Homework Assignment 3

Due date: Sunday, April 26.

1. Meiss, Ch 6, Exc. 5,6:

   5. Suppose a flow φ has a reversor S and an orbit $\Gamma = \{\varphi_t(x) : t \in \mathbb{R}\}$.

      (a) Show that $\tilde{\Gamma} = \{S \circ \varphi_{-t}(x) : t \in \mathbb{R}\}$ is also an orbit of φ.

      (b) Show that the saddle equilibria of (6.28) are a symmetry-related pair.

      (c) Suppose the orbit Γ is symmetric: $\Gamma \cap \mathrm{Fix}(S) \neq \emptyset$. Show that Γ and $\tilde{\Gamma}$ coincide.

      (d) Suppose γ is a symmetric periodic orbit of φ. Show that γ has at least two points on $\mathrm{Fix}(S)$.

      (e) Suppose that x^* is a symmetric equilibrium. Show that $S(W^s(x^*)) = W^u(x^*)$.

   6. (a) Show that if x^* is a symmetric equilibrium of a reversible system, then whenever λ is an eigenvalue of the linearization at x^*, so is $-\lambda$.

      (b) Suppose $x \in \mathbb{R}^3$. Using the result of (a), find the most general form of the characteristic polynomial of any symmetric equilibrium.

      (c) Show that the three-dimensional system

      $$\begin{aligned} \dot{x} &= y + bz + ax(y - bz), \\ \dot{y} &= cx + x^2 + 2yz, \\ \dot{z} &= b^{-1}\left(cx - x^2 - 2yz\right) \end{aligned}$$

      is reversible with the reversor $S(x, y, z) = (-x, bz, b^{-1}y)$.

      (d) Find the fixed sets of S. Are there symmetric equilibria? Verify the eigenvalue property from (a) for each symmetric equilibrium.

2. What are the rescaling transformations that preserve the canonical Hamiltonian structure of an n d.o.f. Hamiltonian system? How does it influence the rescaling in time. Provide an example.

3. Consider the forced Duffing's oscillator:

   $$\dot{x} = ap, \quad \dot{p} = bx - cx^3 + \epsilon \sin \omega t \tag{1}$$

   (a) Write the Hamiltonian of this system. Rescale the equation, keeping its canonical structure. Find the minimal number of parameters needed to fully analyze the system. Write the Hamiltonian of the rescaled coordinates and explain the relation to the rescaling of time. Draw the phase portrait for $\epsilon = 0$.

   (b) Find the symmetries of the system for $\epsilon = 0$ and for $\epsilon \neq 0$. Conclude what are the symmetries expected from the stable and unstable manifolds. Which of these symmetries are preserved if one takes into account a damping term (i.e. adding a $(-\delta p)$ term to the second term of the forcing)?