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An Analog of Tate's Conjecture Over Local and Finitely Generated Fields

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§1 Introduction

Let K be a field complete with respect to a non-Archimedean valuation (which is not assumed to be nontrivial), and let $\mathcal{X}$ be a separated scheme of finite type over K. Recall that the K-analytic space $\mathcal{X}^{an}$ in the sense of [Ber1] and [Ber2], which is associated with $\mathcal{X}$, is locally compact, countable at infinity, and locally arc-wise connected. Furthermore, $\mathcal{X}^{an}$ is compact if and only if $\mathcal{X}$ is proper; it is arc-wise connected if and only if $\mathcal{X}$ is connected, and the topological dimension of $\mathcal{X}^{an}$ is equal to the dimension of $\mathcal{X}$. In [Ber7], it was proven that $\mathcal{X}^{an}$ is locally contractible if $\mathcal{X}$ is smooth and the valuation on K is nontrivial. In the course of the proof, the homotopy type of $\mathcal{X}^{an}$ is described for a broad class of schemes, and one of the consequences of that description (see [Ber7, Theorem 10.1]) states that the cohomology groups $H^i(|\mathcal{X}^{an}|, \mathbf{Z})$ are always finitely generated and that there exists a finite separable extension K' of K such that for any non-Archimedean field K'' over K', one has $H^i(|(\mathcal{X} \otimes K')^{an}|, \mathbf{Z}) \xrightarrow{\sim} H^i(|(\mathcal{X} \otimes K'')^{an}|, \mathbf{Z})$. Moreover, the same facts are also true for the cohomology groups with compact support $H^i_c(|\mathcal{X}^{an}|, \mathbf{Z})$. Here, $|X|$ denotes the underlying topological space of a K-analytic space X. Recall that, if the valuation on K is nontrivial, the above cohomology groups (without support) coincide with the cohomology groups of the associated rigid analytic space (see [Ber2, §1.6]). Notice also that for any abelian group A flat over $\mathbf{Z}$, one has $H^i(|X|, \mathbf{Z}) \otimes A \xrightarrow{\sim} H^i(|X|, A)$ and $H^i_c(|X|, \mathbf{Z}) \otimes A \xrightarrow{\sim} H^i_c(|X|, A)$.

Assume now that K is one of the following fields: (a) a local non-Archimedean field, (b) a field finitely generated over $\mathbf{Z}$ (i.e., it is finitely generated as a field over the

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image of $\mathbf{Z}$ in it), or (c) the field of complex numbers $\mathbf{C}$. (The fields (b) and (c) are provided with the trivial valuation.) We set $\overline{\mathcal{X}} = \mathcal{X} \otimes K^a$, where K^a is an algebraic closure of K, and in the case (c), we denote by $\mathcal{X}(\mathbf{C})$ the associated complex analytic space. Let l be a prime integer different from char(K). The purpose of this paper is to relate the above groups in the cases (a) and (b) with the étale cohomology groups $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$ and $H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)$, provided with the continuous action of the Galois group $G = \mathrm{Gal}(K^a/K)$, and in the case (c) with the singular cohomology groups $H^i(\mathcal{X}(\mathbf{C}), \mathbf{Q})$, provided with the mixed Hodge structure. The form of the results obtained is similar to that of the Tate (resp., Hodge) conjecture which describes, in the case when K is finitely generated over $\mathbf{Z}$ (resp., $K = \mathbf{C}$) and $\mathcal{X}$ is projective smooth, the Galois invariant part of $H^{2i}(\overline{\mathcal{X}}, \mathbf{Q}_l(i))$ (resp., the (i,i)-part of $H^{2i}(\mathcal{X}(\mathbf{C}), \mathbf{Q}(i))$) in terms of the group of algebraic cycles of $\mathcal{X}$ of codimension i.

Let $\overline{\mathcal{X}}^{an} = (\mathcal{X} \otimes \widehat{K^a})^{an}$. The canonical homomorphisms

$$H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Z}) \longrightarrow H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Z}/l^n\mathbf{Z}) \longrightarrow H^i(\overline{\mathcal{X}}^{an}, \mathbf{Z}/l^n\mathbf{Z})$$

and the isomorphism $H^i(\overline{\mathcal{X}}, \mathbf{Z}/l^n\mathbf{Z}) \xrightarrow{\sim} H^i(\overline{\mathcal{X}}^{an}, \mathbf{Z}/l^n\mathbf{Z})$ of the comparison theorems [Ber2, Theorem 7.5.4] and [Ber4, Theorem 3.2] give rise to a homomorphism $H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l) \to H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$. Since it is Galois equivariant, its image is contained in $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)^{sm}$, where for an l-adic representation V of G we denote by V^{sm} the subspace consisting of the elements with an open stabilizer in G. The above homomorphism composed with the canonical map $H^i(|\mathcal{X}^{an}|, \mathbf{Q}_l) \to H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l)$ gives rise to a homomorphism $H^i(|\mathcal{X}^{an}|, \mathbf{Q}_l) \to H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$ whose image is contained in $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)^G$. In the same way, one constructs a canonical Galois equivariant homomorphism of cohomology groups with compact support $H^i_c(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l) \to H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)$. To formulate the main result of the note, we introduce additional notation for each of the types (a)–(c) of the field K.

First, assume that K is a local non-Archimedean field, and let p be the characteristic of its residue field $\widetilde{K}$. (Since $l \neq \mathrm{char}(K)$, the equality $l = p$ may happen only in the case when K is a finite extension of $\mathbf{Q}_p$.) Let F be a fixed element of G that lifts the geometric Frobenius automorphism of the algebraic closure of $\widetilde{K}$ over $\widetilde{K}$. It is known that in the case $l \neq p$ the eigenvalues of F on $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$ and $H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)$ are Weil numbers of weights greater than or equal to zero (see [deJ] or §5). For an l-adic representation V of G, let V_0 denote the maximal F-invariant subspace of V, where all eigenvalues of F are Weil numbers of weight zero. One evidently has $V^{sm} \subset V_0$.

