Final Exam
Nonlinear dynamics and chaos
Fall 2010

Your name here: ________________________________.

Instructions: You may solve all five questions. Note that the credit for all questions sums to 155 points, so you get additional credit for solving more than the number of questions needed for a 100 grade.

Duration: please plan to finish and hand your exam in 3 hours.

Good luck!

1. (50 pts) For the below list of models, using **only elementary tools** (dimensionality consideration and simple geometric consideration - no need to construct the phase space diagrams or compute fixed points and their stability) classify the systems as type **(O)**,**(PC)** or **(C)** and explain your choice shortly;

   (O) Ordered: All solutions can have only simple asymptotic behavior - explain what kinds of behaviors are possible for the corresponding class of systems.

   (C) Chaotic: the system has an invariant set on which the motion is chaotic.

   (PC) Possibly chaotic: the system may have complicated and possibly chaotic dynamics or it may be ordered. Further analysis is needed to find out which of the two cases occurs; suggest methods for verifying whether the system is chaotic or ordered.

   (a) $\frac{dx}{dt} = 2x - y + x^2y^3$, $\frac{dy}{dt} = -y + xy^3$

   (b) i. $\frac{dx}{dt} = x - y + x^2y^3$, $\frac{dy}{dt} = -y + xy^3$, $\frac{dz}{dt} = -z$

   ii. $\frac{dx}{dt} = 4x - y + x^2y^3 + z$, $\frac{dy}{dt} = -y + xy^3$, $\frac{dz}{dt} = -z + x^2$

   iii. $\frac{dx}{dt} = x - y + x^2y^3 + 0.1\sin(t)$, $\frac{dy}{dt} = -y + xy^3$

   (c) i. $x_{n+1} = 5x_n + 0.1\sin(x_n)$, $x_n \in \mathbb{R}$

   ii. $x_{n+1} = -0.1x_n + 5\sin(x_n)$, $x_n \in [0, \pi]$

   (d) $\frac{dx}{dt} = p$, $\frac{dp}{dt} = x - x^3 + 0.1\sin(t)$

(e) i. $x_{n+1} = 5x_n + y_n + z_n$, $y_{n+1} = 2x_n - y_n$, $z_{n+1} = -3z_n + x_n - y_n$

ii. $x_{n+1} = 2x_n + y_n$, $y_{n+1} = 2x_n^2 - y_n$

(f) $x_{n+1} = 4x_n \mod 1, x_n \in [0, 1]$

2. (30) The following system describes the approximate dynamics in the blood of the neutrophils N (one of the main components of the white blood cells) and the growth factor G for a time scale of a few days:

$$\frac{dN}{dt} = a\frac{1 + \epsilon G}{1 + \epsilon G/10} - N \tag{1}$$

$$\frac{dG}{dt} = \frac{1}{1 + \epsilon N} - (1 + \frac{N}{N + \epsilon})G \tag{2}$$

(a) Find the form of the nullclines of the system in the positive quadrant (use the fact that $\epsilon \ll 1$, notice the behavior at large G and at large N).

(b) Find the fixed points and their stability.

(c) What values should the parameter a assume so that the equilibrium fixed point will have a neutrophil level beyond the critical value $N_c = 0.1$ at which the risk of infections dramatically increases?

(d) To help patients with very small a, one can supply G by infusion. We can model this infusion by adding a constant term c to the right hand side of the G equation (you can assume that $c \gg 1$). When such a treatment works?(namely, when can you find a c such that the fixed point has $N > N_c$ and how does this value depend on a).

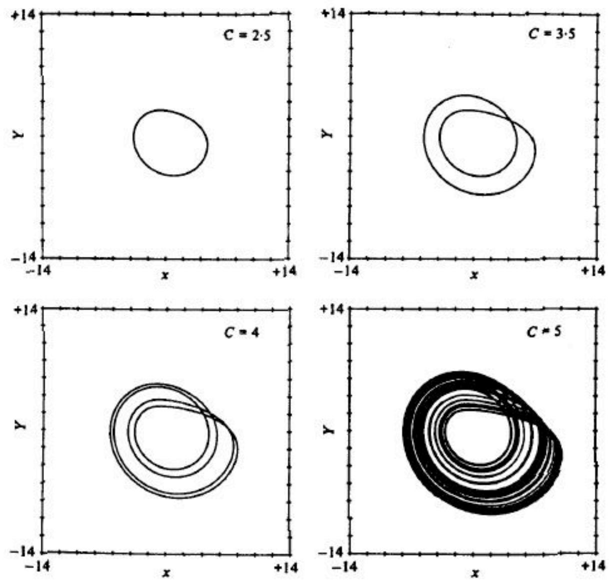
3. (30 pts) Analyze the fixed points, bifurcations, and stability of the following systems. Find the location of the bifurcation point(s!), find the value of the bifurcation parameter r at the bifurcation point(s); plot the vector field in the (r, x) plane; indicate what type of bifurcation(s) is/are involved; mark the stable and unstable branches explicitly.

(a) $\dot{x} = r(x - 1) - \ln(x)$

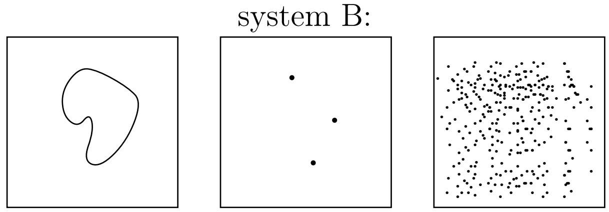
(b) $\dot{x} = r + x/2 - x/(1 + x)$

4. (30 pts) Experimental time series from two different systems were plotted in reconstructed phase space as a function of some parameter C. The values of C are indicated in the plots for system A and are equal to three values $C1, C2, C3$ in system B. The results are shown in the following two figures.

System A:



system B:



C1 C2 C3

answer the following questions:

(a) (i) How would you plot an experimental time series in a reconstructed phase space. (ii) What specific issues would you need to pay attention to when reconstructing phase space in order to maximize the information obtained from such a plot. (iii) What exactly was plotted in reconstructed phase space for system B?

(*Hint:* Note that one sees only 3 dots in the reconstructed phase space plot for the C2 case in system B).

(b) Which routes to chaos do these two systems undergo?

(c) What would you expect to see in a time series from these two systems as the same parameter is varied?

(d) What would you expect to see in the spectrum of the two time series as the same parameter is varied?

(e) For each system, what simple map describes the same route to chaos? Write the expression for the appropriate maps.

5. (15 pt) Consider the fractal set whose first step of construction is illustrated in the Figure. What is the box-counting dimension of the set?