REGULAR ARTICLES

Islands of accelerator modes and homoclinic tangles

Vered Rom-Kedar[a)]
The Department of Applied Mathematics and Computer Science, The Weizmann Institute of Science, P.O. Box 26, Rehovot 76100, Israel

George Zaslavsky
Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012 and Department of Physics, New York University, 2-4 Washington Place, New York, New York 10003

(Received 23 March 1999; accepted for publication 25 May 1999)

Islands are divided according to their phase space structure—resonant islands and tangle islands are considered. It is proved that in the near-integrable limit these correspond to two distinct sets, hence that in general their definitions are not trivially equivalent. It is demonstrated and proved that accelerator modes of the standard map and of the web map are necessarily of the tangle island category. These islands have an important role in determining transport—indeed it has been demonstrated in various works that stickiness to these accelerator modes may cause anomalous transport even for initial conditions starting in the ergodic component. © *1999 American Institute of Physics.* [S1054-1500(99)02203-X]

Typical Hamiltonian dynamics of low-dimensional systems is not ergodic and the domain of chaotic motion contains an infinite number of islands embedded into the area of chaos. The islands strongly influence transport of particles and this feature is important for applications. One type of islands is related to resonances. Another type of islands appears in the chaotic area and is associated with the so-called ballistic or accelerated modes of particle motion. These may have a distinct influence on transport.

I. INTRODUCTION

The action of a time periodic two-dimensional Hamiltonian flow or equivalently of a two-dimensional area and orientation preserving maps on a set of initial conditions is rather complicated. In real systems it is highly sensitive to the initial location of this set. Typically such flows have a mixed dynamics—they have both chaotic and ordered regions. A stable periodic orbit of a two-dimensional area preserving map is typically surrounded by invariant tori which define an area of stability around it.[1] These areas of stability are called islands, and correspond to practically ordered motion. Practically—since even in these islands, near any subresonance, tiny chaotic regions appear.[2] Describing the chaotic region is more problematic since it is still an open question whether, in real systems, there can be a chaotic component with a positive measure. In fact even in seemingly highly chaotic phase space regions islands may appear (e.g., the standard map with large K[3,4,5] or smooth approximations of scattering billiards[6]). In particular, it is unknown whether a patch (a connected set of positive Lebesgue measure) of initial conditions can be contained in the chaotic component. Nonetheless, it follows that one may separate between the ordered and chaotic components up to a certain resolution.

In these chaotic systems, tracking a single solution is quite meaningless, and one is usually interested in either qualitative description of significant phase space structure or information about some averaged quantities, the observables. Examples of observables are correlation functions, moments (and in particular pair separation rate-i.e., diffusion), residence time distribution from some specific regions, Poincaré recurrences, line stretching rates etc. Such quantities are influenced by the presence of islands; even if the contribution from the major islands is subtracted, the stickiness to their boundaries and the inclusion of tiny islands of high period biases the averaged quantities.

Here we propose to start classifying the stability islands which appear in two dimensional chaotic flows.[2,7,8] Classification to the various possible dynamical behaviors of the islands will then serve as a first step in understanding their influence on the space-averaged observables, or the kinetics of particles (see Ref. 4 and references therein). In particular, since chaotic trajectories may stick around islands for very long times, the time averages on the chaotic component depend on the islands behavior as well. Sticking around oscillatory resonant islands could cause subdiffusive behavior (though this has not been numerically observed), whereas sticking around accelerator mode islands or ballistic islands causes superdiffusion. As both types of islands may co-exist,

[a)]Electronic mail: vered@wisdom.weizmann.ac.il

switching between subdiffusive and superdiffusive behavior may appear.

We consider nonlinear area-preserving maps. Furthermore, the two examples for which we demonstrate and substantiate our claims, the standard map and the web map are also twist maps. For such maps, a beautiful theory developed by Aubry,[9] Mather[10] and MacKay and Meiss[11] (see the review[12] and references therein), asserts that there exist an action functional (sum over the generating function evaluated along the orbit) from which many properties of the orbits may be exerted. The beauty of the theory is that no assumption on closeness to integrable system is needed. We will see that one type of islands—the resonance islands—are identified with the usual islands which were investigated in the above works and in Ref. 13 using the generating function formulation, whereas the second type of islands, the tangle islands, were not described there. The appearance of tangle islands follows from Newhouse work.[7] Self similarity properties of the tangle islands were predicted by Melnikov.[8] We will give a geometrical characterization of these islands and prove that there are indeed different from the resonance islands in the near integrable limit. It is possible that these have also some variational characterization (we prove that they cannot be characterized as minimizing)—we leave this question to future studies.

The paper is ordered as follows: in Sec. II we define resonance and tangle islands and prove that these definitions are not equivalent, in Sec. III we demonstrate that tangle islands may give rise to ballistic modes and in Sec. IV we prove and demonstrate that accelerator modes of the standard map and web map are tangle islands. Section V is devoted to discussion.

II. RESONANCE VS TANGLE ISLANDS

Consider an area preserving, orientation preserving mapping $\hat{T}$ defined on the cylinder. The resonance islands of $\hat{T}$ are the large islands seen in typical phase portraits of two dimensional near integrable area preserving maps. For near integrable flows, these correspond to the resonant response of the neighborhood of the unperturbed periodic motion to the perturbation frequency. These islands can be of a rotational type, namely corresponding to periodic motion which monotonically traverses the cylinder, or to an oscillatory type, namely, to a monotone periodic motion about some central periodic motion.

Below we mathematically formulate the definition of resonance islands, following the review paper of Meiss,[12] which, in this part, is based upon.[9,10,11] Let

$$\hat{T}:\hat{p}\rightarrow\hat{p},\quad \hat{p}=(\hat{x},y)\in S\times\mathbf{R}. \tag{2.1}$$

Let T denote a lift of $\hat{T}$ so that T acts on $p=(x,y)\in\mathbf{R}\times\mathbf{R}$. Denote the projection to the angle variable by Π: $\Pi(p)=x$.

Recall the definitions of monotone sets and of rotational (class 0) periodic orbits. An invariant set M is said to be monotone if for any $p_1,\ p_2\in M$, $\Pi(p_1)<\Pi(p_2)$ implies $\Pi(T(p_1))<\Pi(T(p_2))$. An orbit is monotone if the set formed from all its translates is monotone.

