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Instabilities and degeneracies of the four-dimensional stochastic web

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Abstract. We discover a fundamental mechanism for the sharp increase in the diffusion rate of the motion of a charged particle in a uniform magnetic field when a time-periodic electric-field propagating orthogonally to the magnetic field is slightly tilted. It is associated with the degeneracies of the underlying motion; in fact, a two-degree of freedom mechanism of instability induces this behaviour. Unfolding of this degeneracy enables the analysis of the separatrix splitting in some limits, giving a power-law rather than the typical exponential estimate of the splitting.

AMS classification scheme numbers: 70K50, 70K05, 58F05, 58F13, 58F14, 34C37

1. Introduction

The motion of a particle in a unidirectional magnetic field and a field of an infinitely broad electrostatic propagating wavepacket has been suggested as a fundamental model for understanding the motion of charged particles in plasma [1–3]. The particle motion is two dimensional when the wavepacket propagates orthogonally to the magnetic field and is three dimensional in the oblique propagation case. The resulting motion can be modelled by a symplectic mapping—two dimensional for the transverse case and four dimensional for the oblique case. If the magnetic field is uniform and a resonance condition between the Larmor frequency and the wavepacket frequency is satisfied, a surprising phenomena occurs; an infinite stochastic web emerges, leading to the existence of unbounded motion of the particle [4, 3].

In the two-dimensional case, the width of the stochastic web is known to be asymptotically exponentially small in the non-dimensional parameter K [4–7] (K is proportional to the amplitude of the transverse component of the electrostatic wave). Thus the probability of a particle belonging to the instability zone is exponentially small in K.

It has been numerically observed that even the slightest inclusion of an oblique component of the electrostatic wave to the model (resulting in a four-dimensional symplectic map) leads to a sharp increase of the diffusion rate and of the measure of the chaotic component of the phase space [3]. One may speculate that this phenomena is simply associated with the addition of another degree of freedom to the system. Here, we show

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that in this case the degeneracies of the model, which seems to be associated with its symmetries, play a crucial role in creating these new instabilities. Picking a specific limit for the degeneracy's unfolding parameters as a function of the small parameter K, enables us to prove a polynomial rather than an exponential splitting distance.

This paper is organized as follows; in section 2 we follow the derivation of Zaslavsky *et al* [3] of the equations of motion, concentrating on the four-fold symmetric case. Then, we suggest a new model which includes two new unfolding parameters, A and θ. We examine the qualitative behaviour of the system—its symmetries and the structure of the energy surfaces as these parameters are varied. In section 3 we describe the behaviour of the system in the limit of fast Larmor rotation, find the solutions of the integrable limit and define the splitting integral and estimate it for $\theta = 0$ in the limit of small A and small K so that averaging may be utilized. Section 4 is devoted to our conclusions and a discussion.

2. The equations of motion

The equations of motion of a particle with charge Q and mass m in the three-dimensional space under the influence of a constant magnetic field and a time-periodic electric field corresponds to a 3.5 degrees of freedom system of the form [3]:

$$\ddot{\boldsymbol{r}} = \frac{Q}{m}\boldsymbol{E}(\boldsymbol{r}, t) + \frac{Q}{mc}[\dot{\boldsymbol{r}}, \boldsymbol{B}] \qquad \boldsymbol{r} \in \mathbb{R}^3 \tag{2.1}$$

where $\boldsymbol{B}$ is along the z-axis. The electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ lies in the x, z plane and is chosen in the following form:

$$\boldsymbol{E} = -\boldsymbol{E}_0 \sum_{n=-\infty}^{\infty} \sin(\boldsymbol{kr} - n\triangle\omega t) = -\boldsymbol{E}_0 \sin(k_x x + k_z z)T \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2.2}$$

namely, the wavepacket is assumed to be homogeneous and of sufficiently large spectral width. The time interval $T = 2\pi/\triangle\omega$ determines the frequency interval between harmonics of the packet. Potential character of the electric field implies that $\frac{k_z}{k_x} = \frac{E_{0z}}{E_{0x}} = \beta$. Thus, equation (2.1) written in components becomes:

$$\begin{aligned}
\ddot{x} &= -\frac{Q}{m} T E_{0x} \sin(k_x x + k_z z) \sum_{n=-\infty}^{\infty} \delta(t - nT) + \omega_0 \dot{y} \\
\ddot{y} &= -\omega_0 \dot{x} \\
\ddot{z} &= -\frac{Q}{m} T E_{0z} \sin(k_x x + k_z z) \sum_{n=-\infty}^{\infty} \delta(t - nT)
\end{aligned} \tag{2.3}$$

where $\omega_0 = \frac{QB}{mc}$ is the Larmor cyclotron frequency. Equations (2.3) do not depend on y, hence, there exists a conserved quantity—the corresponding generalized momentum. Indeed, the second equation in (2.3) can be integrated yielding

$$\dot{y} + \omega_0 x = \text{constant} = \omega_0 x_c / k_x \tag{2.4}$$

where x_c is the coordinate of the centre of the cyclotron orbit [8]. Note that *if* $k_z \neq 0$ this constant can be set to 0 by a corresponding shift in z: setting $\bar{x} = x - x_c/k_x$ and $\bar{z} = z + x_c/k_z$ corresponds to taking $\bar{x}_c = 0$ without changing the equations of motion (however, see [8, 9, 7] for the non-trivial influence of x_c when $k_z = 0$).

The existence of the conserved quantity enables us to reduce the number of degrees of freedom by 1. Thus, this model corresponds to a 2.5 degrees of freedom Hamiltonian system defined by

$$\mathcal{H} = \frac{p_x^2 + p_z^2}{2m} + \frac{m}{2}\omega_0^2 x^2 - QT\phi_0 \cos(k_x x + k_z z) \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2.5}$$

where p_x and p_z are kinetic momenta and $\phi_0 = E_{0x}/k_x$ is the amplitude of the electric potential. Defining the dimensionless variables (u, v, w, Z) by

$$u = k_x \dot{x}/\omega_0 \qquad v = -k_x x \qquad Z = k_z z \qquad w = k_z \dot{z}/\omega_0 \tag{2.6}$$

the Poincaré map corresponding to the flow of the Hamiltonian (2.5) can be written as follows:

$$\begin{cases} u_{n+1} = v_n \sin\alpha + (u_n + K\sin(v_n - Z_n))\cos\alpha \\ v_{n+1} = v_n \cos\alpha - (u_n + K\sin(v_n - Z_n))\sin\alpha \\ Z_{n+1} = Z_n + \pi w_{n+1}/2 \\ w_{n+1} = w_n + K\beta^2 \sin(v_n - Z_n) \end{cases} \tag{2.7}$$

where the cross section is chosen at $nT - 0$, and

$$K = \frac{QE_{0x}k_x}{m\omega_0} T \qquad \beta = \frac{k_z}{k_x} \qquad \alpha = \omega_0 T = \frac{QB}{mc} T. \tag{2.8}$$

