivalent" to electrical networks (composed of resistors), and this "physical" interpretation gives alternative ways to prove things. We first introduce the basic concept.

Given an undirected graph $G = (V, E)$, we think of it as an electrical circuit with unit resistors. The basic property of electrical circuits is that current flows when there is a potential difference (e.g., between the endpoints of a resistor).

What happens when two vertices are connected to the positive and negative terminals of a battery? We create a "potential difference" between these two vertices, which induces a current (or electrical flow) in the network, which satisfies the following laws:

**Kirchhoff's Current Law (KCL):** At every vertex, the total incoming flow equals the total outgoing flow.

We include here also flow that is injected to/extracted from the network. For example, injecting one unit at $s \in V$ and extracting one at $t \in V$, means that we ship a unit of flow from $s$ to $t$.

In fact, this is just the well-known flow conversation constraint, and a function $f$ that satisfies it is called a *flow*.

**Kirchhoff's Voltage Law (KVL):** The sum of potential differences along every (directed) cycle is zero.

Remark: (KVL) explains why we call it "potential difference". It implies that we can assign a potential to each vertex, i.e., define $\phi' : V \to \mathbb{R}$, such that $\phi_{uv} = \phi'_u - \phi'_v$ for every edge. Obviously, this map is unique up to translation (if $G$ is connected).

**Ohm's Law:** The current flowing from $u$ to $v$ through an edge $\{u, v\}$ of resistance $r_{uv}$ is exactly $\frac{\phi_{uv}}{r_{uv}}$, where $\phi_{uv}$ is the potential difference on (the endpoints of) the resistor.

We assumed unit resistors, but in general, if $G$ has edge weights, then each edge $e$ would have

*These notes summarize the material covered in class, usually skipping proofs, details, examples and so forth, and possibly adding some remarks, or pointers. The exercises are for self-practice and need not be handed in. In the interest of brevity, most references and credits were omitted.

resistance $r_e = 1/w_e$ (i.e., its conductance is $c_e = 1/r_e = w_e$), and this corresponds to a random walk according to the edge weights, i.e., each outgoing edge is picked with probability proportional to $w_e$.

**Example:** Suppose $G$ is a path on 3 vertices $u, w, v$, and we create potential difference $\phi_{uv}$. Then by Ohm's Law, then KCL, then Ohm's Law,

$$\phi_{uw} = f_{uw} r_{uw} = f_{wv} r_{wv} = \phi_{wv}.$$

Since the LHS and RHS sum up to $\phi_{uv}$, each of them is exactly $\frac{1}{2}\phi_{uv}$, and thus $f_{uw} = f_{wv} = \frac{1}{2}\phi_{uv}$ is the amount of flow.

**Observation:** The amount of flow shipped from $u$ to $v$ scales linearly with $\phi_{uv}$.

**Observation:** In fact, we can also add two potential-difference functions, and the flows will add up (and vice versa).

**Theorem 1 (Thomson's Principle of minimum energy):** Let $f$ be a flow that ships a unit flow from $s$ to $t$, and has minimum total energy dissipation

$$\mathcal{E}(f) = \sum_{uv \in E} f_{uv}^2 r_{uv}$$

among all such flows. Then $f$ is an electrical flow.

**Proof:** Was outlined in class (you are encouraged to complete the details). Given a flow $f : E \to \mathbb{R}$ that minimizes the energy dissipation, define $\phi_{uv} = f_{uv} r_{uv}$ for every edge $uv \in E$. KVL and Ohm's Law hold by construction, and it remains to prove KCL. Now consider an arbitrary cycle, and add $\delta$ units of flow along this (directed) cycle, to create another flow $f'$. Now write $\mathcal{E}(f') - \mathcal{E}(f) \geq 0$ in terms of $\delta$ to conclude KCL.

**Exer:** Show that the minimizer $f$ (flow that minimizes the energy) is attained uniquely.