Let $p^{(m,n)}$ denote a periodic orbit of type (m,n): $p_{i+n}^{(m,n)}=p_i^{(m,n)}+(m,0)$. Then, $p^{(m,n)}$ is a rotational periodic orbit (or class 0 periodic orbit) if the orbit of $p^{(m,n)}$ is monotone. Namely, the set $M^{(m,n)}=\{\{(x_i^{(m,n)}+j,y_i^{(m,n)}\}_{i=0}^{n-1}\}_{j=0}^{m-1}$ is monotone. For an area preserving twist map, for any coprime (m,n) there exist a pair of monotone (m,n) periodic orbits (modulo translations, i.e., $2n$ such orbits), one which minimizes the action and the other is a minimax (more than two may exist, for simplicity of presentation we will assume that precisely two, modulo translations, exist). Moreover these orbits are well ordered with respect to each other.[10] Finally, it follows that the minimizing orbit (in the nondegenerate case) is a saddle and that the minimax orbit is either elliptic or hyperbolic with reflection.[11] Hence the stable and unstable manifolds of the minimizing orbit may be used to define the resonance zone associated with (m,n), even when the minimax periodic orbit is unstable.[12]

To define the oscillatory resonance islands, consider the motion around the (m,n) minimax periodic orbit. Transform T^n to action angle coordinates near this orbit. Since the map is nonlinear, it is expected that in most regions it will be a twist map, hence, the above theory applies, and minimizing and minimax periodic orbits of type (m_1,n_1) exist, defining the "class 1" island chain. Clearly this procedure may be carried on, defining "class N" subislands as the Nth level of the island around island chain. If there exist an elliptic fixed point of T (a stable (0, 1) minimax orbit), the class 1 subislands about this fixed point, and all of their higher level subislands correspond to motion which never traverses the cylinder. These are the oscillatory resonance islands. Oscillatory motion about an (m,n) periodic orbit with $m\neq 0$ corresponds to oscillation about a rotational motion, and thus, we will refer to it as an oscillating rotational motion.

To summarize, resonance island chains are defined by a pair of relatively ordered monotone periodic orbits, the minimum and minimax of an action functional. The stable and unstable manifolds of the minimizing orbit are used to define the resonance region, and the minimax orbit is called the center of the island. The coordinate system and dynamics used in the definition of the action functional may be one of two kinds: The rotational coordinate system—this corresponds to the original coordinate system defined on the cylinder, and the action is defined by the original map T. Denote the projection to the first coordinate in this system by Π_r. The (m,n) oscillatory coordinate system—corresponds to the local action-angle coordinates defined about an (m,n) periodic orbit of T, and the action is defined for T^n. Denote the projection to the first coordinate in this system by Π_o.

Rotational island chains are defined using the rotational coordinate system, oscillating rotational motion is defined using (m,n) oscillatory coordinate system about a rotational orbit with $m\neq 0$ or about an oscillating rotational orbit, and oscillating island chains are defined using the $(0,\ n)$ oscillating coordinate system, or oscillating orbits about such oscillatory orbits.

A few remarks are now in order. Note that when the minimax orbit is unstable, the island's invariant region may, but is not necessarily, be of zero measure. Also note that if the twist condition is locally violated, the theory still applies

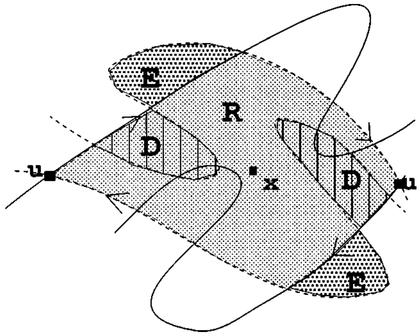

FIG. 1. Resonance region and turnstile lobes. *R*-resonance region of the period one periodic orbit *u*. *E*-incoming lobes, *D*-outgoing lobes.

to regions which are bounded away from this degenerate curve. Near this curve nontwist resonance structure appears.[14] Finally, clearly the definition of oscillatory/rotational orbits depends on the observer frame of reference. Therefore, we can think of our original cylindrical coordinates as action-angle coordinates around a base periodic orbit p, and all the above definitions can then be applied locally near any periodic orbit.

Now, consider an island chain, and the resonance region, R which is defined by segments of the stable and unstable manifolds of the (m,n) minimizing periodic orbits.[13] Two consecutive primary intersections of the manifolds define a lobe, and the lobes are responsible for the flux through the resonance,[13,15] see Fig. 1. In the resonance geometry, it is possible to have islands inside the lobes[8]—we call such islands tangle islands. More precisely, let u denote the minimizing (unstable) periodic orbit with stable and unstable manifolds emanating from it and first intersecting at the primary homoclinic point r_0 creating the resonance region R. Let x denote the minimax (central-elliptic or hyperbolic with reflection) periodic orbit in R (thus x, u have the same period and there exist coordinates in which the first coordinate of these orbits are ordered, i.e., $\Pi_r(x_i)<\Pi_r(u_i)$, $i=0,\ldots,n-1$). The entraining and detraining turnstile lobes E,D are defined by the segments of the stable and unstable manifolds connecting the primary intersection point r_0 to its image. In the near integrable geometry, where the turnstile lobes are small compared with R and do not intersect each other, these lobes are defined by: $R-(F(R)\cap R)=F(E)$ and $R-(F^{-1}(R)\cap R)=D$, see Fig. 1.

Definition 2.1. An $(\overline{n},\overline{m})$ R-tangle island is an invariant stability region C with positive Lebesgue measure satisfying $C\cap E=C_0$ and $C=\cup_{i=0}^{\overline{n}-1}\hat{T}^i C_0$, $T^{\overline{n}}C_i=C_i+\overline{m}$, where E denotes one of the out-going turnstile lobes of R.

It follows that C_0 lies in the interior of E. Furthermore, the condition $C_0\subset E$ and its invariance implies that $C_0\subset\hat{T}^{\overline{n}}(E)\cap E$ and that there exist a k, $0<k\leq\overline{n}$ such that $C_k=\hat{T}^k(C_0)\subset D$. In the near integrable case, $\overline{n}\gg n$, where n is the period of the periodic orbit which defines R, and $0<k<\overline{n}$.

Theorem 2.2. Consider a twist map on the cylinder with a rotational resonance island chain R. Assume this map has a near integrable limit (i.e., in this limit the turnstile lobes of R are much smaller in both width and length w.r.t. the width and length of R). Then, in this limit the set of Tangle islands and the set of Resonance islands are disjoint sets.

Proof. We will prove that in the near integrable limit the tangle island orbits are nonmonotone in both the rotational and the oscillatory coordinate systems. Since the resonance islands centers are monotone, this proves the theorem.

Let x denote the central periodic orbit of the resonance region R, and let t denote an orbit in the (n,m) R-tangle island. With no loss of generality, for simplicity of presentation, let us assume that x is in fact a fixed point. Divide the cylindrical phase space to three regions, the $+1$ region above R, the -1 region below R and 0 for points in R.