If the resonant condition between the eigenfrequency ω_0 and the frequency of the perturbation is satisfied, i.e. $\alpha = 2\pi/q$, where q is some integer, then the map (2.7) generates an infinite stochastic web in phase space. Zaslavsky *et al* [3] analysed in detail the mapping (2.7). A resonant Hamiltonian was obtained by reordering the infinite sum of the δ functions in tuples of q, corresponding heuristically to some averaging procedures. Then, the resulting Hamiltonian was separated into a mean time-independent part and a time-dependent perturbation.

One of the most interesting cases, which will be considered henceforth, is the case of the fourfold symmetry, when $q = 4$. In this case, the heuristically derived mean Hamiltonian was found to be integrable [3]. For $q = 4$, (2.7) has the form

$$\begin{cases} u_{n+1} = v_n \\ v_{n+1} = -u_n - K\sin(v_n - Z_n) \\ Z_{n+1} = Z_n + \pi w_{n+1}/2 \\ w_{n+1} = w_n + K\beta^2 \sin(v_n - Z_n). \end{cases} \tag{2.9}$$

The parameter β, which couples the (u, v) dynamics to the (Z, w) dynamics, plays a critical role in determining the diffusion rate in the u, v plane [3]. Even very small β values lead to a vast increase in the instability (hence 'diffusion') of the particles motion.

In section 3.1, it is shown that the map (2.9) is highly degenerate in the limit of small K. We introduce two parameters which unfold the degeneracy and break some of the symmetries of the map (see also the discussion section). Based on this mathematical consideration, we suggest a new model, in which we add a second electrostatic wave, $\boldsymbol{E}_1$, which is parallel to the magnetic field $\boldsymbol{B}$, so that (2.2) changes to:

$$\boldsymbol{E} = -(\boldsymbol{E}_0 \sin(k_x x + k_z z) + \boldsymbol{E}_1 \sin(k_z z + \theta))T \sum_{n=-\infty}^{\infty} \delta(t - nT). \tag{2.10}$$

Here θ plays the role of the phase shift introduced by the initial cyclotron centre position x_c, $\theta = x_c/\beta$, and here it cannot be omitted by translation of z (see the remark after (2.4)). The resulting Hamiltonian of the system is:

$$\mathcal{H} = \frac{p_x^2 + p_z^2}{2m} + \frac{m}{2}\omega_0^2 x^2 - qT(\phi_0 \cos(k_x x + k_z z) + \phi_1 \cos(k_z z + \theta)) \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2.11}$$

and the corresponding Poincaré map for the case $q = 4$ can be written as

$$\mathcal{F}_4 : \begin{cases} u_{n+1} = v_n \\ v_{n+1} = -u_n - K\sin(v_n - Z_n) \\ Z_{n+1} = Z_n + \pi w_{n+1}/2 \\ w_{n+1} = w_n + K\beta^2(\sin(v_n - Z_n) - A\sin(Z_n + \theta)) \end{cases} \tag{2.12}$$

where $A = E_{1z}/E_{0z}$ is the ratio of the amplitudes of the electrostatic waves. Symmetries of this map depend on the value of the parameters A and θ. For $\theta = 0$ (or $A = 0$) there is an exact Z_2 symmetry

$$Z_2 : (u, v, w, Z) \to -(u, v, w, Z). \tag{2.13}$$

For $A = 0$ (and arbitrary value of θ) there is a shift symmetry in u, v, Z:

$$(u, v, Z, w) \to (u + \pi, v + \pi, Z + \pi, w) \tag{2.14}$$

which introduces a 2π shift in v: in the new variables the right-hand side of the second equation of the map (2.12) has an additional $+2\pi$ term. Introducing a non-zero A breaks the shift symmetry (2.14) and introducing a non-zero θ breaks the Z_2 symmetry (2.13). In the analysis, we consider the unfolding with respect to A, taking $\theta = 0$. We leave the detailed (and, as will be apparent later on, the more complicated) analysis of the influence of the unfolding with respect to θ to future research.

For asymptotically small K there are various limits and cases for which the mapping (2.12) may be analysed. First, if $\beta = 0$ and $w = 0$, (2.12) reduces to 1.5 degrees of freedom system corresponding to a transverse propagation of the wave, where the initial value of Z plays the role of the cyclotron centre. In [7] we have carefully analysed the structure and properties of the stochastic web for a general fixed Z (x_c in [7]), showing that the reduced system depends very sensitively on the value of Z.

If $\beta = 0$ and w is rational, i.e. $w = m/n$, then each iteration of $\mathcal{F}_4$ may introduce a substantial change in Z. However, the 4nth iterate of $\mathcal{F}_4$ leaves Z unchanged and produces a near identity two-dimensional diffeomorphism in the (u, v) plane. It may be studied in the limit $K \to 0$ with similar techniques to those used in [7]. Further information may be obtained by expanding $\mathcal{F}_4^{4n}$ in w near its rational value $w = m/n + \triangle w$, and consider the limit $(K, \triangle w \to 0)$. From the physical point of view this case corresponds to the cyclotron resonance between the frequencies of longitudinal and transversal motions. The case $m = n = 1$ has been studied in [3].

For small values of w, the change in Z is small at each iteration, and thus the fourth iterate $\mathcal{F}_4^4$ produces a *near identity* symplectic diffeomorphism. Physically, this case corresponds to a fast Larmor rotation with respect to the longitudinal motion. As for the two-dimensional case, considering $\mathcal{F}_4^4$ corresponds heuristically to averaging with respect to this fast rotation [4].

In the rest of this paper the last case will be analysed. The resulting approximating flow will be considered as some perturbation of an integrable 2 degrees of freedom Hamiltonian system. Analysis of the energy surfaces reveals peculiarities and degeneracies of the unperturbed motion. Further, Melnikov-type analysis for the limiting flow and averaging results in estimates for the separatrix splittings.

