Homework Assignment 4

Due date: Sunday, May 31.

1. From Meiss book:

   8.15. Exercises 331

   22. Find a formula that generalizes (8.89) for the Melnikov function $M(\theta)$ when $\mathrm{tr}(Df) \neq 0$.

   23. Why is the Melnikov function (8.89) independent of t?

2. Find a paper that computes the Melnikov function or one of its generalizations (use for example "MathSciNet" or "Google Scholar" to search for "Melnikov +" your favorite application. Recommended journals: Physica D, SIAM Dynamical Systems, Chaos, Phys. Rev. series). Write what are the unperturbed and perturbed equations that are considered and repeat/complete the Melnikov function calculation. Explain briefly how the authors used this calculation in their paper (optional: would you draw similar conclusions? any additional insights?)

3. Consider the asymmetrically forced Duffing's oscillator:

$$\dot{x} = p, \quad \dot{p} = x - x^3 + \epsilon\gamma x(1 - \beta x)\sin\omega t - \epsilon\delta p \tag{1}$$

   (a) Write the Hamiltonian of the perturbed system when $\delta = 0$.

   (b) Compute the form of the Melnikov function for this system (here, with some calculations, the analytic form of all terms may be found - you may challenge yourself and compute them/ look up in the literature to find them/ write the expression for the Melnikov function in a form that depends at most on some integrals that may depend on ω).

   (c) For given $\omega, \beta > 0$, draw the homoclinic bifurcation curves in the (γ, δ) space and draw schematically the manifolds of the Poincare map at each regime.

   (d) Bonus: Find the symmetries of the Poincare map at $\theta = 0$ for $\epsilon \neq 0, \beta = 0$. Conclude what are the symmetries expected from the stable and unstable manifolds under these conditions.

   (e) Bonus: Write a code that computes these manifolds.

4. Bonus question (for those who want to play with maps): Consider the Hénon map:

$$x_{n+1} = a + by_n - x_n^2$$
$$y_{n+1} = x_n$$

(a) Find its fixed points and their stability (as a function of the parameters a,b).

(b) Draw, in the parameter space ((a, b) plane) the bifurcation curves of this map (recall that for maps bifurcations occur when eigenvalues cross the unit circle). In particular, denote the saddle-node bifurcation curve by $a_0(b)$ (you may use the program henon.m to check your results).

(c) Let R be the larger root of:

$$\rho^2 - (|b| + 1)\rho - a = 0.$$

Denote by S the square centered at the origin with vertices $(\pm R, \pm R)$. Examine the image of the square S under the Henon map. Show that there exists a curve $a_2(b)$ such that for $a > a_2(b)$ the image of S under the Hénon map creates a horseshoe map.

(d) Play with the Henon map simulation, henon.m which also plots the stable and unstable manifolds of the saddle point. Can you find the famous Henon attractor? Examine its structure and the structure of the manifolds.

(e) Follow the rest of section 2.9 in Devaney's book "An introduction to Chaotic Dynamical Systems" and prove that the stretching and contraction conditions are indeed satisfied for $a > a_2(b)$, so that the dynamics of the Hénon map in this regime on S is the same as that of the horseshoe map.

(f) Compare the method for computing the manifolds in henon.m with the straight forward techniques to find the manifolds (iterating points placed in the direction of the stable/unstable manifold). What is the advantage/disadvantage of this algorithm and what are possible ways to improve it?

(g) Can you follow the lobes numerically?