

Homework #1

Due: April 1 (really..), 2019

1. (Strogatz 2.4) Use linearization to classify the fixed points of the following systems. If linearized stability fails, use graphical/ geometric approach:

$$\dot{x} = ax - x^3 \text{ for all possible values of } a \quad (1)$$

$$\dot{x} = x(1-x)(2-x) \quad (2)$$

$$\dot{x} = x^2(6-x) \quad (3)$$

$$\dot{x} = \ln x \quad (4)$$

$$\dot{x} = 1 - e^{-x^2} \quad (5)$$

2. Consider a Hamiltonian system:

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H(x,p)}{\partial p} \\ \frac{dp}{dt} = -\frac{\partial H(x,p)}{\partial x} \end{cases} \quad (6)$$

- (a) Prove that $\frac{dH}{dt}(x(t), p(t)) = 0$ where $(x(t), p(t))$ denotes a solution to (6).
- (b) What is the dynamical significance of the level sets of the Hamiltonian $H(x, p) = h$?
- (c) Consider the Hamiltonian of a particle in a double well potential:

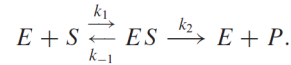
$$H(x, p) = p^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4, \quad (x, p) \in \mathbb{R}^1 \times \mathbb{R}^1$$

draw the level curves of H in the (x, p) plane (the phase space), add time arrows, and explain the different types of motion of the particle.

- (d) Think: how would the above answers change if $H = H(x, p, t)$?

3. (Meiss, pg 24, Q. 3):

3. The Michaelis–Menton mechanism describes the catalysis of a reaction by an enzyme (Michaelis and Menten 1913). The chemical notation for this reaction is



Here the enzyme E combines with the substrate S to make an intermediate complex, ES , that is converted into the product P , releasing the enzyme for another reaction. The notation $A \xrightarrow{k} B$ refers to the elementary system $\dot{b} = ka$, $\dot{a} = -ka$, where b and a are the concentrations of species A and B , and k is the rate constant. A binary reaction, such as $A + B \xrightarrow{k} C$, corresponds to the nonlinear system $\dot{c} = -\dot{a} = -\dot{b} = kab$. Note that these elementary reactions have conservation laws that reflect the conversion of one species into another. For example, in the latter case $c(t) + a(t) = \text{constant}$ and $c(t) + b(t) = \text{constant}$.

- Convert the Michaelis–Menton reaction into a system of four ODEs for the concentrations e , s , c , and p of the enzyme, substrate, complex, and product, respectively. Each arrow in the reaction diagram above refers to an elementary reaction that adds to the rates.
- There are two conservation laws for your system. Assuming that the initial product, $p(0)$, and complex, $c(0)$, concentrations are zero, these two laws can be thought of as conservation of enzyme, $e(0) = e_o$, and substrate, $s(0) = s_o$. Use these two laws to eliminate $p(t)$ and $e(t)$ from your four equations, leaving a system of two ODEs.
- Define new variables $\tau = k_1 e_o t$, $S = s/K_s$, $C = c/e_o$, where $K_s = (k_{-1} + k_2)/k_1$, and rescale the two equations. Show that they can be written

$$\begin{aligned} \frac{dS}{d\tau} &= -S + (1 - \eta + S)C, \\ \varepsilon \frac{dC}{d\tau} &= S - (1 + S)C \end{aligned}$$

with the dimensionless parameters $\varepsilon = e_o/K_s$ and $\eta = k_2/(k_{-1} + k_2)$.

- Often the parameter $\varepsilon \ll 1$, which indicates that the complex evolves much more rapidly than the substrate. Consider the limit $\varepsilon = 0$, and reduce your system to a single equation for S . The saturating nonlinearity in this ODE is typical of catalytic reactions.

4. Bonus I: Read chapters 1+3 of Meiss and solve additional ex. on pg 23-27.

5. Bonus II: find a research paper with a simple model in your field of interest (look for "toy model of .."/ "populations dynamics model of .."/ "bifurcations in ..") and describe the model: what are the dependent and independent variables, what effects were neglected and how was this justified.