3. The limit of fast Larmor rotation

Here we analyse the limit of the fast Larmor rotation:

$$w \ll 1 \qquad K \ll 1. \tag{3.1}$$

Which, using the dimensional parameters, corresponds to

$$\omega_0 \gg \max(k_z \dot{z}, (Q\phi_0 k_x^2/m)^{1/2}). \tag{3.2}$$

This implies $\Omega_\perp \ll 1$, where $\Omega_\perp = (Q\phi_0 k_x^2/m)^{1/2}$ is the transverse bounce frequency corresponding to the frequency of small oscillations of a particle in the field of the plane wave with amplitude ϕ_0 [3]. Thus, we rescale $w = \delta W$, where the new small parameter $\delta \equiv \sqrt{K\beta^2}$ has been defined, and W is of the order of 1. After expanding the fourth iterate of $\mathcal{F}_4$ in K and δ, $\mathcal{F}_4^4$ is given, to the first order in these small parameters, by:

$$\begin{cases} u' = u + 2K\cos Z \sin v \\ v' = v - 2K \cos Z \sin u \\ Z' = Z + 2\pi W\delta, \\ W' = W - 2\delta((\cos u + \cos v)\sin Z + 2A\sin(Z+\theta)) \end{cases} \tag{3.3}$$

where Z is taken mod 2π.

3.1. The limiting flow and its degeneracies

The mapping (3.3) is a near-identity analytic mapping, which, in the limit of asymptotically small K and δ, can be approximated by a flow of the following autonomous Hamiltonian

$$\mathcal{H} = \frac{K\pi}{4}W^2 - \frac{K}{2}(\cos Z(\cos u + \cos v) + 2A\cos(Z+\theta)) \tag{3.4}$$

with the corresponding Hamiltonian equations:

$$\begin{aligned} \dot{u} &= \frac{\partial \mathcal{H}}{\partial v} = \frac{1}{2}K\cos Z \sin v \\ \dot{v} &= -\frac{\partial \mathcal{H}}{\partial u} = -\frac{1}{2}K\cos Z\sin u \\ \dot{Z} &= \frac{\delta}{K}\frac{\partial \mathcal{H}}{\partial W} = \frac{\delta\pi}{2}W \\ \dot{W} &= -\frac{\delta}{K}\frac{\partial\mathcal{H}}{\partial Z} = -\frac{\delta}{2}((\cos u + \cos v)\sin Z + 2A\sin(Z+\theta)). \end{aligned} \tag{3.5}$$

Note that W and Z are not canonical variables as the scaling factor δ/K is included in the above equations, i.e. in the corresponding non-canonical Poisson brackets. We believe the expression (3.5) corresponds to the *rigorous* limiting flow which is realized in the singular limit $K, \delta \to 0$. Namely, that there exists a flow for which the time-1 map is given by (3.3) to the order of $K\delta$, K^2, and (3.5) is the leading-order approximation to this flow. Moreover, normally hyperbolic sets of the flow and their local stable and unstable manifolds persist and are C^r close (for C^r mapping) to the corresponding sets and manifolds of the mapping (see [10] for the exact formulation in the two-dimensional case). Heuristically, (3.4) corresponds to the Hamiltonian (2.11) expressed in dimensionless variables (2.6) and averaged over the period of the cyclotron rotation; thus it can be called the mean Hamiltonian, and indeed, for $A = \theta = 0$ it coincides with the mean Hamiltonian found in [3].

The Hamiltonian system (3.5) is integrable and possesses a second integral of motion $C(u, v) = \cos u + \cos v$. In fact, the solutions to (3.5) may be explicitly calculated. Notice that the equations in the (Z, W) plane depend on (u, v) only through $C = C(u, v)$, and they may be expressed in terms of the solutions of the pendulum equations. Indeed, introducing new time $\tau = \delta t/2$, then,

$$\begin{cases} Z' = \pi W \\ W' = -C(u, v)\sin Z - 2A\sin(Z + \theta). \end{cases} \tag{3.6}$$

Defining the constants of motion μ and ψ via:

$$\begin{cases} \mu^2 = (C(u, v) + 2A\cos\theta)^2 + (2A\sin\theta)^2 \\ \cos\psi = (C(u, v) + 2A\cos\theta)/\mu \qquad \sin\psi = (2A\sin\theta)/\mu \end{cases} \tag{3.7}$$

the (Z, W) equations have the form of a standard pendulum

$$Z' = \pi W \qquad W' = -\mu\sin(Z + \psi).$$

For $\mu > 0$, choosing the initial time τ_0 such that $Z(\tau_0) = -\psi$, $W(\tau_0) = W_0$, and using again the regular time t, the rotational solution $(Z(t; t_0, W_0, \psi), W(t; t_0, W_0, \psi))$ is given by

$$W(t; t_0, W_0, \psi) = W_0 \,\mathrm{dn}\left(\frac{\pi}{4} W_0\delta(t - t_0), \kappa\right) \tag{3.8}$$

$$Z(t; t_0, W_0, \psi) = -\psi + 2\arcsin\left(\mathrm{sn}\left(\frac{\pi}{4} W_0\delta(t - t_0), \kappa\right)\right) \tag{3.9}$$

where

$$\kappa^2 = \frac{4\mu}{\pi W_0^2} < 1 \tag{3.10}$$

defines rotational orbits with period

$$T_0 = \frac{8}{\pi\delta W_0}\mathcal{K}(\kappa). \tag{3.11}$$

For future calculations, notice that ξ, the average of $\cos(Z(t))$ over one rotational period T_0 is given by:

$$\xi(\kappa, \psi) = \frac{1}{T_0}\int_0^{T_0} \cos Z(t)\,\mathrm{d}t = \left(1 - 2\frac{1}{\kappa^2}\left(1 - \frac{\mathcal{E}(\kappa)}{\mathcal{K}(\kappa)}\right)\right)\cos\psi \tag{3.12}$$

where $\mathcal{K}, \mathcal{E}$ are the complete elliptic integral of the first/second kind. For $\psi \neq \pi/2, \kappa < 1$, ξ is monotonically decreasing with κ: for large W_0 (namely $\kappa \to 0$) $\xi(\kappa)$ tends to zero, whereas near the separatrices ($\kappa \to 1$) it approaches -1.

