Randomized Algorithms 2021A – Problem Set (done in class)

Robert Krauthgamer and Moni Naor

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1. Suppose the matrix $S \in \mathbb{R}^{s\times n}$ is $(\varepsilon', \delta', b')$-JLT for $b' = 3$. Show that for every $x, y \in \mathbb{R}^n$ with $\|x\| = \|y\| = 1$, with probability at least $1 - \delta'$,

$$|\langle Sx, Sy\rangle - \langle x, y\rangle| \le 9\varepsilon'. \tag{1}$$

   Hint: Write $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2$.

2. Let $\varepsilon > 0$ and $A, B \in \mathbb{R}^{n\times m}$ be an input for AMM. Suppose the matrix $S \in \mathbb{R}^{n\times s}$ is $(\varepsilon', \delta', b')$-JLT, where the parameters satisfy $\varepsilon' = \varepsilon/9$, $\delta' = \delta$, and $b' = O(m^2)$. Show that with probability at least $1 - \delta$, the matrix $(SA)^\top(SB)$ solves AMM, i.e.,

$$\|(SA)^\top(SB) - A^\top B\|_F \le \varepsilon\|A\|_F\|B\|_F.$$

   Hint: Use the previous question to reduce the dimension n (in A, B) to dimension s.

3. Show that given n points $x_1, \ldots, x_n \in [m]^d$ for $m = d = n/10$ as input, the radius of the point set (under ℓ_2-distance) can be $(1 + \varepsilon)$-approximated faster than the naive computation in time $O(n^2 d) = O(n^3)$. Here, the radius is defined as

$$r := \min_{i\in[n]} \max_{j\in[n]} \|x_i - x_j\|_2,$$

Bonus question:

4. (a) Let $x_1, \ldots, x_n \in \mathbb{R}^d$ and suppose the linear map $L : \mathbb{R}^d \to \mathbb{R}^t$ preserves all pairwise distances within factor $1 \pm \varepsilon$, i.e.,

$$\forall i, j \in [n], \quad \|L(x_i - x_j)\| \in (1 \pm \varepsilon)\|x_i - x_j\|.$$

   Prove that the area of every right-angled triangle $\{x_i, x_j, x_k\}$ (i.e., whenever $\langle x_j - x_i, x_k - x_i\rangle = 0$) is preserved by L within factor $1 + O(\varepsilon)$.

   Hint: Use (1).

   (b) Show there is a random map $L : \mathbb{R}^d \to \mathbb{R}^t$ for $t = O(\varepsilon^{-2} \log n)$, such that for every n points $y_1, \ldots, y_n \in \mathbb{R}^d$, with high probability, L preserves the area of every triangle $\{y_i, y_j, y_k\}$ within factor $1 + \varepsilon$.

   Hint: For every triangle, use one extra point to "break" it into two right-angle triangles.