Let us divide the orbit of t to segments according to the region they belong to—denote by $s_i\in\pm 1,0$ the sequence of the regional location of t_i ($s_i=1$ indicates that t_i is in the region above R). Clearly the sequence $\{s_i\}$ is n-periodic. If $s_i\neq s_{i+1}$ it follows that $t_i\in E\cup D$, namely t_i belongs to a turnstile lobe. In the near-integrable limit, the lobes are small, hence $\{s_i\}$ is composed of long strings of identical values. We assume, for definiteness, that $s_0=1$ and $s_1=0$ ($s_0=-1$, $s_1=0$ can be treated similarly. All other possibilities can be transformed, by shifting the origin, to one of these two cases). Let $n_1\gg 1$ denote the first $i>1$ for which $s_i\neq 0$. Notice that, by periodicity of s and near-integrability, $n_1\leq n-1$. It follows that $\{t_i\}_{i=1}^{n_1-1}$ encircle x in an oscillatory fashion. In particular, we now show that in rotational coordinate system this segment is nonmonotone. Furthermore, the same argument shows that this segment is nonmonotone in the oscillatory coordinate system associated with any monotone (rotational) periodic orbit which is not in R. Denote by i^* the first ''rotational turning point'' along the orbit: then $\Pi_r(t_{i^*-1})<\Pi_r(t_{i^*})$ whereas $\Pi_r(t_{i^*})>\Pi_r(t_{i^*+1})$. Such a turning point, at which monotonicity is broken, necessarily exists if $\max\Pi_r(E)<\min\Pi_r(TE)$ (where $\Pi_r(E)=\{\Pi_r(x)|x\in E\}$) and either $\max\Pi_r(D)<\min\Pi_r(TE)$ or $\max\Pi_r(TD)<\min\Pi_r(TE)$. These conditions clearly hold in the near-integrable case because the lobes are close to the limiting separatrix and the separatrix is monotone w.r.t. the cylindrical coordinate system.

Now, assume there exists an oscillatory coordinate system for which this segment of t is monotone (if such system does not exist then the theorem is proved). We will prove next that there exist another segment of the orbit of t for which this coordinate system is nonmonotone. For simplicity of presentation consider only the upper turnstile lobes. Since $t_0\in E$, t may not be confined to only one cell—in particular it must have a rotational segment $t_{n-n_2},\ldots,t_{n-1}$ which extends to at least one more cell to the left of the origin. This segment, in the near integrable limit, follows very closely the separatrices associated with R, and thus cannot be monotone in the oscillatory coordinate system of the central periodic orbit of R or in the oscillatory coordinate system associated with any oscillatory periodic orbit in R. In particular, let D_l denote the upper D lobe on the cell to the left of the oscillatory coordinates central point. Then, an oscillatory turning point must exist if there exist an n such that $\max\Pi_o(T^nD_l)<\min\Pi_o(TD_l)$ and $\max\Pi_o(T^nD_l)<\min\Pi_o(E)$. Such an n exists in the near integrable case because of the closeness to the limiting rotational invariant circles which are necessarily nonmonotonic in any of the oscillatory systems about monotone periodic orbits since the width of R has been assumed to be positive in this limit. □

It follows that the definition of tangle islands and of resonance islands are not equivalent. Furthermore, since minimizing periodic orbits are monotone,[12] it follows that all periodic orbits contained in the tangle island are not minimizing in the rotational and oscillatory coordinates systems. We conjecture that even in the far-from integrable limit the two families of islands are distinct. The resonance islands are the ones which evolve continuously from the near-integrable limit until their stability zone diminishes whereas the tangle islands are the islands which appear via homoclinic bifurcations, thus have no near-integrable continuation. It follows that in the near integrable limit the majority of the stability zone is governed by resonance islands whereas in the strong chaos case tangle islands are the main contributers to the stability regions. It is still an open question whether these two categories are exhaustive, see discussion.

III. BALLISTIC MODES

Ballistic modes correspond to stable periodic motion on the cylinder with rotation rate which is different then the rotation rate of the central periodic orbit x: denote by b the ballistic trajectory, then

$$\liminf_{n\to\infty} \frac{1}{n}|T^n b - T^n x| = v > 0.$$

Such a motion influences transport as the separation is larger than vt with some constant v, as opposed to the ''diffusive'' behavior which is observed for nonsticky chaotic orbits. We may identify now that the anomalous transport observed in Ref. 16 for the *ABC* flow and in Ref. 17 for the standard map was due to the orbits stickiness to ballistic modes.

Clearly ballistic modes can be created by regular resonance islands. For these, v, the ''velocity'' of the ballistic motion, simply correspond to the rotation number. However, tangle islands may also create ballistic motion with rotation numbers which are different then their central periodic orbits, see the example below.

A. Ballistic island near the separatrix

The dynamics near a separatrix can be described by the so-called separatrix map (Ref. 18 or in more contemporary form Refs. 19–23). For example, for the perturbed pendulum

$$\ddot{x} + \sin x = \epsilon \sin(x - \nu t) \tag{3.1}$$

with the perturbation parameter ϵ and perturbation frequency ν, the separatrix map can be written using the dimensionless energy h and phase ϕ:

$$\begin{aligned} h_{n+1} &= h_n + \epsilon K_n \sin\phi_n, \\ \phi_{n+1} &= \phi_n + \nu \ln(32/|h_{n+1}|) \quad (\mathrm{mod}\, 2\pi), \end{aligned} \tag{3.2}$$

where

$$K_n = \frac{4\pi\nu^2}{\sinh \pi\nu}\left(\exp(\pi\sigma_n\nu/2) - \sigma_n \frac{\sinh \pi\nu/2}{\nu^2}\right) \tag{3.3}$$

and $\sigma_n = \pm 1$ is a dynamic sign function:

$$\sigma_{n+1} = \sigma_n \cdot \mathrm{sign}\, h_{n+1}. \tag{3.4}$$

On the separatrix $h=0$ (see Ref. 22 for more precise formulation). Consider the limit $\nu \gg 1$, $\epsilon \ll 1$: then K_n is exponentially small in ν and the separatrix splitting is given by the Melnikov function provided $\epsilon \leqslant O(\nu^{-p})$: this has been recently proven for $p>0$ (see Ref. 24 for the general formulation and other references). In this limit, simplified version of (3.3) is

$$K_n = \begin{cases} K = 8\pi\nu^2 \exp(-\pi\nu/2), & \sigma_n = 1 \\ 0, & \sigma_n = -1 \end{cases} \tag{3.5}$$

namely, to $O(K/\nu^2)$ only the upper separatrix breaks. In this approximation, a simple ballistic trajectory can be defined by the initial conditions:

$$\sigma_0 = 1, \quad h_0 = \epsilon K/2, \quad \phi_0 = 3\pi/2 \tag{3.6}$$

if ϵ, ν are such that

$$|h_0^*| = \frac{1}{2}\epsilon^* K^* = 32 \exp\left(-\frac{\pi}{2\nu^*}(2m+1)\right) \tag{3.7}$$

with integer m. In this case

$$\begin{aligned} &h_4^* = h_0^*, \quad \phi_4^* = \phi_0^* + (4m+2)\pi = \phi_0 (\mathrm{mod}\, 2\pi), \\ &\sigma_4^* = \sigma_0^* \end{aligned} \tag{3.8}$$

i.e. four is a characteristic period of the ballistic propagation along ϕ, and Eq. (3.7) defines a specific value (ϵ^*, ν^*) where the ballistic motion exists.

