Final Exam
Dynamical Systems and modeling
Fall 2011

Your name here: ______________________________.

Duration: The exam is planned for at most 2 hours, yet, it is not a competition on speed and I will allow extensions if needed.

Good luck!

1. (40 pts) For the below list of models, using **only elementary tools** (dimensionality consideration and simple geometric consideration - no need to construct the phase space diagrams or compute fixed points and their stability) classify the systems as type **(O)**,**(PC)** or **(C)** and explain your choice shortly (1-3 lines);

   (O) Ordered: All solutions for all parameter values can have only simple asymptotic behavior - explain what kinds of behaviors are possible for the corresponding class of systems.

   (C) Chaotic: For some parameter values the system has an invariant set on which the motion is chaotic (if possible, indicate when such a behavior is expected in terms of parameters and initial conditions).

   (PC) Possibly chaotic: the system may have complicated and possibly chaotic dynamics or it may be ordered. Further analysis is needed to find out which of the two cases occurs; suggest methods for verifying whether the system is chaotic or ordered.

   In all below items x, y, z are in $\mathbb{R}^1$ and $n \in \mathbb{Z}$ (integer).

   (a) $\frac{dx}{dt} = 2x - y$, $\frac{dy}{dt} = -y + 3x$, $\frac{dz}{dt} = z - y + 5x$

   (b) $\frac{dx}{dt} = x - y + x^2y^3$, $\frac{dy}{dt} = -y + xy^3$, $\frac{dz}{dt} = -z$

   (c) $\frac{dx}{dt} = 4x - y + x^2y^3 + z$, $\frac{dy}{dt} = -y + xy^3$, $\frac{dz}{dt} = -z + x^2$

   (d) $\frac{dx}{dt} = -x + z + \epsilon xy$, $\frac{dy}{dt} = -y + \epsilon zy^3$, $\frac{dz}{dt} = -z + \epsilon xy^3$, $|\epsilon| \ll 1$

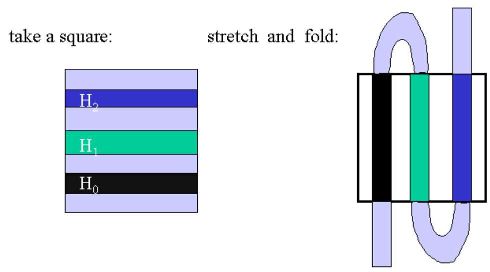
   (e) $\frac{dx}{dt} = p$, $\frac{dp}{dt} = x - x^3 + 0.1\sin(t)$

   (f) $x_{n+1} = 5x_n + y_n + z_n$, $y_{n+1} = 2x_n - y_n$, $z_{n+1} = -3z_n + x_n - y_n$

(g) $x_{n+1} = 2x_n + y_n$, $y_{n+1} = 2x_n^2 - y_n$

(h) $\frac{dx}{dt} = -x + x^2 + 5\sin x$

2. (30 pts) Consider the following 3-fold horseshoe map:



(a) Construct its invariant set and its corresponding symbolic sequence.

(b) How many fixed points does it have? What are their symbolic sequence? Where are they located in the square (draw them).

(c) Repeat b for the period 2 periodic orbits.

(d) Is the map sensitive to i.c. ? Why (sketch a proof)?

3. (30 pts) Analyze the fixed points, bifurcations, and stability of the following model:

$$\dot{x} = r(x-1) - \frac{(x-1)^3}{1+x}, \quad r, x \geqslant 0$$

Find the location of the bifurcation points, find the value of the bifurcation parameter r at the bifurcation points; plot the vector field in the (r, x) plane; indicate what type of bifurcations is/are involved; mark the stable and unstable branches explicitly.