Homework #2

Due: November 21, 2012

1. Consider a populations that grows according to the logistic law with constant harvesting:

$$\frac{dN}{dt} = rN(1 - N/k) - h \qquad x, t \in \mathbb{R}^+ \tag{1}$$

   (a) Determine the dimension of the parameters and their biological interpretation.

   (b) Non-dimensionalise the system and rescale it to obtain an equation for the non-dimensional population concentration u depending on a single parameter $\mu = \mu(r, k, h)$ of the effective harvesting rate (verify by the Pi theorem that indeed a single non-dimensional parameter should emerge).

   (c) Can we have distinct biological situations having the same control parameter? If so, explain the consequence of this observation on designing experiments.

   (d) Find the bifurcation diagram and discuss its biological implications on the harvesting strategy. Find the parameter value at the bifurcation point μ_b. Which kind of bifurcation occurs at μ_b?

   (e) Discuss the biological implications if the harvesting rate is allowed to change slowly with time: can we have an outbreak? hysteresis? extinction?

   (f) Simulate the dynamics:

      i. Fixed parameter values: choose four values of $\mu : \mu_1 = 0, \mu_2 = \mu_b/2, \mu_3 = \mu_b, \mu_4 = 2\mu_b$, and several typical initial conditions and plot time plots $((t, u(t))$ for the four cases. Check consistency with the bifurcation diagram.

      ii. Varied harvesting rate: allow μ to vary slowly with t and demonstrate numerically the non-trivial phenomenon predicted in (e).

      iii. **Bonus:** Replace the constant harvesting term with an harvesting function $hH(N)$ (that may depend on additional parameters), and define the yield till time T to be $Y(T) = h\int_0^T H(N(t); h)dt$. What are the conditions on H so that no extinction is possible? Alternatively, what are the conditions on H so that the bifurcation diagram persist? Can you optimize the yield (assume, for example, that H is linear in N)? Can seasonal changes in the parameters may help in increasing the yield while avoiding extinction? Demonstrate your claims numerically.

2. Consider a general one dimensional family of maps $F_\mu(x) : R \to R$. Find conditions under which $F_\mu(x)$ undergoes a saddle-node bifurcation at some value $(x,\mu) = (x^*,\mu^*)$: define $G(x,\mu) = F_\mu(x) - x$, and using the implicit function theorem find conditions under which $G(x,\mu) = 0$ has a unique parabola like solution $G(x,\mu(x))$ for $\mu > \mu^*$ (and $|\mu - \mu^*|$ and $|x - x^*|$ small).

3. Show that the flow:
$$f_\mu : \dot{x} = \mu x - x^2 \qquad x \in R$$
undergoes a transcritical bifurcation at $(x,\mu) = (0,0)$; show that the stability of the two fixed points is interchanged at the origin. Draw the bifurcation diagram for f_μ. Such a bifurcation occurs when symmetry/modeling constraints imply that $x = 0$ is always a solution.

   Bonus: You may want to verify that by breaking this symmetry assumption we are back to the saddle-node case. You may want to derive general conditions for a flow f_μ having this symmetry to have a transcritical bifurcation. Discuss the difference between the two cases $f_\mu^\pm : \dot{x} = \mu x \pm x^2, x \in R$.