Now, considering a small $\Delta\epsilon$ deviation from the exact ϵ^*, K^* values, we find that the separatrix map has a (small) stability island around this periodic orbit, as is seen in Fig. 2. This stable fixed point undergoes a period doubling bifurcation at $\epsilon = 0.520\,95 \pm 0.000\,05$. A similar picture appears near $\nu = 4$, $\epsilon = 0.2359$ where $\epsilon^*(\nu=4, m=7) = 0.235\,704$.

Indeed, the tangle island structure for the ballistic mode can be derived in a straightforward way. Let us define

$$\begin{aligned} &\Delta h_k = h_k - h_k^*, \quad \Delta\phi_k = \phi_k - \phi_k^*, \quad \sigma_k = \sigma_k^*, \\ &\delta = \Delta\epsilon/\epsilon^* = (\epsilon - \epsilon^*)/\epsilon^* \quad (k = 0,1,2,3,4), \end{aligned} \tag{3.9}$$

where the ballistic trajectory $(h_k^*, \phi_k^*, \sigma_k^*)$ is given in (3.6), (3.7), (3.8). Applying the map (3.2) with (3.5) four times for the values (h_k, ϕ_k, σ_k) in the vicinity of the ballistic trajectory, we find

$$\begin{aligned} \Delta\psi_4 &= \Delta\psi_0 - 4\nu\delta + 2\frac{\nu^3}{h^{*2}}(\Delta h_0)^2, \\ \Delta h_4 &= \Delta h_0 + 2h^*\Delta\psi_4, \end{aligned} \tag{3.10}$$

where the new variable

$$\Delta\psi_k = \Delta\phi_k + \frac{\nu}{h^*}\Delta h_k \tag{3.11}$$

has been introduced and terms of the order of $(\delta^2, \delta\Delta h_k, \delta\Delta\psi_k, (\Delta\psi_k)^2, \Delta\psi\Delta h, \ldots)$ are neglected, namely

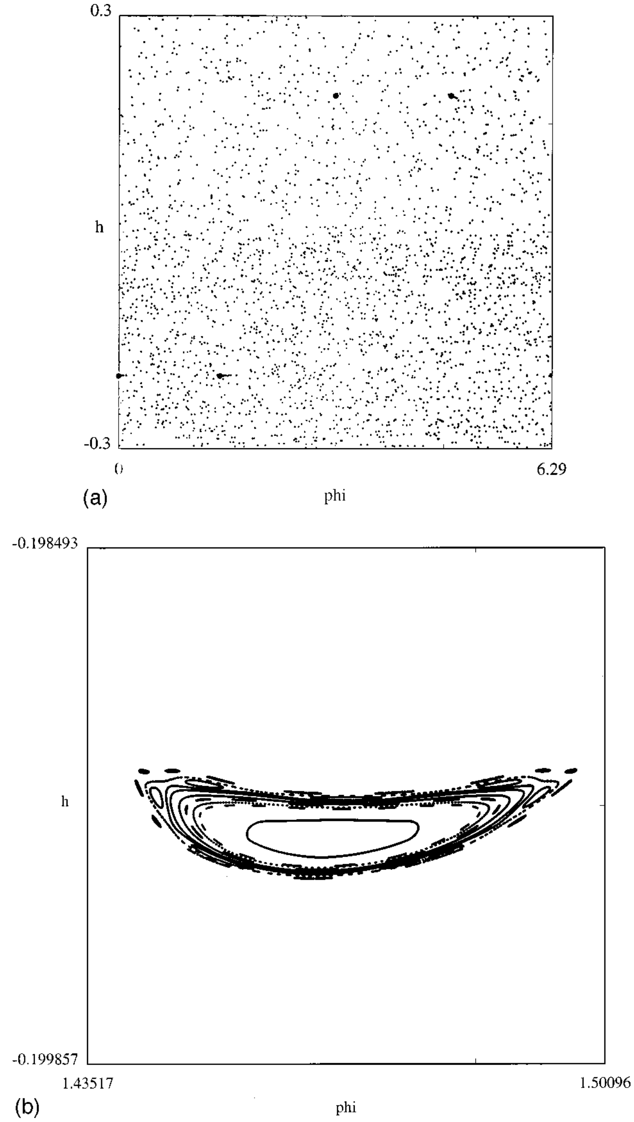

FIG. 2. Stability island around the ballistic tangle periodic orbit. The tangle islands of the separatrix map (3.2) at $\nu=4$, $\epsilon=0.52$ [where $\epsilon^*(\nu=4,m=6)=0.516\,965$]. (*a*) Stochastic layer with islands (4 black dots). (*b*) Zoom on one island zone.

these equations are consistent for orbits with $(\Delta h,\Delta\psi)=(O(\sqrt{\delta}),O(\delta))$. The map (3.10) is area preserving (for this purpose the new variable ψ was introduced) and for infinitesimal $(\Delta\psi_k,\Delta h_k)$ can be written in a Hamiltonian form

$$\frac{d\Delta\psi}{d\tau}=-\frac{\partial H_{\mathrm{bal}}}{\partial\Delta h},\quad \frac{d\Delta h}{d\tau}=\frac{\partial H_{\mathrm{bal}}}{\partial\Delta\psi} \tag{3.12}$$

with the Hamiltonian

$$H_{\mathrm{bal}}=h^*(\Delta\psi)^2+4\nu\delta\cdot\Delta h-\frac{2}{3}\frac{\nu^3}{h^{*2}}(\Delta h)^3 \tag{3.13}$$

and dimensionless time $\tau=t/4$. It follows from (3.13) and (3.7) that the island exists for $\delta>0$ in the domain

$$0\leqslant\Delta h\leqslant(6\delta h^{*2}/\nu^2)^{1/2}=16\sqrt{3\pi\Delta\epsilon}\exp\frac{-\pi}{4}\left(\frac{2m+1}{\nu}+\nu\right). \tag{3.14}$$

Similar Hamiltonian structure was obtained in Ref. 8 in a different way [see also the accelerator mode case (4.3)]. Here the Hamiltonian (3.13) is related to the ballistic mode island and its parameters are correspondingly specified.