Now, examine the behaviour in the (Z, W) plane for $\theta = 0$. Above, we have shown that for $A, \theta \neq 0$ the motion in the (Z, W) plane corresponds to a nonlinear pendulum. The position of the pendulum's fixed points are at $Z = -\psi, -\psi + \pi$, where ψ depends on the values of $C(u, v)$, A and θ via (3.7). The stability of these fixed points is determined by the sign of μ. For $A, \theta \neq 0$, μ is bounded away from zero for all C, thus no bifurcation of the fixed points occur as C is varied: changing C simply corresponds to a smooth change in the location of the fixed points. On the other hand, if $\theta = 0$ then $\psi = 0$, and μ changes sign at $A = -2C$. Indeed, for $C > -2A$, $Z = \pm\pi$ is hyperbolic and $Z = 0$ is elliptic, whereas for $C < -2A$ their stability is exchanged. Namely, there is no continuous transition for C going through $-2A$. Moreover, for $\theta = 0$, at $C(u, v) = -2A$, the pendulum oscillatory orbits degenerate to a *circle of fixed points* at $W = 0$, $Z \in [-\pi, \pi]$—a highly degenerate situation which, as will be shown, may produce strong instabilities when perturbed.

Now, consider the motion in the (u, v) plane. Given $Z(t; t_0, W_0, \psi)$ which solves the pendulum's equations, the solutions in the (u, v) plane may be found by reparametrization of the time variable. Indeed, let

$$u(t) = u_0(g(t)) \qquad v(t) = v_0(g(t))$$

where $u_0(t), v_0(t)$ satisfy

$$\dot{u_0} = \sin v_0 \qquad \dot{v_0} = -\sin u_0$$

and the function $g(t) = g(t; t_0, W_0, \psi)$ satisfies the following differential equation

$$\dot{g}(t) = \tfrac{1}{2} K \cos Z(t; t_0, W_0, \psi) \qquad g(0) = 0 \tag{3.13}$$

where $Z(t)$ is a solution of the pendulum equation with $C = C(u_0(0), v_0(0))$. Then $(u(t), v(t), Z(t), W(t))$ clearly satisfy (3.5) (we thank the referee for pointing this out). In particular, the four separatrix solutions in the (u, v) plane (i.e. solutions belonging to the limiting web, on which $C(u, v) = 0$) satisfy:

$$\begin{aligned} &u^{i,\pm} = iv \pm \pi \qquad i = \pm 1 \\ &\sin u^{i,\pm}(t; t_0, W_0, \theta) = -i \sin v^{i,\pm} = \pm \frac{1}{\cosh \frac{1}{2} K \int_0^t \cos Z(t; t_0, W_0, \theta)}. \end{aligned} \tag{3.14}$$

These solutions satisfy the correct boundary conditions provided $\liminf_{t\to\infty} g(t) = \infty$ and similarly $\limsup_{t\to-\infty} g(t) = -\infty$, and these are fulfilled if $\xi(\kappa, \theta)$ of (3.12) is non-zero.

Summarizing, if ξ, the average of $\cos Z$ along the orbits in the (Z, W) plane, is non-zero, then the flow projected to the (u, v) plane is described by a uniform diamond-web with nested invariant web-tori inside each cell, as in the two-dimensional setting [4]. $C(u, v) = 0$ defines the initial conditions which belong to the web, whereas $C(u, v) \neq 0$ defines the initial conditions which belong to the web-tori.

When $\xi \approx 0$, $(\bar{\dot{u}}, \bar{\dot{v}}) = 0$ to leading order, and then the motion in the (u, v) plane is governed by higher-order terms. For example, for $\theta = -\pi/2$ and $C(u, v) = 0$, ξ approximately vanishes near the pendulum elliptic fixed point at $Z = \pi/2$. This phenomenon is demonstrated in figure 1, where projections of the phase portrait of the mapping (2.12) on the (u, v) plane are depicted. For $\theta = 0$ there is a clear diamond lattice, whereas for $\theta = -\pi/2$ the web resembles a square lattice. Moreover, for all θ, large values of W_0, for which $\xi \approx 0$, produce extremely complicated and unstable behaviour, sometimes leading to structures reminiscent of the square web. These two essentially different patterns with correspondingly different scaling of the separatrices splitting, appear in the two-dimensional setting for the corresponding cyclotron centre values [7, 8]: the (fixed) cyclotron centre, x_c, plays the role of Z in this setting, thus $\xi = \cos x_c$. In [7] it has been shown that as x_c is varied, there exists a sequence of bifurcations which change the web's shape and width from a diamond web with a width of the order of $\exp(-\pi^2/2K\xi)$ for $|\xi| > c > 0$, to a square web with smaller cells and with width of the order of $\exp(-\pi^2/K^2)$ when $\xi \approx 0$. In this paper we analyse the simpler case for which ξ is bounded away from zero.

Combining the information gathered on the motion in the (Z, W) plane and in the (u, v) plane, the following structure of the integrable motion emerges: for general $A, \theta \neq 0$ and most initial conditions the integrable motion essentially corresponds to a cross product of

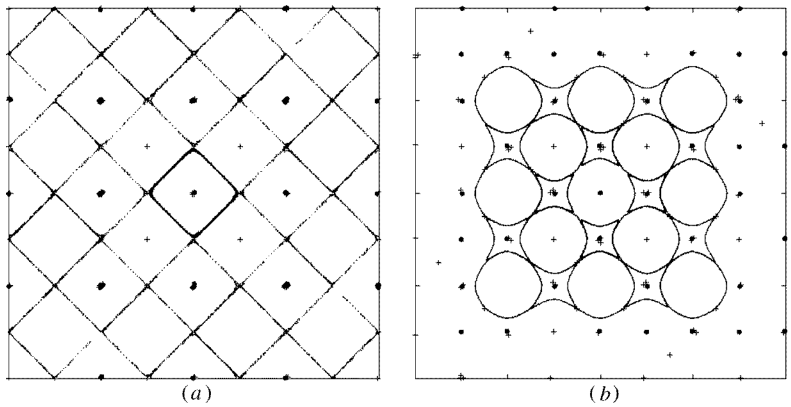

Figure 1. The four-dimensional stochastic web projected to the (u, v) plane $K = 0.1, \beta^2 = 0.1, A = 1$ and $W_0 = 0$; the size of the square is $4\pi \times 4\pi$. (a) $Z_0 = \theta = 0$; (b) $Z_0 = -\theta = \pi/2$.

the web in the (u, v) plane and a pendulum in the (Z, W) plane. However, when $\theta = 0$ and $C(u, v) = -2A$ the motion in the (Z, W) plane degenerates to a free motion. In particular, the energy surface $\mathcal{H} = 0$ contains, for $A \neq 0$, a torus filled with invariant circles: the web-tori crossed with the circle of fixed points $W = 0, Z \in [-\pi, \pi]$. When $A = 0$ this circle of fixed points occurs exactly at $C = 0$. Thus, the value $A = 0$ corresponds to a *double degeneracy*: the energy surface of $\mathcal{H} = 0$ includes (at $C = 0$) the direct product of a uniform web in the (u, v) plane and a circle of fixed points in the (Z, W) plane.