Furthermore, assume that K is finitely generated over $\mathbf{Z}$. Let $\mathcal{S}$ be a model of K, that is, an irreducible normal scheme of finite type over $\mathbf{Z}$ whose field of rational functions coincides with K. We fix, for each closed point $s \in \mathcal{S}$, a closed point $\overline{s}$ over s in the spectrum of the normalization of $\mathcal{O}_{\mathcal{S},s}$ in the separable closure of K in K^a. Let $k(\overline{s})$ be the residue field of $\bar{s}$. (It is an algebraic closure of the finite field $k(s)$, the residue field of $\mathcal{O}_{\mathcal{S},s}$.) Let $D_{\bar{s}}$ be the decomposition group of $\bar{s}$, that is, the stabilizer of $\bar{s}$ in G. Notice that the canonical homomorphism from $D_{\bar{s}}$ to the Galois group of $k(\bar{s})$ over $k(s)$ is surjective. Let $F_{\bar{s}}$ be a fixed element of $D_{\bar{s}}$ that lifts the geometric Frobenius automorphism of $k(\bar{s})$ over $k(s)$, and, for an l-adic representation V of G, let $V_{\bar{s},0}$ denote the maximal $F_{\bar{s}}$-invariant subspace of V, where all eigenvalues of $F_{\bar{s}}$ are Weil numbers of weight zero. One evidently has $V^{sm} \subset V_{\bar{s},0}$.

Finally, assume that $K = \mathbf{C}$. In this case, the singular cohomology groups $H^i(\mathcal{X}(\mathbf{C}), \mathbf{Q})$ are endowed with a (rational) mixed Hodge structure whose Hodge numbers $h^{p,q}$ are zero for $(p, q) \notin [0, i] \times [0, i]$. In particular, the weight filtration W_n is zero for $n < 0$ (see [Del3]). For a mixed Hodge structure H with $W_n = 0$ for $n < 0$, let H_0 denote the subspace W_0.

Theorem 1.1. (a) Suppose that K is a local non-Archimedean field.

(a′) If $l \neq p$, then $H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l) \xrightarrow{\sim} H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)_0$ and $H^i_c(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l) \xrightarrow{\sim} H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)_0$.

(a″) If $l = p$, then $H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_p) \xrightarrow{\sim} H^i(\overline{\mathcal{X}}, \mathbf{Q}_p)^{sm}$ and $H^i_c(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_p) \xrightarrow{\sim} H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_p)^{sm}$.

(b) If K is finitely generated over $\mathbf{Z}$, then, given a model $\mathcal{S}$ of K, there is a dense open subset $\mathcal{S}' \subset \mathcal{S}$ such that $H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l) \xrightarrow{\sim} H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)_{\bar{s},0}$ and $H^i_c(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l) \xrightarrow{\sim} H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)_{\bar{s},0}$ for every closed point $s \in \mathcal{S}'$.

(c) If $K = \mathbf{C}$, there is a canonical isomorphism $H^i(|\mathcal{X}^{an}|, \mathbf{Q}) \xrightarrow{\sim} H^i(\mathcal{X}(\mathbf{C}), \mathbf{Q})_0$. □

A canonical map from the first group to the second in (c) will be constructed in the proof. Recall that, in the case of positive characteristic of K, it is not yet known that the dimensions of the groups $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$ and $H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_l)$ do not depend on l.

Furthermore, since $H^i(|(\mathcal{X} \otimes K')^{an}|, \mathbf{Z}) \xrightarrow{\sim} H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Z})$ for a finite Galois extension K' of K, and since there is a canonical homeomorphism $\mathrm{Gal}(K'/K)\backslash|(\mathcal{X} \otimes K')^{an}| \xrightarrow{\sim} |\mathcal{X}^{an}|$, it follows from [Gro, §5.3] that $H^i(|\mathcal{X}^{an}|, \mathbf{Q}_l) \xrightarrow{\sim} H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l)^G$.

Corollary 1.2. In the situation of Theorem 1.1(a) and (b), one has

$$H^i\left(|\mathcal{X}^{an}|, \mathbf{Q}_l\right) \xrightarrow{\sim} H^i\left(\overline{\mathcal{X}}, \mathbf{Q}_l\right)^G \quad \text{and} \quad H^i_c\left(|\mathcal{X}^{an}|, \mathbf{Q}_l\right) \xrightarrow{\sim} H^i_c\left(\overline{\mathcal{X}}, \mathbf{Q}_l\right)^G. \qquad \square$$

The proof of the four cases of Theorem 1.1 is done in a parallel way. First of all, using the construction of P. Deligne from [Del3] together with the results of J. de Jong from [deJ] (and some extra arguments), the situation is reduced to the case of a scheme $\mathcal{X}$ of a very special type. The main ingredients of the proof in that case are, on the one hand, results from [Ber7] on the homotopy description of the space $\mathcal{X}^{an}$ and, on the other hand, the following well-known results: P. Deligne's Hodge theory [Del2] (see Theorem 1.1(c)), the Weil conjecture proven by P. Deligne [Del4] (see Theorem 1.1(b)), the description of vanishing cycles sheaves in the semistable reduction case by M. Rapoport and T. Zink [RaZi] (see Theorem 1.1(a′)), and the results of T. Tsuji [Tsu1] and G. Faltings [Fal] on the Fontaine-Jannsen conjecture (see Theorem 1.1(a″)).