Will the stability island we have found for the separatrix map appear in the Hamiltonian flow? Indeed, (3.7) defines $\epsilon^*=\epsilon(\nu,m)$, and as shown in Ref. 24, Melnikov analysis is expected to be applicable only if $\epsilon=o(1/\nu)$, hence the behavior of the map near the periodic orbit (h^*,ϕ^*) reflects the dynamics of the flow provided m is sufficiently large, namely m must satisfy:

$$m\geqslant m_{\min}=\frac{1}{2}(\nu^2-1)-a\frac{2}{\pi}\nu\log\nu,\quad a\ll 1. \tag{3.15}$$

In Fig. 2 we take $\nu=4$ and $m=6$ [where by (3.15) with $a=1$, $m_{\min}=4$], hence we expect these (tiny) islands to appear in the Hamiltonian flow (3.1) as well.

The schematic structure of this periodic orbit in the Hamiltonian flow is shown in Fig. 3. From (3.8), it follows that this period four orbit in (ϕ,h) corresponds to a period $T_m=(4m+2)\pi/\nu$ orbit in $x(t)$. Moreover, since $h_0^*>0$, $h_1^*<0$, $h_2^*<0$, $h_3^*>0$, $h_4^*=h_0^*>0$ it follows that $x(T_m)=x(0)+4\pi$ (see Fig. 3). Thus, the periodic orbit has a rotation number (''velocity'') $2\nu/(2m+1)$ which is of $O(1/\nu)$ for large ν, corresponding to nonmonotone ballistic motion.

IV. ACCELERATOR MODES

Accelerator modes correspond to islands which are periodic on the torus ($p\in S$) yet are nonperiodic on the cylinder ($p\in\mathbf{R}$). As p represents momentum, such motion corresponds to unbounded increase in the kinetic energy. Usually, such islands are found for very specific (typically large) parameter values (e.g., see Ref. 4). However, using the separatrix map approximation for the web map, it has been recently established that for the web-map such islands may appear also for very small parameter values.[25] Below, in Secs. 4 A and 4 B we present numerical examples for our main result, which is subsequently formulated in Theorem 4.1 of Sec. 4 B: we establish that for both the standard map and the web map, accelerator islands are necessarily contained in turnstile lobes, namely they are tangle islands.

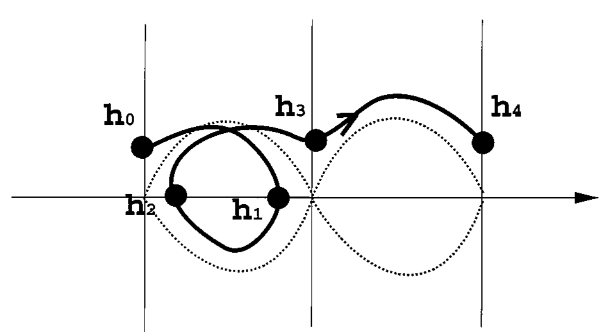

FIG. 3. Schematic phase space trajectory of separatrix islands. The stable periodic orbit of Fig. 1 as a perturbed pendulum orbit shown in the (x,p) plane. Dashed lines correspond to unperturbed pendulum separatrices.

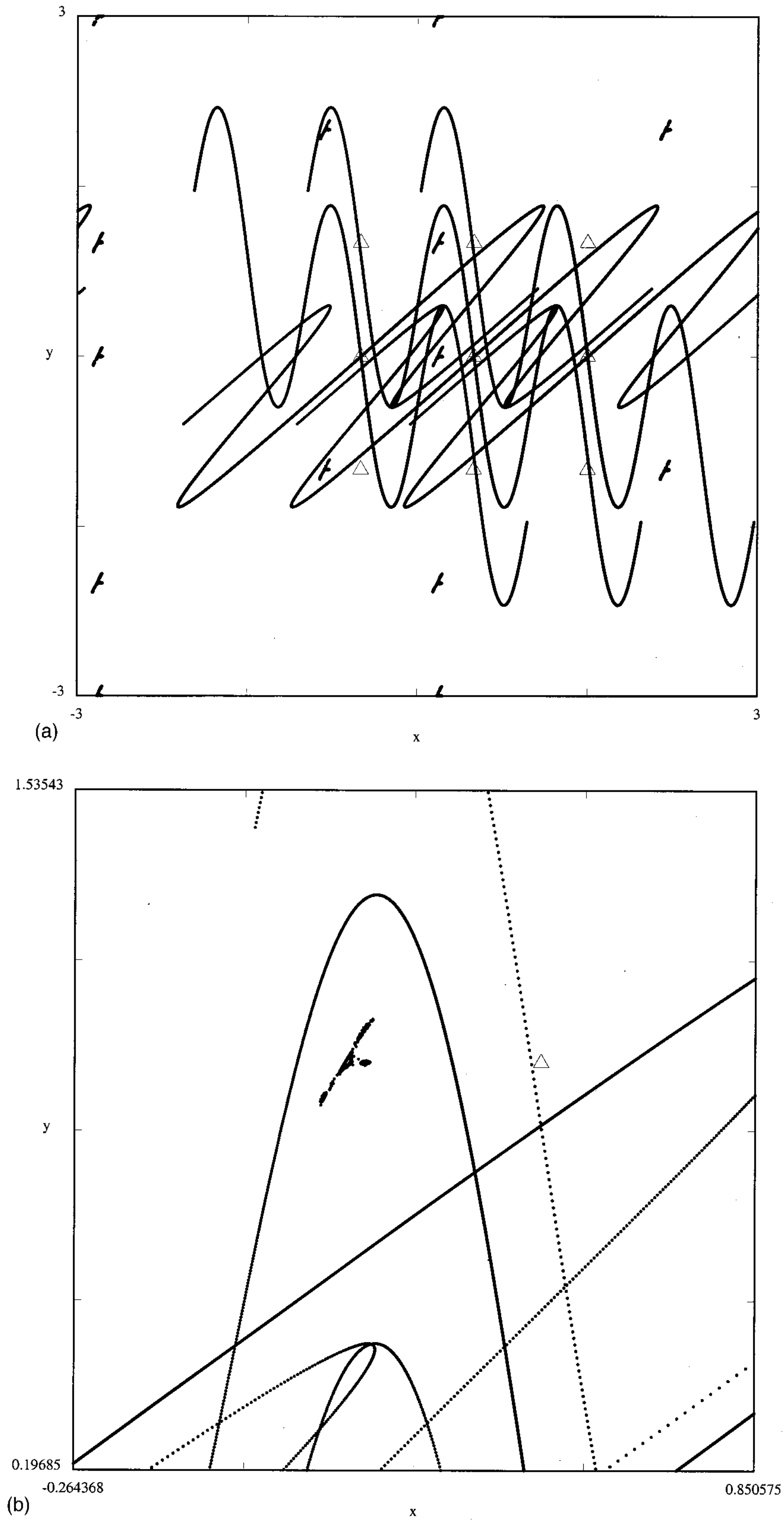

FIG. 4. Accelerator islands in the Standard map, $K=6.908745$. $\triangle$ denotes the unstable periodic orbits at $1/2\pi(x,p) = 1/2\pi((2n+1)\pi, 2m\pi)$ (a) Stable and unstable manifolds (solid lines) and period-one (mod 2π) stability island (birdlike shape) inside the lobes. (b) Zoom on the stability island.