The perturbation to the flow (3.5), created by the next-order terms in the expansion for the fourth iterate of $\mathcal{F}_4^4$, is expected to destroy the conservation of the second integral $C(u, v)$. Hence, the resonant structure which appears at $C = -2A$ and the sudden changes in the structure of the integrable Hamiltonian flow for nearby C's is expected to become significant under these perturbations. Introducing the parameters A and θ employs the standard tool for understanding the behaviour near such sudden changes in phase space. Using the 'unfolding parameters', the corresponding families of systems may be analysed by standard tools, producing information regarding the singular system in the appropriately chosen limits.

From the above considerations, it follows that the role of θ in the unfolding is of crucial importance in the vicinity of the pendulum separatrices and near the pendulum's oscillatory orbits. In what follows, we analyse only the behaviour for rotational orbits which are bounded away from the pendulum separatrices, for which the role of θ seems to be insignificant. Generally, the analysis for $\theta \neq 0$ is expected to be more delicate; first, the limit as $A \to 0$ of the oscillatory orbits is singular as they cease to exist in this limit. Second, the effect of θ is introduced to the approximating Hamiltonian flow only through $O(K^2)$ terms, whereas $O(K, K\delta)$ are sufficient for unfolding the dependence on A. Thus, *we henceforth set θ to zero*. Then $\psi = 0, \mu = C + 2A$ and $Z, W, g(t; t_0, W_0, \psi = 0) \equiv Z, W, g(t; t_0, W_0)$.

3.2. Perturbations and separatrix splittings

In the previous section we have found the explicit solutions of the limiting flow (3.5) which is completely integrable. Here we analyse the influence of the next-order terms on the heteroclinic solutions to specific invariant circles of the unperturbed flow. More precisely, we notice that the hyperbolic fixed points of the web in the (u, v) plane correspond to one-parameter families of invariant circles of the four-dimensional phase-space: $\mathcal{C}(W_0; n, m) = \{(u, v, Z, W)|(u, v, Z, W) = (n\pi, m\pi, Z(t; 0, W_0), W(t; 0, W_0)), n + m = 2\ell + 1, t \in [0, T_0(W_0)]\}$, where $(Z(t; 0, W_0), W(t; 0, W_0))$ are given by (3.9) with $\mu = 2A, \psi = 0$, and $T_0(W_0)$ by (3.11). W_0 is the parameter which distinguishes between the different circles belonging to the family. These circles are normally hyperbolic on the three-dimensional energy surface provided $\xi(\kappa) \neq 0$ (see (3.12)), since this condition implies that their Flouqet multipliers are of a hyperbolic nature. Thus, for $\xi \neq 0$, these invariant circles and their two-dimensional stable and unstable manifolds persist under small perturbations. For the integrable flow (3.5), these manifolds coincide—the solutions $q^0(t; t_0, W_0) = (Z(t; t_0, W_0), W(t; t_0, W_0), u(t; t_0, W_0)), v(t; t_0, W_0)))$ of (3.9) and (3.14) parametrized by $(t; t_0, W_0)$ span these manifolds: t parametrizes the motion along the solutions, t_0 parametrizes the initial phase in Z and W_0 labels the invariant circle and thus the energy surface. The case $t_0 = 0$ corresponds to the symmetric solutions for which $W, \sin(u), \sin(v)$ are even functions of t whereas $Z, \cos(u), \cos(v)$ are odd functions of t. From general principles, we expect that the manifolds will split and in fact intersect transversely when higher-order terms are added. The rest of this paper is dedicated to proving such an assertion, and moreover, to estimate the splitting distance.

Before starting on the calculations we will describe the general strategy. To find the splitting distance we first find the higher-order terms (expanding in powers of K, δ) of the fourth iterate of the mapping, from which we obtain, in the limit, the perturbation to the integrable flow. We discuss the delicate nature of this step below. Then, we measure $\Delta C(t_0; W_0)$, the difference between the values of the unperturbed first integral $C(u, v)$ on the stable and unstable manifolds near the point $q^0(0; t_0, W_0)$. If ΔC has simple zeros in t_0, it implies that the stable and unstable manifolds of the invariant circle corresponding to W_0 intersect transversely, see [11, 12]. In fact, $\Delta C(t_0, W_0)$ measures the component of the distance between $q^s(0; t_0, W_0)$ and $q^u(0; t_0, W_0)$ in the ∇C direction. ΔC is found by integrating $\dot{C}$ along the stable and unstable manifolds, using the first variational equations for the perturbed solutions q^s, q^u, [12]. Such a Melnikov-type calculation gives a complicated integral, which depends on (t_0, W_0) in a non-trivial way, see (3.18) below (though it is easy to observe that the symmetric solutions produce zeros of this integral). In the special limit in which some terms may be considered as fast oscillating (which, as shown in the appendix, corresponds to the limit in which averaging with respect to the motion in the (Z, W) plane is legitimate) it is possible to estimate this integral. Such an estimate produces a 'leading order' term (in powers of K, δ) which has an exponentially small factor in $\sqrt{K}$. This problematic phenomenon is not surprising—it is the usual consistency problem one encounters in applying a simple-minded perturbational method to the singular problem of rapid oscillating forcing (see [13, 14] and [10, 15] and references therein). However, it is shown here that, by taking a special limit in A, it is possible to eliminate the exponentially small dependence on K. Then, in this limit, the expansion in K, δ is a valid asymptotic expansion, and the results prove that in this limit ΔC is polynomial in K. It follows from the form of C that this implies that the distance between q^s, q^u projected to the u, v plane must be polynomial in K.