In a previous version of the manuscript Theorem 1.1(a″) was proven under the assumption that $\mathcal{X}$ is proper. I am very grateful to M. Kisin for explaining to me that G. Faltings's results from [Fal] can be used to withdraw the assumption.

§2 Proof of Theorem 1.1(c)

By the construction from [Del3, §6.2] and Hironaka's theorem on resolution of singularities (or [deJ, Theorem 4.1]), there exists a proper hypercovering $\mathcal{X}_\bullet \to \mathcal{X}$ such that each $\mathcal{X}_n$ is isomorphic to an open subset of a projective smooth scheme over $\mathbf{C}$. By [SGA4, Exp. V bis], it gives rise to spectral sequences ${}'E_1^{p,q} = H^q(|\mathcal{X}_p^{an}|, \mathbf{Q}) \Rightarrow H^{p+q}(|\mathcal{X}^{an}|, \mathbf{Q})$ and ${}''E_1^{p,q} = H^q(\mathcal{X}_p(\mathbf{C}), \mathbf{Q}) \Rightarrow H^{p+q}(\mathcal{X}(\mathbf{C}), \mathbf{Q})$. By [Ber7], the connected components of each $\mathcal{X}_p^{an}$ are contractible. It follows that ${}'E_1^{p,q} = 0$ for $q \geq 1$ and, therefore, there are canonical isomorphisms ${}'E_2^{i,0} \xrightarrow{\sim} H^i(|\mathcal{X}^{an}|, \mathbf{Q})$. On the other hand, by [Del2, Corollary 3.2.15(ii)], the mixed Hodge structure on $H^i(\mathcal{X}_p(\mathbf{C}), \mathbf{Q})$ has the property that $W_n = 0$ for $n < i$. Since the functor $H \mapsto H_0 = W_0$ on the category of (rational) mixed Hodge structures H with $W_n = 0$ for $n < 0$ is exact (see [Del2, Theorem 2.3.5(iv)]), the latter implies that $({}''E_1^{p,q})_0 = 0$ for $q \geq 1$ and, therefore, there are canonical isomorphisms $({}''E_2^{i,0})_0 \xrightarrow{\sim} H^i(\mathcal{X}(\mathbf{C}), \mathbf{Q})_0$. Thus, the canonical isomorphisms ${}'E_1^{i,0} \xrightarrow{\sim} ({}''E_1^{i,0})_0$ give rise to an isomorphism $H^i(|\mathcal{X}^{an}|, \mathbf{Q}) \xrightarrow{\sim} H^i(\mathcal{X}(\mathbf{C}), \mathbf{Q})_0$. That the latter does not depend on the choice of a proper hypercovering follows from the fact that for any pair of proper hypercoverings of $\mathcal{X}$ there exists a proper hypercovering of the above form that refines them.

§3 Proof of Theorem 1.1(b)

First of all, since the functors $V \mapsto V_{\bar{s},0}$ on the category of l-adic representations of G are exact, the five-lemma reduces the statement for cohomology with compact support to that for cohomology without support. Let $\mathcal{S}$ be a model of K. Shrinking it, we may assume that l is invertible in $\mathcal{S}$. By the construction from [Del3, §6.2] and [deJ, Theorem 4.1], there exists a proper hypercovering $\mathcal{X}_\bullet \to \mathcal{X}$ such that each $\mathcal{X}_n$ is a finite disjoint union of schemes $\mathcal{X}'$, and, in turn, each $\mathcal{X}'$ is an open subscheme of a projective smooth scheme $\mathcal{Y}'$ over a finite extension K' over K such that $\mathcal{Y}'\backslash\mathcal{X}'$ is a strict normal crossings divisor in $\mathcal{Y}'$. It gives rise to spectral sequences ${}'E_1^{p,q} = H^q(|\overline{\mathcal{X}}_p^{an}|, \mathbf{Q}_l) \Rightarrow H^{p+q}(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l)$ and ${}''E_1^{p,q} = H^q(\overline{\mathcal{X}}_p, \mathbf{Q}_l) \Rightarrow H^{p+q}(\overline{\mathcal{X}}, \mathbf{Q}_l)$. Using again the exactness of the functors $V \mapsto V_{\bar{s},0}$, we see that in order to prove that $H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l) \xrightarrow{\sim} H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)_{\overline{s},0}$ for all closed points s from a dense open subset $\mathcal{S}'$ of S, it suffices to check that one can find $\mathcal{S}'$ such that ${}'E_1^{p,q} \xrightarrow{\sim} ({}''E_1^{p,q})_{\overline{s},0}$ for all $0 \le p \le n_0$, $q \ge 0$ and all closed points $s \in \mathcal{S}'$. We also notice that to check the required fact for $\mathcal{X}'$, considered as a scheme over K, and the model $\mathcal{S}$ of K, it suffices to check it for $\mathcal{X}'$, considered as a scheme over K', and the normalization of $\mathcal{S}$ in K'. This reduces the statement to the case where $\mathcal{X}$ is an open subset of a projective smooth scheme $\mathcal{X}'$ over K such that $\mathcal{X}'\backslash\mathcal{X}$ is a strict normal crossings divisor in $\mathcal{X}'$. In this case, it follows from [Ber7] that the connected components of $\overline{\mathcal{X}}^{an}$ are contractible and, therefore, $H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Q}_l) = 0$ for all $i \ge 1$. This means that we have to check that one can shrink $\mathcal{S}$ so that $H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)_{\overline{s},0} = 0$ for all $i \ge 1$ and all closed points $s \in \mathcal{S}$.