A. Accelerator islands for the standard map

Consider the standard map:

$$p_{n+1}=p_n-K\sin x_n, \quad x_{n+1}=x_n+p_{n+1}, \quad (\mathrm{mod}\, 2\pi). \tag{4.1}$$

An example of an accelerator mode is

$$x_0=\pi/2, \quad p_0=0, \quad K^*=2\pi m \tag{4.2}$$

with integer m. Conditions (4.2) define linearly growing p_n

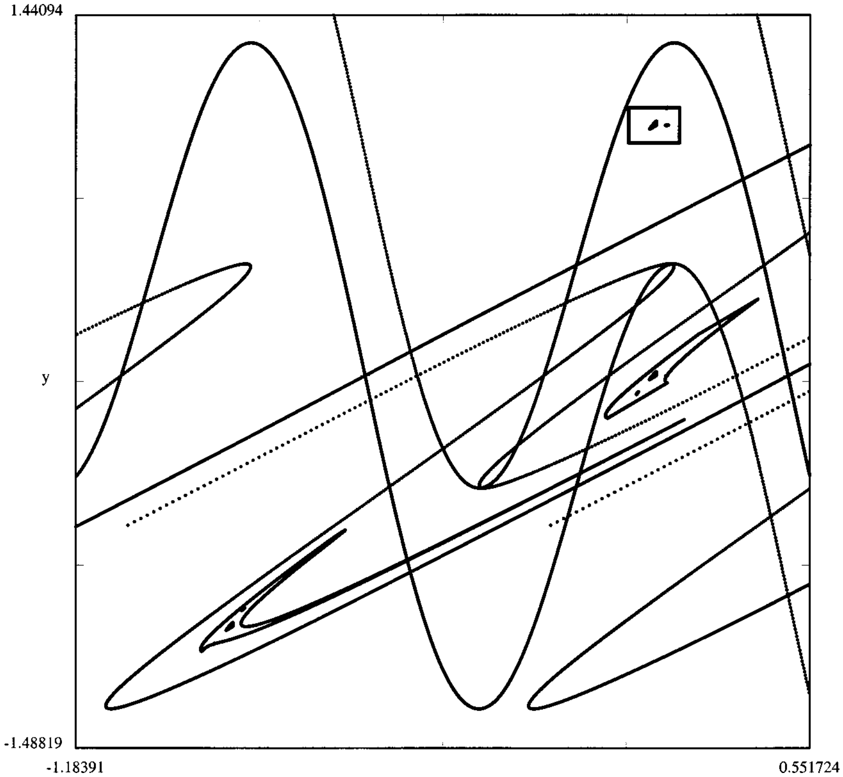

FIG. 5. Motion of initial conditions around accelerating island. Same parameter and island as in Fig. 4. A box of initial conditions around the island is iterated twice.

(momentum) which means growth of the energy. Then, analysis of small deviations around this accelerator mode produces the following effective Hamiltonian (see Ref. 4):

$$H_{\rm ef}=\frac{1}{2}(\Delta p)^2+\Delta K\Delta x-\frac{\pi}{3}(\Delta x)^3 \tag{4.3}$$

with the equations of motion

$$\frac{d}{d\tau}\Delta p=-\frac{\partial H_{ef}}{\partial\Delta\phi},\quad \frac{d}{d\tau}\Delta x=\frac{\partial H_{ef}}{\partial\Delta h} \tag{4.4}$$

which define the dynamics inside the accelerator island if $\Delta K>0$.

Indeed, we establish the existence of accelerator mode islands inside the turnstile lobes of a fundamental region of the standard map; For $K=6.908\,45$, we find the stable and unstable manifolds of the hyperbolic fixed points at $(x,p)=(0.5,0)\ (\mathrm{mod}\ 1)$ and the location of the accelerator mode island. In Fig. 4 it is demonstrated that the island is located inside the turnstile lobe. In Fig. 5 we put a box of initial conditions which surround the island and resides within the turnstile lobe. The dynamics of this box clearly demonstrates how a finite area around the accelerating mode island is dragged along and stretched with the lobes.

Usually, one expects tangle islands to be quite small. However, when the lobes are large, large islands may be created. This is the situation with the accelerating modes of the standard and web map, as shown in Fig. 4.

B. Accelerator modes for the web map

Consider the web map

$$u_{n+1}=v_n,\quad v_{n+1}=-u_n-K\sin v_n,\quad (\mathrm{mod}\ 2\pi). \tag{4.5}$$

The near-integrable limit of this map, corresponding to small K values, consists of a diamond shape web, created by the stable and unstable manifolds of the hyperbolic periodic orbits at $(u,v)=(m\pi,(2n+m+1)\pi)$. These hyperbolic periodic orbits persist for all K and so are the cells defined by them. Below we refer to these cells as ''resonance cells.'' Theorem 2.2 may be easily modified for this case, showing that in the near integrable limit tangle islands cannot be monotone with respect to the centers of these cells (here all monotone orbits are of oscillatory type).

An accelerator mode is found for

$$K=2\pi,\quad u=3\frac{\pi}{2},\quad v=\frac{\pi}{2}. \tag{4.6}$$

A corresponding Hamiltonian is[4]

$$H_{ef}=\frac{1}{2}\Delta K(\Delta v-\Delta u)-\frac{\pi}{6}[(\Delta v)^3-(\Delta u)^3], \tag{4.7}$$

which together with the equations of motion

$$\frac{d}{d\tau}\Delta u=\frac{\partial H_{ef}}{\partial\Delta v},\quad \frac{d}{d\tau}\Delta v=-\frac{\partial H_{ef}}{\partial\Delta u} \tag{4.8}$$

provides a description of the accelerator islands in the degenerate unperturbed case. Indeed, in Fig. 6 we see that this accelerator island is contained in turnstile lobes of the hyperbolic periodic orbit consisting of the points $\{(0,\pm\pi),(\pm\pi,0)\}$.