We begin the analysis by finding the higher-order terms in the expansion for $\mathcal{F}_4^4$, and consider the resulting limiting flow as $K \to 0$. The next-order terms in K, δ are proportional to $K\delta$ in the equations for u, v and to δ^2 in the equations for W, Z. In general, one would hope to obtain, to any given order in the expansion, a symplectic map with a natural limiting Hamiltonian flow. However, the straightforward expansion does not produce here a symplectic map to the order of $K\delta$: while the $K\delta$ terms which appear in the equations for u, v are of the form $(\frac{\partial H_1}{\partial v}, -\frac{\partial H_1}{\partial u})$ with $H_1 = \mathrm{O}(K\delta)$, the $\mathrm{O}(\delta^2)$ terms in the (Z, W) equations do not appear to be of the form $\frac{\delta}{K}(\frac{\partial H_1}{\partial W}, -\frac{\partial H_1}{\partial Z})$. Notice that there exists some freedom in the expansion since in the limiting flow there is no difference between q_n and q_{n+1}, whereas for the mapping expansion such differences change higher-order terms, and thus the symplectic nature of the equations. Henceforth, we *assume* that it is also possible to rearrange the equations, without changing the terms in the (u, v) direction, so that the limiting equations are Hamiltonian to the order of† $K\delta$. Then, we obtain:

$$\begin{cases} \dot{u} = \frac{1}{2}K\cos Z\sin v + \frac{K\delta\pi W}{4}\cos(v+Z) + \mathrm{O}(K^2) \\ \dot{v} = -\frac{1}{2}K\cos Z\sin u + \frac{K\delta\pi W}{8}(-\cos(u+Z) + 3\cos(u-Z)) + \mathrm{O}(K^2) \\ \dot{Z} = \frac{\delta\pi}{2}W + \mathrm{O}(\delta^2, K\delta^2) \\ \dot{W} = -\frac{\delta}{2}((\cos u + \cos v)\sin Z + 2A\sin(Z+\theta)) + \mathrm{O}(\delta^2, K\delta^2). \end{cases} \tag{3.15}$$

Thus the additional $K\delta$ and δ^2 terms of (3.15) can be interpreted as a perturbation of the integrable flow (A.3). Since the $\mathrm{O}(K^2)$ terms‡ are not included in the perturbation, we require for consistency that $K\delta \gg K^2$, or alternatively

$$\beta \gg \sqrt{K}. \tag{3.16}$$

Notice that the $\mathrm{O}(\delta^2)$ terms are not neglected in the analysis, rather, as will be apparent below, their explicit form is not important.

Now, we find ΔC by integrating $\dot{C}$ along the stable and unstable solutions of the perturbed flow. By the stable and unstable manifold theorem it follows that $q^u(t; t_0, W_0) = q^0(t; t_0, W_0) + \mathrm{O}(\delta)$ on the semi-infinite time interval $(-\infty, 0)$ and similarly for $q^s(t; t_0, W_0)$. Since C is an integral of the unperturbed flow, it follows that

$$\begin{aligned} \Delta C(t_0, W_0) &= \int_{-\infty}^{0} \dot{C}|_{q^u(t;t_0,W_0)}\,\mathrm{d}t + \int_{0}^{\infty} \dot{C}|_{q^s(t;t_0,W_0)}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} (\nabla C Y)_{q^0(t;t_0,W_0)}\,\mathrm{d}t + \cdots \\ &= M(t_0, W_0) + \cdots \end{aligned} \tag{3.17}$$

where Y is the vector field corresponding to the perturbation of the flow (3.5), $q^0(t; t_0, W_0)$ denotes the unperturbed separatrix solution of this flow and '...' stand for higher-order terms (which are polynomial in K, δ). Since C is a function of u and v coordinates only, the change in C along the separatrix, depends, to leading order, only on the force acting on the (u, v) components and the explicit form of the $\mathrm{O}(\delta^2)$ terms in (3.15) are not needed. Moreover, the form of C is such that the above integrals are absolutely converging, and not just conditionally converging as in the usual case. Substituting the explicit expressions for Y and ∇C in (3.17) we find (after some manipulations):

$$\begin{aligned} M(t_0, W_0) = -\frac{K\delta\pi}{4}\int_{-\infty}^{\infty} W(t, t_0)(&\cos Z\cos v\sin u + \cos Z\cos u\sin v \\ &+ \sin Z\sin u\sin v)\,\mathrm{d}t. \end{aligned} \tag{3.18}$$

† Unfortunately, we are not aware of any existing algorithm which produces symplectic truncations of the symplectic map $\mathcal{F}_4^4$ to a given order.

‡ These are responsible for the appearance of the square-like pattern of the web [7].

Using the unperturbed solution $q^0(t; t_0, W_0)$, we obtain for the four separatrices $(i, \pm)$, $i = \pm 1$ of (3.14):

$$\begin{aligned}
M^{i,\pm}(t_0, W_0) &= -\frac{K\delta\pi}{4}\int_{-\infty}^{\infty} \pm(\mathrm{i}+1)W(t,t_0)\frac{2}{K}\frac{\mathrm{d}\sin u^{1,+}}{\mathrm{d}t} - \mathrm{i}W\sin Z\sin u^{1,+}\sin v^{1,+}\,\mathrm{d}t \\
&= \frac{K\delta\pi W_0}{4}\int_{-\infty}^{\infty}\frac{\mathrm{dn}(\frac{\pi W_0\delta}{4}t,\kappa)}{\cosh^2(g(t+t_0))} \\
&\quad\times\left(\pm(\mathrm{i}+1)\sinh(g(t+t_0)\left(2\,\mathrm{cn}^2\left(\frac{\pi W_0\delta}{4}t,\kappa\right)-1\right)\right. \\
&\quad\left.+\,2\mathrm{i}\,\mathrm{cn}\left(\frac{\pi W_0\delta}{4}t,\kappa\right)\mathrm{sn}\left(\frac{\pi W_0\delta}{4}t,\kappa\right)\right)\mathrm{d}t.
\end{aligned}\tag{3.19}$$

Notice that the integral vanishes identically for $t_0 = 0$ by symmetry. Moreover, it follows from (3.13) that

$$g(t) = \tfrac{1}{2}K\xi t + \tfrac{1}{2}K\int_0^t(\cos Z(t; t_0, W_0, \theta) - \xi) = \tfrac{1}{2}K\xi t + \tilde{g}(t)\tag{3.20}$$

where $\tilde{g}(t)$ is a periodic function, with average zero, and with period T_0. The integral (3.19) seems to involve quite complicated expressions. None the less, it is possible to estimate it in some limits. In particular, notice that the arguments of the elliptic functions are of the form $\frac{\pi W_0\delta}{4}t$ whereas, the argument of the cosh and sinh functions are essentially of the form $\frac{1}{2}K\xi t$. Thus, fixing κ, the elliptic terms which appear in the integral may be considered as fast oscillating if their period $T_0 \propto \frac{1}{\delta W_0}$ is much smaller than the characteristic time appearing in the cosh and sinh arguments, namely $1/K$. Since we keep κ fixed, $W_0 \propto \sqrt{A}$, thus the elliptic terms are fast oscillating provided:

$$K \ll \delta W_0 \propto \beta\sqrt{KA} \Leftrightarrow A \gg \frac{K}{\beta^2}.\tag{3.21}$$