Assume first that $\mathcal{X} = \mathcal{X}'$. Shrinking $\mathcal{S}$, we may assume that $\mathcal{X}$ is the generic fiber $\mathcal{Y}_\eta$ of a scheme $\mathcal{Y}$ projective and smooth over $\mathcal{S}$. We claim that the required fact is true for every closed point $s \in \mathcal{S}$. Indeed, let φ denote the canonical morphism $\mathcal{Y} \to \mathcal{S}$. By the specialization theorem for proper smooth morphisms (see [SGA4, Exp. XVI, 2.2]), the constructible sheaves $R^i\varphi_*(\mathbf{Q}_l)$ are smooth, and, in particular, there is a specialization isomorphism $H^i(\overline{\mathcal{Y}}_s, \mathbf{Q}_l) \xrightarrow{\sim} H^i(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_l) = H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)$ equivariant with respect to the surjective homomorphism from $D_{\overline{s}}$ to the Galois group of $k(\overline{s})$ over $k(s)$. From the Weil conjecture proven by P. Deligne (see [Del4]), it follows that the eigenvalues of $F_{\overline{s}}$ on $H^i(\overline{\mathcal{Y}}_s, \mathbf{Q}_l)$ are Weil numbers of weight i, and the required fact follows.

In the general case, let $Z_1, \dots, Z_n$ be the irreducible components of the divisor $\mathcal{X}'\backslash\mathcal{X}$. Shrinking $\mathcal{S}$ we may assume that $\mathcal{X}'$ and all of the intersections $\cap_{j\in J} Z_j$ for $J \subset \{1, \dots, n\}$ are the generic fibers of projective smooth schemes over $\mathcal{S}$, and we claim again that the required fact is true for every closed point $s \in \mathcal{S}$. For this, we set $\mathcal{X}_j = \mathcal{X}'\backslash(Z_1 \cup \cdots \cup Z_j)$, $0 \le j \le n$. (One has $\mathcal{X}_0 = \mathcal{X}'$ and $\mathcal{X}_n = \mathcal{X}$.) Consider the Gysin exact sequence

$$\cdots \longrightarrow H^i(\overline{\mathcal{X}}_j, \mathbf{Q}_l) \longrightarrow H^i(\overline{\mathcal{X}}_{j+1}, \mathbf{Q}_l) \longrightarrow H^{i-1}(\overline{Z}_{j+1} \cap \overline{\mathcal{X}}_j, \mathbf{Q}_l(-1)) \longrightarrow \cdots .$$

We know that the eigenvalues of $F_{\overline{s}}$ on $H^i(\overline{\mathcal{X}}_0, \mathbf{Q}_l)$ and $H^i(\overline{Z}_1, \mathbf{Q}_l(-1))$ are Weil numbers of weight i and $i+2$, respectively. Since $Z_{j+1} \cap \mathcal{X}_j = Z_{j+1}\backslash(Z_1 \cup \cdots \cup Z_j)$, the induction on j shows that the eigenvalues of $F_{\overline{s}}$ on $H^i(\overline{\mathcal{X}}_j, \mathbf{Q}_l)$ are Weil numbers of weights greater than or equal to i, and the required fact follows.

§4 Topological vanishing cycles for formal schemes

Let k be a non-Archimedean field. Using a construction from [Ber3], one can define as follows for any formal scheme $\mathfrak{X}$ locally finitely presented over k°, the ring of integers of k, a functor θ from the category of sheaves on the underlying topological space of the generic fiber $\mathfrak{X}_\eta$ to the category of sheaves in the Zariski topology of the closed fiber $\mathfrak{X}_s$. Given a sheaf F on $\mathfrak{X}_\eta$, one sets $\theta(F)(\mathcal{U}) = F(\pi^{-1}(\mathcal{U}))$ for each open subset $\mathcal{U} \subset \mathfrak{X}_s$, where π denotes the reduction map $\mathfrak{X}_\eta \to \mathfrak{X}_s$, and, for a subset $V \subset \mathfrak{X}_\eta$, $F(V)$ denotes the set of global sections of the pullback of F to V. It follows from [Ber3, §4] that the correspondence $\mathcal{U} \mapsto \theta(F)(\mathcal{U})$ is a sheaf in the Zariski topology of $\mathfrak{X}_s$, and, if F is an abelian soft sheaf, then $\theta(F)$ is a flabby (cohomologically trivial) sheaf. It follows that for each $i \geq 0$, the sheaf $R^i\theta(F)$ is associated with the presheaf $\mathcal{U} \mapsto H^i(\pi^{-1}(\mathcal{U}), F)$.

Lemma 4.1. If $\mathfrak{X}$ is a strictly polystable formal scheme over k°, then $R^i\theta(\Lambda_{\mathfrak{X}_\eta}) = 0$ for all $i \geq 1$ and all abelian groups Λ, and, in particular, $H^i_{\mathrm{Zar}}(\mathfrak{X}_s, \Lambda) \xrightarrow{\sim} H^i(|\mathfrak{X}_\eta|, \Lambda)$ for all $i \geq 0$. □

Strictly polystable schemes and formal schemes are introduced in [Ber7, §1]. They include strictly semistable schemes (over k° with discrete valuation) and the formal completions of the latter along their closed fibers. (Only such schemes and formal schemes are actually used in §5 and §6.)

