Final Exam - Dynamical Systems 2008

Please answer all questions;

Total score is 125 (with a maximum grade of 100).

1. (50 pts) For the below list of models, using **only elementary tools** (dimensionality consideration and simple geometric consideration - no need to construct the phase space diagrams or compute fixed points and their stability) classify the systems as type **(O)**,**(PC)** or **(C)** and explain your choice;

   (O) Ordered: All solutions can have only simple asymptotic behavior - explain what kinds of behaviors are possible for the corresponding class of systems.

   (C) Chaotic - the system has an invariant set on which the motion is chaotic.

   (PC) Possibly chaotic - the system may have complicated and possibly chaotic dynamics or it may be ordered. Further analysis is needed to find out which of the two cases occurs; suggest methods for verifying whether the system is chaotic or ordered.

   (a) $x_{n+1} = 5x_n + 0.1\sin(x_n),\ \ x_n \in \mathbb{R}$

   (b) $x_{n+1} = -0.1x_n + 5\sin(x_n),\ \ x_n \in [0, \pi]$

   (c) $\frac{dx}{dt} = x - y + x^2y^3,\ \frac{dy}{dt} = -y + xy^3$

   (d) $\frac{dx}{dt} = x - y + x^2y^3,\ \frac{dy}{dt} = -y + xy^3,\ \frac{dz}{dt} = -z$

   (e) $\frac{dx}{dt} = x - y + x^2y^3 + z,\ \frac{dy}{dt} = -y + xy^3,\ \frac{dz}{dt} = -z + x^2$

   (f) $\frac{dx}{dt} = x - y + x^2y^3 + 0.1\sin(t),\ \frac{dy}{dt} = -y + xy^3$

   (g) $\frac{dx}{dt} = y,\ \frac{dy}{dt} = x - x^3 + 0.1\sin(t)$

   (h) $x_{n+1} = 2x_n + y_n,\ \ y_{n+1} = 2x_n - y_n$

   (i) $x_{n+1} = 2x_n + y_n,\ \ y_{n+1} = 2x_n^2 - y_n$

   (j) $x_{n+1} = 4x_n \bmod 1.$

2. (25 pts) For one of the systems that you have classified as chaotic, provide a geometrical and symbolic dynamics description of the invariant set and the dynamics on it (i.e. explain how the invariant set is constructed by geometrical means, explain how you code solutions on it with symbolic dynamics, outline the proof that the dynamics on the invariant set is chaotic, and provide an example of an orbit exhibiting a nontrivial behavior on the invariant set).

3. (50pts) Consider the forced and damped motion of a particle in a slightly asymmetric double well potential (so $0 < a \ll 1$, $\delta \geq 0$, $\varepsilon \geq 0$):

$$\frac{dx}{dt} = p$$
$$\frac{dp}{dt} = x - ax^2 - x^3 + \varepsilon \sin \omega t - \delta p$$

(a) For $\delta = \varepsilon = 0$ show that this unperturbed system is Hamiltonian, find the Hamiltonian function $H(x, p)$, draw schematically the potential and the level curves of H in the (x, p) plane (the phase space) and explain what are the possible types of motion the particle may exhibit.

(b) For $\varepsilon = 0$ (and $\delta \geq 0$) find the fixed points and their stability.

(c) For $\varepsilon = 0$, $\delta > 0$ and x **sufficiently large**, find a ν such that the function $V(x, p) = H(x, p) - \nu xp$ is a Lyapunov function. Explain and draw the implications on the dynamics.

(d) Find the form of the Melinikov functions $M_{L,R}(t_0; a, \omega, \delta)$; Denote by $q_{L,R}(t; a) = (x_{L,R}(t; a), p_{L,R}(t; a))$ the left/right homoclinic orbit of the unperturbed system satisfying $p_{L,R}(0; a) = 0$. "Compute" the Melnikov function by using the solutions symmetries up to the unknown, yet computable expressions (denote such expressions by parameter dependent functions, e.g. $C_{L,R}(a) = \int_{-\infty}^{\infty} (p_{L,R}(t; a))^2 \, dt$ etc.).

(e) Assuming that all the parameter-dependent functions are positive for some fixed a, ω values, find for what values of (ε, δ) the Melnikov functions may have zeroes and draw schematically the corresponding phase portraits in the Poincaré map. Draw schematically the bifurcation diagram in the (ε, δ) space for $a = 0$ and for a fixed $a > 0$ and explain the qualitative difference between these two cases.