As shown in the appendix, this condition essentially guarantees that the integrable motion may be averaged with respect to the motion in the (Z, W) plane. Then, we estimate (3.19) by neglecting the effect of $\tilde{g}(t)$ on the integral and by Fourier expanding the other elliptic functions to find†:

$$\begin{aligned}
M^{i,\pm}(t_0, W_0) &\approx \sin\left(\frac{\pi^2}{4\mathcal{K}(\kappa)}W_0\delta t_0\right)\frac{W_0^2\delta^2}{K}\exp\left(-\frac{\pi^3}{4\mathcal{K}(\kappa)\xi(\kappa)}\frac{W_0\delta}{K}\right) \\
&\quad\times\frac{\pi^4}{2\mathcal{K}(\kappa)\xi(\kappa)^2}\left(\mp(\mathrm{i}+1)\left(a_1(\kappa)-\frac{1}{2}d_1(\kappa)\right)-\mathrm{i}a_2(\kappa)\right)
\end{aligned}\tag{3.22}$$

where $a_1(\kappa), d_1(\kappa)$, (resp. $a_2(\kappa)$) are the coefficients of the $\cos(\pi u/\mathcal{K})$ (resp. $\sin(\pi u/\mathcal{K})$) in the Fourier expansion of $\mathrm{dn}(u)\,\mathrm{cn}^2(u), \mathrm{dn}(u)$ (resp. $\mathrm{dn}(u)\,\mathrm{cn}(u)\,\mathrm{sn}(u)$) respectively. To leading order in the nome of the elliptic functions $q \equiv \exp(-\pi\mathcal{K}'(\kappa)/\mathcal{K}(\kappa))$, they are given by:

$$a_1(\kappa) \approx \frac{\pi^3}{\kappa^2\mathcal{K}^3}\frac{q}{(1+q)^2} \qquad d_1(\kappa) \approx \frac{2\pi}{\mathcal{K}}\frac{q}{1+q^2} \qquad a_2(\kappa) \approx \frac{\pi^3}{\kappa^2\mathcal{K}^3}\frac{q}{1-q^2}.$$

The expression in (3.22) is 'leading order' in K, δ, in a regular perturbation series in these parameters. To restore the double degeneracy present in the original system (and avoid

† Such an approach is justified in the appendix by the averaging method. Direct asymptotic justification of this approach using, for example, the residue method, may be possible yet non-trivial since the elliptic functions themselves have poles and the function $\tilde{g}(t)$ (which includes integrals of the elliptic functions) alters the position of the poles of $\mathrm{sech}(K\xi t/2)$.

the exponential smallness of the leading term) consider the limit $A \to 0$ and $W_0 \to 0$. After introducing a new scaled coordinate, $\bar{W}_0 = W_0/\sqrt{8A/\pi} = 1/\kappa$, the condition of fixed $\kappa < 1$, corresponding to the condition that the rotary motion is bounded away from the separatrix in the (Z, W) plane, implies $\bar{W}_0 > 1$. The other condition of fast oscillations (3.21) of the elliptic function in the integral (3.19) (which corresponds to requiring slower motion in the (u, v) plane than in the (Z, W) plane) is satisfied asymptotically in K if, for example, β is fixed and $A \gg K$, e.g. we may take:

$$\sqrt{A} = \frac{\sqrt{K}\ln 1/K^{\alpha}}{\beta} \qquad \alpha > 0 \tag{3.23}$$

Then, for any positive α there exists $K_0(\alpha)$ such that (3.23) implies (3.21) for any K satisfying $K < K_0(\alpha)$. It follows that in this limit the maximal separatrix splitting is given, to leading order, by

$$\triangle C^{i,\pm} \approx f^{i,\pm}(\bar{W}_0) K^{1+\alpha\frac{\pi^2\sqrt{2\pi}\bar{W}_0}{2\mathcal{K}(\kappa)\xi(\kappa)}} \left(\ln\frac{1}{K^{\alpha}}\right)^2 \tag{3.24}$$

where $f^{i,\pm}(\bar{W}_0)$ may be found directly from (3.22) (with $\kappa = 1/\bar{W}_0$), and constitute (not identically zero) functions of $\bar{W}_0$. Note that for all $K < K_0 < 1$, $\ln^2 1/K^{\alpha}$ can be bounded from below by a constant ($\ln^2 1/K_0^{\alpha}$) and from above by an arbitrarily small positive power of K. Thus, for sufficiently small asymptotic values of K the separatrix splitting $\triangle C$ is *polynomially* and not exponentially small in the perturbation parameter K:

$$\triangle C \propto K^{1+\epsilon}.$$

4. Conclusions

We have established that in the fast Larmor rotation limit with unbounded motion in the longitudinal direction (i.e. rotary motion in Z), one mechanism which causes a substantial increase in the diffusion due to a longitudinal component is a degeneracy of the underlying motion. To analyse this behaviour, we have constructed a limiting flow in the small K limit and found it corresponds to a highly degenerate 2 degrees of freedom system. The phase space for zero values of two integrals of motion is given by a direct product of an infinite web and a circle of fixed points. Unfolding this degenerate structure and then analysing the unfolded system in specific limits, using the Melnikov technique, enabled us to prove that the separatrix splitting becomes *polynomially* and not exponentially small in the perturbation parameter K. This shows that one mechanism for the observed strong instability stems from a degenerate 2 degrees of freedom mechanism, and is not governed by higher-dimensional phenomena such as Arnold diffusion.

Note that a rigorous connection between near-identity symplectic maps and their limiting flows has been established only in the two-dimensional setting [10], in special volume-preserving maps [15] and in some classes of symplectic maps [18, 19]. None the less, we have assumed in our analysis that a similar connection may be established in the higher-dimensional case, and moreover that the symplectic structure of the mappings induces, in the limit, Hamiltonian flows. Proving such an assertion and finding an algorithm which produces such Hamiltonian flows is an interesting open problem.

Preliminary numerical experiments seem to confirm the observation that the instability in β is far more significant in the degenerate case (i.e. when $A = \theta = 0$) than it is for non-degenerate systems ($A, \theta = \mathrm{O}(1)$). Quantitative study of this phenomena is yet to be performed.