Proof. It suffices to check that each point of $\mathfrak{X}_s$ has an open neighborhood $\mathcal{U}$ such that the space $\pi^{-1}(\mathcal{U})$ is contractible. Since $\mathfrak{X}$ is strictly polystable, we can shrink it and assume that $\mathfrak{X}_s$ is elementary; that is, the set of strata of $\mathfrak{X}_s$ has a unique maximal element (see [Ber7, §2]). In this case, the geometric realization of the nerve $N(\mathfrak{X}_s)$ of the set of strata is contractible. On the other hand, by [Ber7, Lemmas 3.10 and 3.12], $|N(\mathfrak{X}_s)|$ is homeomorphic to the geometric realization of the polysimplicial set $C(\mathfrak{X}_s)$ associated with $\mathfrak{X}_s$. It remains to apply [Ber7, Theorems 5.2 and 5.4], which imply that $\mathfrak{X}_\eta$ is homotopy equivalent of $|C(\mathfrak{X}_s)|$. ■

Notice also that the constructions of the above functor θ and the nearby cycles functor Θ from [Ber3] are compatible so that for any abelian torsion group Λ the following diagram is commutative:

$$\begin{array}{ccc} H^i_{\mathrm{Zar}}(\mathfrak{X}_s, \Lambda) & \longrightarrow & H^i(|\mathfrak{X}_\eta|, \Lambda) \\ \downarrow & & \downarrow \\ H^i(\mathfrak{X}_s, \Lambda) & \longrightarrow & H^i(\mathfrak{X}_\eta, \Lambda) \end{array}$$

where the horizontal arrows are specialization morphisms (induced by the spectral sequences for the functors θ and Θ), and the vertical arrows are the canonical maps.

For example, let $\mathfrak{Y}$ be a scheme proper and strictly polystable over k°. The above construction, applied to the formal completion of $\mathfrak{Y}$ along the closed fiber $\mathfrak{Y}_s$, gives rise to a commutative diagram

$$\begin{array}{ccc} H^i_{\mathrm{Zar}}(\mathcal{Y}_s, \mathbf{Z}/l^n\mathbf{Z}) & \longrightarrow & H^i(|\mathcal{Y}^{an}_\eta|, \mathbf{Z}/l^n\mathbf{Z}) \\ \downarrow & & \downarrow \\ H^i(\overline{\mathcal{Y}}_s, \mathbf{Z}/l^n\mathbf{Z}) & \longrightarrow & H^i(\overline{\mathcal{Y}}_\eta, \mathbf{Z}/l^n\mathbf{Z}) \end{array}$$

where $l \neq \mathrm{char}(k)$. Since $\mathcal{Y}$ is strictly polystable and proper, Lemma 4.1 implies that the high arrow is an isomorphism and, therefore, $H^i_{\mathrm{Zar}}(\mathcal{Y}_s, \mathbf{Q}_l) \widetilde{\to} H^i(|\mathcal{Y}^{an}_\eta|, \mathbf{Q}_l)$. On the other hand, the left and low arrows give rise to a canonical homomorphism $H^i_{\mathrm{Zar}}(\overline{\mathcal{Y}}_s, \mathbf{Q}_l) \to H^i(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_l)$.

§5 Proof of Theorem 1.1(a′)

As in §3, the exactness of the functor $V \mapsto V_0$ reduces the statement for cohomology with compact support to that for cohomology without support. By the construction from [Del3, §6.2] and [deJ, Theorem 6.5], there exists a proper hypercovering $\mathcal{X}_\bullet \to \mathcal{X}$ such that each $\mathcal{X}_n$ is a finite disjoint union of schemes $\mathcal{X}'$. In turn, each $\mathcal{X}'$ is an open subscheme in a scheme $\mathcal{Y}'$ projective over K'°, where K' is a finite extension of K, such that the pair $(\mathcal{Y}', \mathcal{Z}')$ with $\mathcal{Z}' = \mathcal{Y}' \backslash \mathcal{X}'$ is strictly semistable over K'° (see [deJ, §6.3]). By [SGA4, Exp. V bis] and the exactness of the functor $V \mapsto V_0$, the situation is reduced to the case where $\mathcal{X} = \mathcal{Y} \backslash \mathcal{Z}$ for a strictly semistable pair $(\mathcal{Y}, \mathcal{Z})$ over K°. By the results from [Ber7], $\overline{\mathcal{X}}^{an}$ is homotopy equivalent to $\overline{\mathcal{Y}}^{an}_\eta$, and the cohomology groups of both spaces coincide with their singular cohomology groups. Since the latter are homotopy invariant, there are canonical isomorphisms $H^i(|\overline{\mathcal{Y}}^{an}_\eta|, \mathbf{Z}) \widetilde{\to} H^i(|\overline{\mathcal{X}}^{an}|, \mathbf{Z})$. Thus, to prove the theorem, it suffices to prove it for $\mathcal{Y}_\eta$ and to show that $H^i(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_l)_0 \widetilde{\to} H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)_0$. By §4, to prove the theorem for $\mathcal{Y}_\eta$, it suffices to check that the specialization morphism induces an isomorphism $H^i_{\mathrm{Zar}}(\overline{\mathcal{Y}}_s, \mathbf{Q}_l) \widetilde{\to} H^i(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_l)_0$.

Consider the spectral sequence of vanishing cycles $E_2^{i,j} = H^i(\overline{\mathcal{Y}}_s, R^j\Psi_\eta(\mathbf{Q}_l)) \Rightarrow H^{i+j}(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_l)$. To prove the theorem for $\mathcal{Y}$, it suffices to verify the following statements:

(1) the eigenvalues of F on $H^i(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_l)$ are Weil numbers of weights greater than or equal to zero;

(2) the specialization morphism induces an isomorphism

$$H^i(\overline{\mathcal{Y}}_s, \mathbf{Q}_l)_0 \widetilde{\longrightarrow} H^i(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_l)_0;$$

(3) there is a canonical isomorphism

$$H^i_{\mathrm{Zar}}(\overline{\mathcal{Y}}_s, \mathbf{Q}_l) \widetilde{\longrightarrow} H^i(\overline{\mathcal{Y}}_s, \mathbf{Q}_l)_0.$$

