Randomized Algorithms 2021A – Lecture 4 (second part)
Effective Resistance*

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1 Reminder: Graphs as Electrical Networks

Recall that in an electrical network, we view a graph as a collection of (undirected) resistors. When we impose a potential difference ϕ_{uv} between vertices u, v, it induces an electrical flow (current), which is (i) a feasible flow in the sense of flow preservation (KCL), and (ii) creates potentials (voltages) on all other vertices (KVL), and (iii) the flow along each edge is inverse proportional to the potential difference, and directed accordingly (Ohm's Law).

Recall our notation $\phi_{uv} = \phi_u - \phi_v$ and f is defined on "directed" edges with $f_{uv} = -f_{vu}$, even though all the edges $uv \in E$ are undirected.

Observation: The amount of flow shipped from u to v scales linearly with ϕ_{uv}.

Observation: In fact, we can also add two potential-difference functions, and the flows will add up (and vice versa).

2 Effective Resistance

Effective Resistance: The *effective resistance* between vertices u, v in an electrical network, denoted $R_{\text{eff}}(u, v)$, is the potential difference ϕ_{uv} we need to create between u and v to induce exactly one unit of current flowing from u to v.

The name comes from the viewpoint that the entire network can be "simulated" by a single resistor between u, v, with resistance $r_{uv} = R_{\text{eff}}(u, v)$, then the current between u, v would be the same. Indeed, if we impose the same potential difference $\phi_{uv} = R_{\text{eff}}(u, v)$ on this single resistor, thus the amount of flow will be $f_{uv} = \phi_{uv}/r_{uv} = 1$, exactly as in G.

Notice that $R_{\text{eff}}(u, v)$ is symmetric (by the linearity observations).

We can now show that the effective resistance is essentially the same as the commute time.

*These notes summarize the material covered in class, usually skipping proofs, details, examples and so forth, and possibly adding some remarks, or pointers. The exercises are for self-practice and need not be handed in. In the interest of brevity, most references and credits were omitted.

Theorem 2 [Chandra, Raghavan, Ruzzo, Smolensky and Tiwari, 1989]: Let $G = (V, E)$ be an undirected graph. Then

$$\forall u, v \in V, \qquad C_{uv} = 2|E| R_{\text{eff}}(u, v).$$

Proof idea: These quantities satisfy the same set of linear equations. For the actual proof it is more convenient to deal with the hitting time.

Lemma 3: Let N_z be the electrical network corresponding to G, when we inject $\deg(u)$ units of flow at every vertex $u \in V$, and extract $\sum_{u \in V} \deg(u) = 2|E|$ units of flow at z. Then the potential differences ϕ^{N_z} satisfy

$$\forall u \in V, \qquad \phi_{uz}^{N_z} = H_{uz}.$$

Proofs of Lemma 3 and Theorem 2: Was seen in class.

Theorem 4 (Thomson's Principle revisited): Let f be a flow that ships one unit from u to v with minimum energy. Then

$$R_{\text{eff}}(u, v) = \mathcal{E}(f).$$

It provides an alternative definition for effective resistance.

Proof: We already saw the minimizer f is just a unit of electrical flow from u to v. By convention, let $f_{xy} = 0$ whenever $\{x, y\} \notin E$, then

$$\begin{aligned}
\mathcal{E}(f) &= \sum_{xy \in E} f_{xy}^2 r_{xy} \\
&= \tfrac{1}{2} \sum_{x} \sum_{y \in N(x)} f_{xy}^2 r_{xy} \\
&= \tfrac{1}{2} \sum_{x} \sum_{y} f_{xy} (\phi_x - \phi_y) \\
&= \sum_{x} \sum_{y} \phi_x f_{xy} && (f \text{ is anti-symmetric}) \\
&= \sum_{x} \phi_x \sum_{y} f_{xy}
\end{aligned}$$

and observe that $\sum_y f_{xy} = 0$ is zero for all $x \notin \{u, v\}$, hence the above is just $\phi_u (+1) f_{uy} + \phi_v (-1) = \phi_u - \phi_v$.

QED

Theorem 5 (Rayleigh's Monotonicity Law): If $\{r(e)\}$ and $\{r'(e)\}$ are sets of resistances on the edges of the same graph G, such that $r(e) \leq r'(e)$ for all $e \in E$,

$$\forall u, v \in V, \qquad R_{\text{eff}}^{(r)}(u, v) \leq R_{\text{eff}}^{(r')}(u, v).$$

The proof follows directly from Thomson's Principle above, as the LHS minimizes the energy over all unit flows, including the flow that attains the RHS.

Corollary 6: For every edge $(u, v) \in E$, we have $R_{\text{eff}}(u, v) \leq 1$ and thus $C_{uv} \leq 2|E|$.

Exer: Prove this.

Hint: Argue that a non-edge is equivalent to having infinite resistance, thus adding an edge is equivalent to decreasing its resistance.

This proves Theorem 3 from the first class (claimed earlier without a proof).

Lemma 7 (Bridge Edge): Suppose edge $uv \in E$ is a *bridge* in G (which means that removing this edge disconnects the graph). Then $R_{\text{eff}}(u, w) = 1$, and thus $C_{uv} = 2|E|$.

Exer: Prove this.

Example A: The path: $C_{1n} = H_{1n} + H_{n1} = 2H_{1n}$ by symmetry. Similarly to Lemma 7 about bridges, $R_{\text{eff}}(1, n) = n - 1$, and thus $C_{1n} = 2(n-1)R_{\text{eff}}(1, n) = 2(n-1)^2$. We conclude that $H_{1n} = (n-1)^2$.

Notice this is also the cover time of that path.

Example B: The lollipop: The "lollipop" graph is a path of $n/2$ edges from u to v, and this last vertex v is part of a clique with $n/2 - 1$ new vertices. It can be easily seen $H_{uv} = (n/2)^2$ while $H_{vu} = \Theta(n^3)$ and also $\text{cov}(G) = \Theta(n^3)$.

Exer: Prove these bounds (it's actually easy to get precise formulas).

Hint: Use the effective resistance formula and our earlier theorem about a spanning tree.

Series Composition: Consider two graphs, G_1 and G_2 on disjoint sets of vertices, and fix in each graph G_i, $i = 1, 2$, a pair of vertices s_i, t_i. Let $\bar{G}$ be their series composition, defined as the graph obtained by taking their union and identifying t_1 with s_2. Then

$$R_{\text{eff}}^{\bar{G}}(s_1, t_2) = R_{\text{eff}}^{G_1}(s_1, t_1) + R_{\text{eff}}^{G_2}(s_2, t_2).$$

Exer: Prove this.

Parallel Composition: Let G_1 and G_2 be as above. Let $\bar{G}$ be now their parallel composition, defined as the graph obtained by taking their union and identifying s_1 with s_2 (denote it $\bar{s}$), and identifying t_1 with t_2 (denote it $\bar{t}$). Then

$$\frac{1}{R_{\text{eff}}^{\bar{G}}(\bar{s}, \bar{t})} = \frac{1}{R_{\text{eff}}^{G_1}(s_1, t_1)} + \frac{1}{R_{\text{eff}}^{G_2}(s_2, t_2)}.$$

Exer: Prove this (using Ohm's Law, or the minimum energy principle).