Finally, we note that the four-dimensional mapping system is very complex and rich. Other limits (e.g. $\theta \neq 0$, or $W_0 \ll 1$) and different symmetries (i.e. $q \neq 4$) are left for future studies. Especially interesting is the singular limiting behaviour of the oscillatory orbits of the pendulum in the limit $A \to 0$. While analysing only one limiting case is a modest achievement in the era of computer simulations, the significance of the analysis lies in revealing a new mechanism for the instability of four-dimensional symplectic maps which stems from a new mechanism of instability of a 2 degrees of freedom Hamiltonian system. Further, the general theme which emerges from this and other studies [16, 4, 3], is that the symmetries, which are only slightly broken in nature, are associated with degeneracies which are responsible for many of the strong instabilities which are observed in low-dimensional systems. Thus, a fundamental study of the implications of symmetries on the emerging degeneracies of the energy surfaces is needed.

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Appendix. The averaged motion

Here, we show that the limit (3.21), corresponds in fact, to the limit in which averaging over the motion in the (Z, W) plane is legitimate. Under the simplifying assumption $\theta = 0$, (3.4) has the form

$$\mathcal{H} = \frac{K\pi}{4} W^2 - \frac{K}{2} \cos Z(\cos u + \cos v + 2A). \tag{A.1}$$

We choose orbits and the appropriate limits in K, δ for which the motion in the (Z, W) plane can be considered fast compared with the motion in the (u, v) plane. First, we consider orbits in the (Z, W) plane which are bounded away from the separatrix so that their period is finite for fixed δ, K. To examine the dependence of the time-scales on δ, K, we rescale time by δ; consider the new time $\tau = \omega t$, where $\omega^2 = \frac{1}{2}\delta^2\pi A$ is the rotation frequency in the the (Z, W) plane and A is assumed to be positive. (The case of negative A produces essentially the same results and will be briefly discussed later.) Then, the rescaled Hamiltonian is given by

$$\mathcal{H} = \frac{K\pi}{4\omega} W^2 - \frac{K}{2\omega} \cos Z(\cos u + \cos v + 2A) \tag{A.2}$$

with the equations

$$\begin{aligned}
u' &= \frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\sqrt{K}}{\beta\sqrt{2\pi A}} \cos Z \sin v \\
v' &= \frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{\sqrt{K}}{\beta\sqrt{2\pi A}} \cos Z \sin u \\
Z' &= \sqrt{\frac{\pi}{2A}}\, W \\
W' &= \sqrt{\frac{1}{2A\pi}} \sin Z(\cos u + \cos v + 2A)
\end{aligned} \tag{A.3}$$

where $\delta = \beta\sqrt{K}$ has been substituted. Now, it is evident that for small K the motion in the u, v plane is slower than that of the (Z, W) plane for any finite β.

Choosing A so that $\sqrt{K/A}$ is a small parameter, new averaged coordinates $\bar{u}$ and $\bar{v}$, such that $u = \bar{u} + \sqrt{K/A}u_1$ and $v = \bar{v} + \sqrt{K/A}v_1$, can be defined [17] so that

$$\begin{aligned} \bar{u}' &= \frac{\sqrt{K}}{\beta\sqrt{2\pi A}} \frac{1}{T_0} \int_0^{T_0} \cos Z(\tau)\,\mathrm{d}\tau\ \sin\bar{v} + \frac{K}{\beta^2 A} f_1(\tau) \\ \bar{v}' &= -\frac{\sqrt{K}}{\beta\sqrt{2\pi A}} \frac{1}{T_0} \int_0^{T_0} \cos Z(\tau)\,\mathrm{d}\tau\ \sin\bar{u} + \frac{K}{\beta^2 A} g_1(\tau) \end{aligned} \tag{A.4}$$

where T_0 is the period of the rotational orbit of Z (in τ units) and $f_1(\tau), g_1(\tau)$ are some periodic functions in τ, which depend on u, v as well and are introduced by the averaging theorem. For the averaging theorem to hold u' and v' must be small. It means that the following condition must be satisfied:

$$\frac{\sqrt{K}}{\beta\sqrt{A}} \ll 1 \Leftrightarrow A \gg \frac{K}{\beta^2} \tag{A.5}$$

which is exactly (3.21). Moreover, averaging over the standard pendulum's rotational orbits brings (A.4) to the form:

$$\bar{u}' = \frac{K\xi}{2\omega} \sin\bar{v} + \mathrm{O}(K) \qquad \bar{v}' = -\frac{K\xi}{2\omega} \sin\bar{u} + \mathrm{O}(K) \tag{A.6}$$

(recall that $\omega = \mathrm{O}(\sqrt{K})$) with $\xi = \xi(\kappa, 0)$ of (3.12). It follows from (A.6) that on the slow time scale $\mathrm{O}(\sqrt{K}\tau)$, for any $\xi \not\approx 0$ the averaged system has, to $\mathrm{O}(\sqrt{K})$ the structure of the two-dimensional web with hyperbolic fixed points connected by separatrices, which are given by:

$$\sin\bar{u} = -\sin\bar{v} = -\frac{1}{\cosh\frac{K\xi\tau}{2\omega}}. \tag{A.7}$$

By the averaging theorem, the hyperbolic fixed points and their stable and unstable manifolds persist as the hyperbolic periodic orbit of the full system, thus it makes sense to speak of the splitting separatrices of the full system in the above limit.

The higher-order terms coming from the averaging may move the manifolds by at most $\mathrm{O}(\sqrt{K})$. Thus, in general, such $\mathrm{O}(\sqrt{K})$ terms may destroy the web, by, for example splitting the C levels of the hyperbolic fixed points. However, recall that the averaging procedure may be applied repeatedly to obtain higher-order corrections to the averaged motion in the (u, v) plane. Since the original motion is integrable, and in particular, the explicit solutions which were found preserve the heteroclinic connections, it follows that, on the nth step of the averaging, $C(u, v)$ is conserved to order $K^{n/2}$ and thus equations (A.6) are essentially correct at any order of the averaging: the ξ term is replaced by a power series in $\sqrt{K}$ with the leading-order term (3.12), and the error term $(\mathrm{O}(K))$ is replaced by an error of $\mathrm{O}(K^{(n+1)/2})$. Thus, for $|\xi| > \mathrm{O}(\sqrt{K})$, the diamond web is approached asymptotically, with errors which can be made as small as needed in K. Given the above arguments, the leading-order term of the averaged flow may be substituted in (3.18), producing, to leading order, the same results as in (3.22).

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