reduced to one or two; these are called the reduced parameters in the table (see Exercise 10). The values of these reduced parameters for which Sprott observed chaotic behavior are listed in the last column.

You, the reader, are encouraged to adopt one of Sprott's systems as your very own. Throughout this text a number of exercises will refer to this system. You can also apply many of the techniques that will be covered in later chapters to the study of your system. Many of these systems have not been completely analyzed and you may discover new phenomena in your study!

## 1.8 Exercises

1. In population dynamics, *depensation* or the *Allee effect* (Allee et al. 1949) corresponds to the reduction in birth rate when a population is small due to the difficulty of finding mates and the harmful effects of inbreeding. A simple model to account for this that generalizes the logistic model (1.7) is

$$\dot{N} = -rN\left(1 - \frac{N}{E}\right)\left(1 - \frac{N}{K}\right),$$

where $0 < E < K$.

(a) Discuss the biological meaning of the variable $N(t)$ and the parameters $r$, $E$, and $K$.

(b) Analyze this system using the methods of §1.3, assuming $r, E, K > 0$.

2. "Habitat conversion from forests to agriculture and then to degraded land is the single biggest factor in the present biological diversity crisis" (Dobson, Bradshaw, and Baker 1997). Let $F$ be the area covered by forest, $A$ the area devoted to agriculture, $U$ the unused land area, and $P$ the human population. A simple model for habitat conversion is

$$\begin{aligned}
\dot{F} &= sU - dPF, \\
\dot{A} &= dPF + bU - aA, \\
\dot{U} &= aA - (b+s)U, \\
\dot{P} &= rP\left(1 - \frac{h}{A}P\right).
\end{aligned} \tag{1.34}$$

(a) Interpret the constants $s$, $d$, $b$, $a$, and $h$ in the model. In particular, what is the assumed carrying capacity of this environment? What is the interpretation of the nonlinear term $dPF$? Why is it reasonable to include the area $U$ in the model?

(b) So that this model makes sense, the total land area, $T$, must be constant. Demonstrate that this is the case for (1.34). Reduce the model to three equations using the fact that $T$ is constant.

(c) Find the equilibrium solution(s) for this model for a given total area $T$.

3. The Michaelis–Menton mechanism describes the catalysis of a reaction by an enzyme (Michaelis and Menten 1913). The chemical notation for this reaction is

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} ES \xrightarrow{k_2} E + P.$$

Here the enzyme $E$ combines with the substrate $S$ to make an intermediate complex, $ES$, that is converted into the product $P$, releasing the enzyme for another reaction. The notation $A \xrightarrow{k} B$ refers to the elementary system $\dot{b} = ka$, $\dot{a} = -ka$, where $b$ and $a$ are the concentrations of species $A$ and $B$, and $k$ is the rate constant. A binary reaction, such as $A + B \xrightarrow{k} C$, corresponds to the nonlinear system $\dot{c} = -\dot{a} = -\dot{b} = kab$. Note that these elementary reactions have conservation laws that reflect the conversion of one species into another. For example, in the latter case $c(t) + a(t) =$ constant and $c(t) + b(t) =$ constant.

   (a) Convert the Michaelis–Menton reaction into a system of four ODEs for the concentrations $e, s, c,$ and $p$ of the enzyme, substrate, complex, and product, respectively. Each arrow in the reaction diagram above refers to an elementary reaction that adds to the rates.

   (b) There are two conservation laws for your system. Assuming that the initial product, $p(0)$, and complex, $c(0)$, concentrations are zero, these two laws can be thought of as conservation of enzyme, $e(0) = e_o$, and substrate, $s(0) = s_o$. Use these two laws to eliminate $p(t)$ and $e(t)$ from your four equations, leaving a system of two ODEs.

   (c) Define new variables $\tau = k_1 e_o t$, $S = s/K_s$, $C = c/e_o$, where $K_s = (k_{-1}+k_2)/k_1$, and rescale the two equations. Show that they can be written

$$\frac{dS}{d\tau} = -S + (1 - \eta + S)C,$$
$$\varepsilon\frac{dC}{d\tau} = S - (1 + S)C$$

   with the dimensionless parameters $\varepsilon = e_o/K_s$ and $\eta = k_2/(k_{-1} + k_2)$.

   (d) Often the parameter $\varepsilon \ll 1$, which indicates that the complex evolves much more rapidly than the substrate. Consider the limit $\varepsilon = 0$, and reduce your system to a single equation for $S$. The saturating nonlinearity in this ODE is typical of catalytic reactions.

4. A system of point masses that are coupled by harmonic springs is defined by the equations

$$m_i\ddot{x}_i = -k_i(x_i - x_{i+1}) - k_{i-1}(x_i - x_{i-1}), \quad i = 0, \dots, n-1,$$

where $x \in \mathbb{R}$, $x_n \equiv x_o$, $x_{-1} \equiv x_{n-1}$, and $k_{-1} \equiv k_{n-1}$.

   (a) Describe the physical system that these equations model.

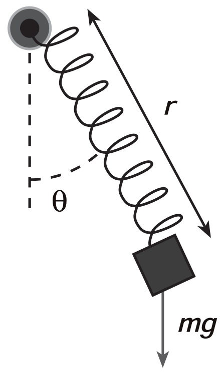


**Figure 1.10.** *Spring-pendulum of Exercise* 5.

(b) Rewrite the system of $n$ second-order equations as a system of $2n$ first-order equations.