Recall the description of the vanishing cycles sheaves which follows from the cohomological purity theorem (see [SGA4]) in the equicharacteristic case and is due to M. Rapoport and T. Zink [RaZi] in the case of mixed characteristic. Let $\{Y_i\}_{i\in I}$ be the family of irreducible components of $\mathcal{Y}_s$, and, for a subset $J \subset I$, let $Y_J = \cap_{i\in J} Y_i$. Furthermore, for $j \geq 0$, let Y^j denote the disjoint union of all Y_J with $\mathrm{card}(J) = j+1$, and let α^j denote the canonical morphism $Y^j \to \mathcal{Y}_s$. Then, for each $j \geq 0$, there is an exact sequence

$$0 \longrightarrow R^j\Psi_\eta(\mathbf{Q}_l) \longrightarrow \overline{\alpha}^j_*(\mathbf{Q}_l)(-j) \longrightarrow \overline{\alpha}^{j+1}_*(\mathbf{Q}_l)(-j) \longrightarrow \cdots$$

of étale sheaves on $\overline{\mathcal{Y}}_s$. It follows that the action of G on the sheaves $R^j\Psi_\eta(\mathbf{Q}_l)$ factors through an action of the Galois group of $\widetilde{K}$. The above exact sequence gives rise to a spectral sequence

$$'E_1^{m,n} = H^n(\overline{Y}^{m+j}, \mathbf{Q}_l(-j)) \Longrightarrow H^{m+n}(\overline{\mathcal{Y}}_s, R^j\Psi_\eta(\mathbf{Q}_l)).$$

Since the schemes Y^j are smooth, the Weil conjecture [Del4] implies that the eigenvalues of F on $'E_1^{m,n} = H^n(\overline{Y}^{m+j}, \mathbf{Q}_l(-j))$ are Weil numbers of weight $n+2j$, and, therefore, the eigenvalues of F on $E_2^{i,j} = H^i(\overline{\mathcal{Y}}_s, R^j\Psi_\eta(\mathbf{Q}_l))$ are Weil numbers of weights greater than or equal to $2j$. The statements (1) and (2) follow. It follows also that $('E_2^{i,0})_0 \widetilde{\rightarrow} H^i(\overline{\mathcal{Y}}_s, \mathbf{Q}_l)_0$. The similar exact sequence

$$0 \longrightarrow \mathbf{Q}_l \longrightarrow \overline{\alpha}^0_*(\mathbf{Q}_l) \longrightarrow \overline{\alpha}^1_*(\mathbf{Q}_l) \longrightarrow \cdots$$

of sheaves in the Zariski topology of $\overline{\mathcal{Y}}_s$ gives rise to a spectral sequence

$$''E_1^{m,n} = H^n_{\mathrm{Zar}}(\overline{Y}^{m+j}, \mathbf{Q}_l) \Longrightarrow H^{m+n}_{\mathrm{Zar}}(\overline{\mathcal{Y}}_s, \mathbf{Q}_l),$$

which, in turn, gives an isomorphism $''E_2^{i,0} \widetilde{\rightarrow} H^i_{\mathrm{Zar}}(\overline{\mathcal{Y}}_s, \mathbf{Q}_l)$. Since

$$H^0_{\mathrm{Zar}}(\overline{Y}^j, \mathbf{Q}_l) \xrightarrow{\sim} H^0(\overline{Y}^j, \mathbf{Q}_l),$$

the statement (3) follows.

Consider now the general case. We have to show that $H^i(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_l)_0 \widetilde{\rightarrow} H^i(\overline{\mathcal{X}}, \mathbf{Q}_l)_0$. For this we use the reasoning from the end of §3. Let $Z_1, \dots, Z_n$ be the irreducible components of $\mathcal{Z}$ that are flat over K°. By the definition of a strictly semistable pair (see [deJ, §6.3]), for each $J \subset \{1, \dots, n\}$ the scheme $\cap_{j\in J} Z_j$ is strictly semistable over K°. For $0 \leq j \leq n$, we set $\mathcal{X}_j = \mathcal{Y}_\eta \backslash (Z_{1,\eta} \cup \cdots \cup Z_{j,\eta})$. (One has $\mathcal{X}_0 = \mathcal{Y}_\eta$ and $\mathcal{X}_n = \mathcal{X}$.) Consider the Gysin exact sequence

$$\begin{aligned}\cdots \longrightarrow H^{i-2}\left(\overline{Z}_{j+1,\eta}\cap\overline{\mathcal{X}}_j,\mathbf{Q}_l(-1)\right) \longrightarrow H^i\left(\overline{\mathcal{X}}_j,\mathbf{Q}_l\right) \longrightarrow H^i\left(\overline{\mathcal{X}}_{j+1},\mathbf{Q}_l\right)\\ \longrightarrow H^{i-1}\left(\overline{Z}_{j+1,\eta}\cap\overline{\mathcal{X}}_j,\mathbf{Q}_l(-1)\right) \longrightarrow \cdots .\end{aligned} \qquad (*)$$

By the property (1), the eigenvalues of F on $H^i(\overline{\mathcal{X}}_0,\mathbf{Q}_l)$ and $H^i(\overline{Z}_{1,\eta},\mathbf{Q}_l(-1))$ are Weil numbers of weights greater than or equal to zero and greater than or equal to 2, respectively. Since $Z_{j+1,\eta}\cap\mathcal{X}_j = (Z_{j+1}\backslash(Z_1\cup\cdots\cup Z_j))_\eta$, the induction on j shows that the eigenvalues of F on $H^i(\overline{\mathcal{X}}_j,\mathbf{Q}_l)$ and $H^i(\overline{Z}_{j+1,\eta}\cap\overline{\mathcal{X}}_j,\mathbf{Q}_l(-1))$ are Weil numbers of weights greater than or equal to zero and greater than or equal to 2, respectively. It follows that $H^i(\overline{\mathcal{X}}_j,\mathbf{Q}_l)_0 \xrightarrow{\sim} H^i(\overline{\mathcal{X}}_{j+1},\mathbf{Q}_l)_0$; therefore, $H^i(\overline{\mathcal{Y}}_\eta,\mathbf{Q}_l)_0 \xrightarrow{\sim} H^i(\overline{\mathcal{X}},\mathbf{Q}_l)_0$.

