Homework Assignment 1

Due date: Sunday, March 22, 2015.

1. Consider the torqued pendulum:

$$\dot{x} = p \qquad \dot{p} = -\omega_0^2 \sin x + b, \qquad (x, p) \in T \times R$$

   (a) Find the Hamiltonian and identify the potential.

   (b) Draw the potential and find the global phase space diagram (schematically).

   (c) Establish that in mechanical systems fixed points are always associated with the potential extremal points.

   (d) Relate, for a one dimensional potential $V(x)$, the type of the extremal points with the stability of the corresponding fixed points (schematically, by geometrical means).

2. Consider the product system:

$$H(x, p) = \sum_{i=1}^{n} (p_i^2/2 + V_i(x_i)), \ (x, p) \in R^n \times R^n. \qquad (1)$$

   Show that each of the individual energies $H_i(x_i, p_i) = p_i^2/2 + V_i(x_i)$ is an integral of motion. List some of the possible solutions of such a system.

3. Execresise 1,2 from Meiss book, Chapter 9

386 Chapter 9. Hamiltonian Dynamics

9.18 Exercises

1. Let $x_i \in \mathbb{R}^3$ represent the positions of a system of N interacting particles with masses m_i and forces that depend only upon the interparticle distances $x_i - x_j$:

$$m_i \ddot{x}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} f(x_j - x_i),$$

   where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is the force.

   (a) Show that the total momentum, $P = \sum_{i=1}^{N} m_i \dot{x}_i$, is an invariant if the force is odd: $f(-x) = -f(x)$.

   (b) Show that the total angular momentum, $L = \sum_{i=1}^{N} m_i \dot{x}_i \times x_i$, is an invariant if the force is directed along the interparticle separation: $f(x) = x g(|x|)$.

2. Show that the system of equations, defining Arnold's ABC flow, (1.16),

$$\dot{x} = A \sin z + C \cos y,$$
$$\dot{y} = B \sin x + A \cos z,$$
$$\dot{z} = C \sin y + B \cos x,$$

   is volume preserving. Show that when $A = 0$ it has an invariant, $\psi(x, y, z) = B \cos x + C \sin y$. Discuss the phase portrait for this case.