(c) Write the system in (b) as a matrix differential equation (see §2.1).

5. The planar spring-pendulum is modeled by the set of equations

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 + mg\cos\theta - k(r-L), \\ r^2\ddot{\theta} &= -2r\dot{r}\dot{\theta} - gr\sin\theta. \end{aligned} \tag{1.35}$$

(a) Describe the physical system (e.g., Figure 1.10) that these equations model and explain each term in the equations.

(b) Define the "angular momentum" by $p_\theta = mr^2\dot{\theta}$ and the radial momentum by $p_r = m\dot{r}$. Rewrite the spring-pendulum system as a set of four first-order ODEs for $x = (r, \theta, p_r, p_\theta)$.

(c) Find the equilibrium solution(s), $x_{eq}$, of the equations, i.e., those solutions for which $x$ is constant.

6. Consider the ABC vector field (1.16).

(a) Show that (1.16) is incompressible: $\nabla \cdot \mathbf{v} = 0$.

(b) Show that (1.16) satisfies the Beltrami property: $\mathbf{v} = \nabla \times \mathbf{v}$.

(c) Show that (1.16) is a solution of the Euler equation

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla P$$

for some suitable pressure $P$. The simplest way to do this is to use the vector identity $\mathbf{v}\cdot\nabla\mathbf{v} = {}^1\!/_2\nabla(\mathbf{v}\cdot\mathbf{v}) - \mathbf{v}\times\nabla\times\mathbf{v}$.

(d) Show that (1.16) is a solution of the Navier–Stokes equations

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = \nu\nabla^2\mathbf{v} + F$$

for some suitable choice of forcing field $F$.

7. The Lotka–Volterra system (1.20) has a number of possible phase portraits depending upon parameters. To investigative these it is first convenient to eliminate as many parameters as possible.

   (a) Rescale time and the variables $x$ and $y$ using the scaling transformations
   $$x = \alpha\xi, \;\; y = \beta\eta, \;\; \text{and} \;\; t = \delta\tau$$
   to obtain the differential equations for the new variables $(\xi(\tau), \eta(\tau))$. Show that the parameters $(\alpha, \beta, \delta)$ can be selected to obtain the simplified model
   $$\begin{aligned}\dot{\xi} &= \xi(1 - \xi - C\eta),\\ \dot{\eta} &= D\eta(1 - E\xi - \eta),\end{aligned}$$
   where $C, D, E > 0$.

   (b) Show there are five distinct possibilities for the nullclines depending upon the values of $C$ and $E$. Sketch the phase portraits for each case.

   (c) Find the set of initial conditions in each case that are asymptotic to each of the equilibria.

8. The principle of competitive exclusion states that if two species occupy the same ecological niche, then one of them will become extinct. For the Lotka–Volterra model (1.20), being in the same "niche" means that $c/b = f/e$, for this implies that the competitive effect of $y$ on $x$ is relatively the same as that of $y$ on itself. (This is the same as $CE = 1$ for the scaling in Exercise 7.) Prove the exclusion principle for the Lotka–Volterra model in this case (with one exceptional value).

9. Derive the Lorenz model (1.33) from the Boussinesq equations (1.30).

   (a) Substitute (1.31) into (1.30) and collect terms with common spatial dependence. Truncate by neglecting all terms that do not depend upon space in the same way as the terms in (1.31) to obtain the three ODEs (1.32).

   (b) Define $x = c_1 A$, $y = c_2 B$, $z = c_3 C$, and $\tau = c_4 t$ to obtain the differential equations for $x(\tau)$, $y(\tau)$, and $z(\tau)$. Choose the constant scaling factors $c_i$ so that the equations simplify to obtain the Lorenz model (1.33).

10. Adopt one of Sprott's quadratic systems from Table 1.1 as your very own ODE model.[6] This model will be referred to in the exercises in each chapter.

   (a) From your variables $(x, y, z)$ and $t$ define a new set of variables $(\xi, \eta, \zeta)$ and $\tau$ using a general scaling transformation
   $$x = \alpha\xi, \;\; y = \beta\eta, \;\; z = \gamma\zeta, \;\; \text{and} \;\; t = \delta\tau$$
   to find a set of differential equations for $(\xi(\tau), \eta(\tau), \zeta(\tau))$ that have the minimum number of parameters. Note that the chain rule gives $\frac{dx}{dt} = \frac{\alpha}{\delta}\frac{d\xi}{d\tau}$, etc. You will need to solve four nonlinear equations to obtain $(\alpha, \beta, \gamma, \delta)$ in terms of

[6] If your system is a single third-order equation, first rewrite it as a system of three first-order equations.

$(a, b, c, \ldots)$ so that all the parameters in your ODEs for $(\xi, \eta, \zeta)$ are 1 except for those listed as "reduced parameters" in the table (keep the same signs in the equations). The nonreduced parameters should be assumed to be nonzero, and in some cases (noted in the table) they may have to be assumed to have a certain sign. Note that the "reduced parameters" will be different from the original ones, e.g., $d \to \hat{d}$.

(b) For your reduced system of ODEs (which you can write as $x, y, z$ again, and drop the "hats" on the reduced parameters) find all the equilibria, i.e., real-valued points $(x, y, z)$ such that $\dot{x} = \dot{y} = \dot{z} = 0$. Is the number of equilibria constant as the (reduced) parameters vary? Do the equilibria ever collide? Discuss.