Remark 5.1. An equivalent way to get the isomorphism $H^i_{\mathrm{Zar}}(\overline{\mathcal{Y}}_s,\mathbf{Q}_l) \xrightarrow{\sim} H^i(\overline{\mathcal{Y}}_\eta,\mathbf{Q}_l)_0$ is to use the weight spectral sequence

$$E_1^{i,j} = \bigoplus_{k=0}^{\infty} H^{j-2k}\left(\overline{Y}^{i+2k},\mathbf{Q}_l\right)(-k) \Longrightarrow H^{i+j}\left(\overline{\mathcal{Y}}_\eta,\mathbf{Q}_l\right)$$

from [RaZi, Theorem 2.10].

§6 Proof of Theorem 1.1(a″)

First of all, we consider the case $\mathcal{X} = \mathcal{Y}_\eta$, where $\mathcal{Y}$ is a strictly semistable projective scheme over K°. By §4, in this case it suffices to prove that the canonical homomorphism $\alpha : H^m_{\mathrm{Zar}}(\overline{\mathcal{Y}}_s,\mathbf{Q}_p) \to H^m(\overline{\mathcal{Y}}_\eta,\mathbf{Q}_p)^{\mathrm{sm}}$ is bijective. We remark that it suffices to check the bijectivity after a finite unramified extension of K.

Let K_0 be the maximal subfield of K unramified over $\mathbf{Q}_p$, W the ring of integers of K_0 (it coincides with the ring of Witt vectors $W(\widetilde{K})$), and σ the Frobenius automorphism of $\widetilde{K}$, W, and K_0. The Fontaine-Jannsen conjecture C_{st}, proven by T. Tsuji (see [Tsu1]), relates the étale cohomology group $H^m(\overline{\mathcal{Y}}_\eta,\mathbf{Q}_p)$, provided with the continuous action of G, and the log-crystalline cohomology group $H^m_{\text{log-crys}}(\mathcal{Y}) = H^m_{\text{log-crys}}(\mathcal{Y}_s/W)\otimes_W K_0$, where $H^m_{\text{log-crys}}(\mathcal{Y}_s/W)$ is a finite W-module provided with a σ-linear endomorphism φ which induces an automorphism on $H^m_{\text{log-crys}}(\mathcal{Y})$, a W-linear endomorphism N with $N\varphi = p\varphi N$, and a filtration $H^m_{\text{log-crys}}(\mathcal{Y}_s/W)\otimes_W K = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 \supset \cdots \supset \mathrm{Fil}^{m+1} = 0$.

Let B_{dR}, B_{crys}, and B_{st} be the rings introduced by J.-M. Fontaine [Fon]. The ring B_{dR} is a complete discrete valuation field over K with residue field $\widehat{K^a}$, and it is provided with a continuous action of G with $B_{\mathrm{dR}}^G = K$. The discrete valuation gives rise to a filtration on B_{dR}. The ring B_{crys} is a G-invariant K_0-subalgebra of B_{dR}, and the ring B_{st} is a G-invariant

B_{crys}-subalgebra of B_{dR} provided with a σ-linear injective endomorphism φ and a B_{crys}-derivation N such that $B_{st}^G = K_0$, $N\varphi = p\varphi N$, $B_{st}^{N=0} = B_{crys}$, and $B_{st}^{N=0,\,\varphi=1} \cap \mathrm{Fil}^0 B_{dR} = \mathbf{Q}_p$. (The ring B_{st} depends on the choice of a prime element in K°.)

By C_{st}, there is a canonical B_{st}-linear isomorphism

$$H^m(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{st} \xrightarrow{\sim} H^m_{\text{log-crys}}(\mathcal{Y}) \otimes_{K_0} B_{st},$$

which preserves the action of G, φ, N and the filtration after tensoring with B_{dR} over B_{st}, and is compatible with some additional structures including the specialization morphism.

Since $B_{st}^G = K_0$, one gets a canonical isomorphism $(H^m(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{st})^G \xrightarrow{\sim} H^m_{\text{log-crys}}(\mathcal{Y})$. Furthermore, since $B_{st}^{N=0,\,\varphi=1} \cap \mathrm{Fil}^0 B_{dR} = \mathbf{Q}_p$, one gets a canonical isomorphism

$$H^m(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_p) \xrightarrow{\sim} \left(H^m_{\text{log-crys}}(\mathcal{Y}) \otimes_{K_0} B_{st}\right)^{N=0,\varphi=1} \cap \mathrm{Fil}^0 \left(H^m_{\text{log-crys}}(\mathcal{Y}) \otimes_{K_0} B_{dR}\right).$$

Finally, since $B_{dR}^{sm} = K^a \subset \mathrm{Fil}^0 B_{dR}$ and $B_{st}^{sm} = K_0^{nr}$, the maximal unramified extension of K_0, one gets a canonical isomorphism

$$\beta : H^m(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_p)^{sm} \xrightarrow{\sim} \left(H^m_{\text{log-crys}}(\mathcal{Y}) \otimes_{K_0} K_0^{nr}\right)^{N=0,\,\varphi=1} = \left(H^m_{\text{log-crys}}(\mathcal{Y}) \otimes_{K_0} K_0^{nr}\right)^{\varphi=1}.$$

The latter equality is explained as follows. Let x be an element of the right-hand side. Then $x \in (H^m_{\text{log-crys}}(\mathcal{Y}) \otimes_{K_0} K')^{\varphi=1}$, where K' is a finite extension of K_0 in K_0^{nr}. It follows that $\varphi^a(Nx) = (1/p^a)Nx$, where p^a is the number of elements in $\widetilde{K}'$, and, therefore, $Nx = 0$ because φ^a is induced by a W-linear endomorphism of the finite W-module $H^m_{\text{log-crys}}(\mathcal{Y}_s/W)$.