Theorem 4.1. The islands corresponding to accelerator modes of the Standard map and the Web map are contained in turnstile lobes, namely they are tangle islands.

Proof. Accelerator modes, by definition, correspond to periodic orbits which jump in momentum (the p variable for

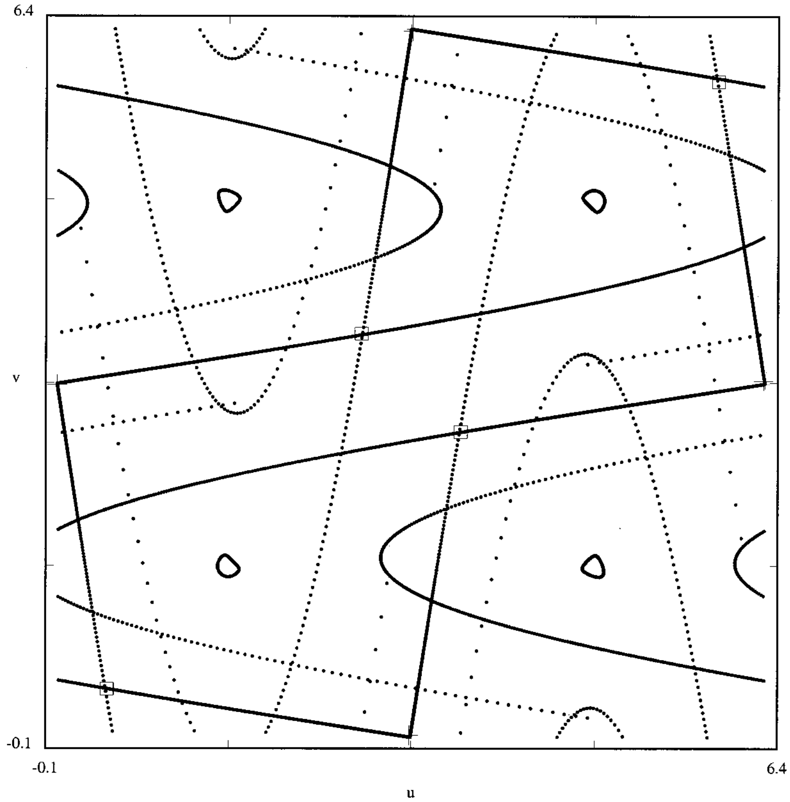

FIG. 6. Accelerator island inside lobes for the web map, $K=6.3$. Stable and unstable manifolds of the period four orbit $(u,v)=\{(\pm\pi,0),(0,\pm\pi)\}$ (crosses) are shown (solid line). Deformed circles correspond to period-4 accelerator island. Boxes denote primary homoclinic points defining the resonance cell.

the standard map and the u or v variable for the Web map). For both maps, there exist well defined resonance cells with well defined turnstile lobes for all parameter values (for the standard map, the cells defined by the stable and unstable manifolds of $p=2n\pi$, $x=\pm\pi$, for the web map, the cells defined by the stable and unstable manifolds of the period-four periodic orbits $u=m\pi$, $v=(2n+m+1)\pi$, m, $n\in Z$). Denote the resonance cells according to their central position: for the standard map R_n for the cell centered at $(0,2n\pi)$ and for the web map $R_{m,n}$ for the cell centered at $(m\pi,n\pi)$. These resonance zones are bounded and disjoint regions. In fact, it can be verified that for all K values they do not extend beyond the basic periodic cell [e.g., $(x,p)\in[-\pi,\pi]\times[(2n-1)\pi,(2n+1)\pi]$ for the standard map], see, for example, Refs. 13 and 12 and references therein. Indeed, for large K the region R_0 asymptotes the parallelogram with vertices $[(-\pi,0),(x_u,p_u),(\pi,0),(-x_u,-p_u)]$, where

$$x_u=\pi\left(1-\frac{2}{K+1}+O\left(\frac{1}{K^2}\right)\right),$$
$$p_u=2\pi\frac{K}{K+1}\left(1-\frac{1}{K+1}+O\left(\frac{1}{K^2}\right)\right), \tag{4.9}$$

and the region R_i corresponds to a translate of R_0 by $2\pi i$ along the p axis. Similarly, the regions $R_{n,m}$ asymptote for large K a double square shape which is bounded by the basic cell unit (see Fig. 6).

For the web map the union of all the regions $R_{m,n}$ supplies a complete partition of $\mathbf{R}^2$ to regions which are separated by partial barriers—the segments of the stable and unstable manifolds. By orientation preservation, the only mechanism of transfer from one cell to the other is through the turnstile lobes.[15] Since accelerator modes, by definition, cannot stay in one cell, they must be contained in a turnstile lobe, hence they are tangle islands.

Now consider the standard map. Here the main resonance zones R_i do not partition the phase space. Denote the gap between cell R_i and cell R_{i+1} by $G_{i,i+1}$. For all K (see asymptotic form of R_i above), it is a cylindrical band which is bounded from below (respectively, from above) by the segments of stable and unstable manifolds of the fixed point at $x=\pi$, $p=2i\pi$ (resp. $p=2(i+1)\pi$) which define the upper (resp. lower) boundary of the region R_i (resp. R_{i+1}). The union of all the regions R_i and $G_{i,i+1}$ for all i does supply a complete partition of the cylinder with a set of regions separated by segments of stable and unstable manifolds. Moreover, the turnstile lobes between region $G_{i,i+1}$ and its neighbors R_i, R_{i+1} are exactly the E,D lobes defined for the R_i's. Repeating the arguments as for the web map, it follows that the accelerator modes must be tangle islands.

V. DISCUSSION

We defined tangle islands as islands contained in turnstile lobes and proved that in the near-integrable limit this definition provides a distinct class of islands which is different from the usual resonance islands. In particular these islands correspond to nonmonotonic motion in both rotational and oscillatory coordinate systems. We demonstrated that tangled islands can be of the ballistic or accelerator type, whereas resonance islands can be either of trapped (oscillatory) or ballistic (rotational) type. We proved that for the standard map and the web map accelerator islands are necessarily tangle islands. It is still an open question whether, in the strongly chaotic regime, where large accelerator modes appear, the tangle islands are necessarily nonmonotone.

Clearly both resonance islands and tangle islands may have themselves their own subislands which again can be of either resonance or tangle type. Since one expects that the width of the separatrix splitting of the sub-island chains falls off exponentially with each sub-structure, one expects that the tangle islands of the second generation will be extremely small (in Ref. 8 it is proved that if these tangle islands exist, then there could be a mechanism for creating self-similar tangle island structures around them).

