Homework #1

Due: April 1 (really..), 2019

1. (Strogatz 2.4) Use linearization to classify the fixed points of the following systems. If linearized stability fails, use graphical/ geometric approach:

$$\dot{x} = ax - x^3 \text{for all possible values of } a \tag{1}$$

$$\dot{x} = x(1-x)(2-x) \tag{2}$$

$$\dot{x} = x^2(6-x) \tag{3}$$

$$\dot{x} = \ln x \tag{4}$$

$$\dot{x} = 1 - e^{-x^2} \tag{5}$$

2. Consider a Hamiltonian system:

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H(x,p)}{\partial p} \\ \frac{dp}{dt} = -\frac{\partial H(x,p)}{\partial x} \end{cases} \tag{6}$$

   (a) Prove that $\frac{dH}{dt}$ $(x(t), p(t)) = 0$ where $(x(t), p(t))$ denotes a solution to (6).

   (b) What is the dynamical significance of the level sets of the Hamiltonian $H(x, p) = h$?

   (c) Consider the Hamiltonian of a particle in a double well potential:

   $$H(x, p) = p^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4, \quad (x, p) \in R^1 \times R^1$$

   draw the level curves of H in the (x, p) plane (the phase space), add time arrows, and explain the different types of motion of the particle.

   (d) Think: how would the above answers change if $H = H(x, p, t)$?

3. (Meiss, pg 24, Q. 3):

3. The Michaelis–Menton mechanism describes the catalysis of a reaction by an enzyme (Michaelis and Menten 1913). The chemical notation for this reaction is

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} ES \xrightarrow{k_2} E + P.$$

Here the enzyme E combines with the substrate S to make an intermediate complex, ES, that is converted into the product P, releasing the enzyme for another reaction. The notation $A \xrightarrow{k} B$ refers to the elementary system $\dot{b} = ka$, $\dot{a} = -ka$, where b and a are the concentrations of species A and B, and k is the rate constant. A binary reaction, such as $A + B \xrightarrow{k} C$, corresponds to the nonlinear system $\dot{c} = -\dot{a} = -\dot{b} = kab$. Note that these elementary reactions have conservation laws that reflect the conversion of one species into another. For example, in the latter case $c(t) + a(t) =$ constant and $c(t) + b(t) =$ constant.

(a) Convert the Michaelis–Menton reaction into a system of four ODEs for the concentrations e, s, c, and p of the enzyme, substrate, complex, and product, respectively. Each arrow in the reaction diagram above refers to an elementary reaction that adds to the rates.

(b) There are two conservation laws for your system. Assuming that the initial product, $p(0)$, and complex, $c(0)$, concentrations are zero, these two laws can be thought of as conservation of enzyme, $e(0) = e_o$, and substrate, $s(0) = s_o$. Use these two laws to eliminate $p(t)$ and $e(t)$ from your four equations, leaving a system of two ODEs.

(c) Define new variables $\tau = k_1 e_o t$, $S = s/K_s$, $C = c/e_o$, where $K_s = (k_{-1}+k_2)/k_1$, and rescale the two equations. Show that they can be written

$$\frac{dS}{d\tau} = -S + (1 - \eta + S)C,$$
$$\varepsilon\frac{dC}{d\tau} = S - (1 + S)C$$

with the dimensionless parameters $\varepsilon = e_o/K_s$ and $\eta = k_2/(k_{-1} + k_2)$.

(d) Often the parameter $\varepsilon \ll 1$, which indicates that the complex evolves much more rapidly than the substrate. Consider the limit $\varepsilon = 0$, and reduce your system to a single equation for S. The saturating nonlinearity in this ODE is typical of catalytic reactions.

4. Bonus I: Read chapters 1+3 of Meiss and solve additional ex. on pg 23-27.

5. Bonus II: find a research paper with a simple model in your field of interest (look for "toy model of .."/ "populations dynamics model of .."/ "bifurcations in ..") and describe the model: what are the dependent and independent variables, what effects where neglected and how was this justified.