In the definition of the resonance islands we have used the monotonicity property which is associated with periodic orbits of twist maps. Two sources for oscillatory islands which are not monotone may arise. First, when the twist condition is violated at a certain frequency nearby resonances have special structure which may provide nonmonotone orbits.[14] The second source may be islands produced by homoclinic bifurcations:[7,8] Notice that tangle islands are produced by homoclinic bifurcations. We now argue that such bifurcations may produce, in addition, oscillatory islands of a different type. Indeed, consider first ''open'' systems—the turnstile lobes of these have no mechanism to self-intersect (like in the Hénon map). In ''open'' systems, homoclinic tangencies may occur only outside of the turnstile lobes and their images—hence if any islands are created they are necessarily of an oscillatory nature. In particular these must have the same rotation number as the base periodic orbit. Are these islands ordered with respect to the base periodic orbit? Hockett and Holmes[26] results show that unstable periodic orbits associated with the horseshoes which are created near the homoclinic points may have arbitrary order. It is an open question whether the stability islands which are created by saddle-center bifurcations of these orbits are necessarily well ordered (knot theory may be useful to investigate such questions[27]). If they are, then they may be included in the category of the resonance islands. If not, then they necessarily create a new category, since tangle islands cannot exist in ''open'' systems. Thus, we conclude that for any two dimensional mapping: Either the Newhouse islands are well ordered, or, there exist a third category of islands which are not resonant nor of a tangle type.

ACKNOWLEDGMENTS

R.K. acknowledges the support of the Jakubskind–Cymerman Prize and the research grant from the Henri Gutwirth fund for research. G.Z. was supported by the U.S. Navy Grants N00014-93-1-0218, N00014-97-1-0426, and U.S. Department of Energy Grant DE-FG02-92ER54184. Part of the work was done during the G.Z. visit of the Weizmann Institute of Science.

[1] V. I. Arnold, *Dynamical Systems, III, Encyclopedia of Mathematical Sciences* (Springer-Verlag, Berlin, 1988).
[2] H. Poincaré, ''Sur les équations de la dynamique et le problème de trois corps,'' Acta Math. Acad. Sci. Hung. **13**, 1–270 (1890).
[3] C. F. F. Karney, ''Long time correlations in the stochastic regime,'' Physica D **8**, 360 (1983).
[4] G. M. Zaslavsky, M. Edelman, and B. Niyazov, ''Self-similarity, renormalization and phase space nonuniformity of Hamiltonian chaotic dynamics,'' Chaos **7**, 159 (1997).
[5] P. Duarte, ''Plenty of elliptic islands for the standard family of area preserving maps,'' Ann. Inst. Henri Poincaré Anal. Non Linaire **11**, 359–409 (1994).
[6] D. Turaev and V. Rom-Kedar, ''Islands appearing in near-ergodic flows,'' Nonlinearity **11**, 575–600 (1998); V. Rom-Kedar and D. Turaev, ''Big islands in dispersing billiard-like potentials,'' Physica D **130**, 187–210 (1999).
[7] S. Newhouse, ''Quasi-elliptic periodic points in conservative dynamical systems,'' Am. J. Math. **99**, 1061–1087 (1977).
[8] V. K. Melnikov, ''On the existence of self-similar structures in the resonance domain,'' in *Transport, Chaos and Plasma Physics, II, Proceedings, Marseille*, edited by F. Doveil, S. Benkadda, and Y. Elskens (World Scientific, Singapore, 1996), pp. 142–153.
[9] S. Aubry, ''The twist map, the extended Frenkel-Kontorova model and the devil's staircase,'' Physica D **7**, 240–258 (1983).
[10] J. N. Mather, ''A criterion for non-existence of invariant circles,'' Publ. Math. I.H.E.S. **63**, 153–204 (1986).
[11] R. S. MacKay and J. D. Meiss, ''Linear stability of periodic orbits in Lagrangian systems,'' Phys. Lett. A **98**, 92–94 (1983).
[12] J. D. Meiss, ''Symplectic maps, variational principles, and transport,'' Rev. Mod. Phys. **64**, 795–848 (1992).
[13] R. S. MacKay, J. D. Meiss, and I. C. Percival, ''Resonances in area preserving maps,'' Physica D **27**, 1–20 (1987).
[14] D. del Castillo-Negrete, J. M. Green, and P. J. Morrison, ''Area-preserving nontwist maps: Periodic orbits and transition to chaos,'' Physica D **61**, 1–23 (1996).
[15] V. Rom-Kedar and S. Wiggins, ''Transport in two-dimensional maps,'' Arch. Ration. Mech. Anal. **109**, 239–298 (1990).
[16] G. M. Zaslavsky, D. Stevens, and M. Weitzner, ''Self-similar transport in incomplete chaos,'' Phys. Rev. E **48**, 1683–1694 (1993).
[17] R. White, S. Benkadda, S. Kassibrakis, and G. M. Zaslavsky, ''Near threshold anomalous transport in the standard map,'' Chaos **8**, 757–767 (1998).
[18] G. M. Zaslavsky and N. N. Filonenko, ''Stochastic instability of trapped particles and conditions of applicability of the quasi-linear approximation,'' Sov. Phys. JETP **25**, 851–857 (1968).
[19] B. V. Chirikov, ''A universal instability of many-dimensional oscillator systems,'' Phys. Rep. **52**, 263–379 (1979).
[20] D. F. Escande, ''Stochasticity in classical Hamiltonian systems: Universal aspects,'' Phys. Rep. **121**, 165–261 (1985).
[21] V. Rom-Kedar, ''Homoclinic tangles—classification and applications,'' Nonlinearity **7**, 441–473 (1994).
[22] V. Rom-Kedar, ''Secondary homoclinic bifurcation theorems,'' Chaos **5**, 385–401 (1995).
[23] G. M. Zaslavsky and S. S. Abdullaev, ''Scaling properties and anomalous transport of particles inside the stochastic layer,'' Phys. Rev. E **51**, 3901–3910 (1995).
[24] A. Delshams and T. M. Seara, ''Splitting of separatrices in Hamiltonian systems with one and a half degrees of freedom,'' Math. Phys. Electron J. **3**, 1–40 (1997).
[25] A. Iomin, D. Gangardt, and S. Fishman, ''Nonlinear dynamics in periodic phase space,'' Phys. Rev. E **57**, 4054–4062 (1998).
[26] K. Hockett and P. Holmes, ''Josephson's junction, annulus maps, Birkhoff attractors, horseshoes and rotation sets,'' Ergod. Th. and Dynam. Sys. **6**, 205–239 (1986).
[27] R. W. Ghrist, P. J. Holmes, and M. C. Sullivan, *Knots and Links in Three-dimensional Flows* (Springer-Verlag, Berlin, 1997).