Recall that $H^m_{\text{log-crys}}(\mathcal{Y}_s/W)$ is a projective limit of the cohomology groups $H^m_{\text{log-crys}}(\mathcal{Y}_s/W_n)$, where $W_n = W/p^nW$. The latter is the cohomology group of the structural sheaf in the log-crystalline topos $(\mathcal{Y}_s/W_n)^{\sim}_{\text{log-crys}}$ of $(\mathcal{Y}_s, M)$ over (S_n, L_n), where M is the log-structure on $\mathcal{Y}_s$ defined by $\mathcal{Y}$ and L_n is the canonical log-structure on $S_n = \mathrm{Spec}(W_n)$ (see [HyKa, §2 and §3]). Let $u^{\log}_{\mathcal{Y}_s/W_n}$ denote the canonical morphism of topoi $(\mathcal{Y}_s/W_n)^{\sim}_{\text{log-crys}} \to (\mathcal{Y}_s)^{\sim}_{\text{ét}}$, and let $\nu^{\log}_{\mathcal{Y}_s/W_n}$ denote the composition of $u^{\log}_{\mathcal{Y}_s/W_n}$ with the canonical morphism of topoi $(\mathcal{Y}_s)^{\sim}_{\text{ét}} \to (\mathcal{Y}_s)^{\sim}_{\text{Zar}}$. The latter induces a homomorphism $H^m_{\text{Zar}}(\mathcal{Y}_s, \mathbf{Q}_p) \to H^m_{\text{log-crys}}(\mathcal{Y})$.

From the isomorphism β, it follows that the action of G on $H^m(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_p)^{sm}$ is unramified. We can therefore replace the field K by a finite unramified extension and assume that $H^m(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_p)^{sm} = H^m(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_p)^G$. Furthermore, let us replace K by a finite unramified extension so that all of the strata of $\mathcal{Y}_s$ (the irreducible components of the intersections of families of irreducible components) are geometrically irreducible. In this case, *the right-hand side of* β *coincides with* $H^m_{\text{log-crys}}(\mathcal{Y})^{\varphi=1}$. Indeed, let $\{Y_i\}_{i\in I}$ be the family of irreducible components of $\mathcal{Y}_s$, $Y_J = \cap_{i\in J} Y_i$ for a subset $J \subset I$, and let Y^j be the disjoint union of all Y_J with $\text{card}(J) = j+1$, $j \geq 0$. Then there is a spectral sequence ${}'E_1^{i,j} = H^j_{\text{Zar}}(Y^i, W) \Rightarrow H^{i+j}_{\text{Zar}}(\mathcal{Y}_s, W)$. Since the strata of $\mathcal{Y}_s$ are geometrically irreducible, the similar spectral sequence for $\overline{\mathcal{Y}}_s$ implies that $H^m_{\text{Zar}}(\mathcal{Y}_s, W) \tilde{\to} H^m_{\text{Zar}}(\overline{\mathcal{Y}}_s, W)$. On the other hand, by a result of A. Mokrane (see [Mok, §3.23 and Theorem 3.32]), there is a weight spectral sequence (similar to that from Remark 5.1):

$$''E_1^{i,j} = \bigoplus_{k=0}^{\infty} H^{j-2k}_{\text{crys}}(Y^{i+2k}/W)(-k) \Longrightarrow H^{i+j}_{\text{log-crys}}(\mathcal{Y}_s/W).$$

Let p^a be the number of elements in $\widetilde{K}$. By K. Katz and W. Messing [KaMe], the eigenvalues of φ^a on $H^{j-2k}_{\text{crys}}(Y^{i+2k}/W) \otimes_W K_0$ are Weil numbers of weight $j-2k$. It follows that the canonical morphism of spectral sequences ${}'E \to {}'' E$, which is induced by the morphisms of topoi $\nu^{\log}_{\mathcal{Y}_s/W_n}$, gives rise to an isomorphism $H^m_{\text{Zar}}(\mathcal{Y}_s, K_0) \tilde{\to} H^m_{\text{log-crys}}(\mathcal{Y})_0$. (The latter is the maximal subspace of $H^m_{\text{log-crys}}(\mathcal{Y})$, where the eigenvalues of φ^a are Weil numbers of weight zero.) The claim follows. We also get an isomorphism $\gamma : H^m_{\text{Zar}}(\mathcal{Y}_s, \mathbf{Q}_p) \tilde{\to} H^m_{\text{log-crys}}(\mathcal{Y})^{\varphi=1}$ and the fact that the eigenvalues of φ^a on $H^m_{\text{log-crys}}(\mathcal{Y})$ are Weil numbers of weights between zero and m.

Thus, the bijectivity of α follows from the equality $\beta \circ \alpha = \gamma$ which, in turn, follows from the compatibility of the isomorphism of C_{st} with the specialization morphism.

Before going further, we make a useful observation. First, consider the following three properties of a p-adic semistable representation V of G:

(1) $\text{Fil}^0(D(V) \otimes_{K_0} K) = D(V) \otimes_{K_0} K$;

(2) there are no eigenvalues of φ^a on $D(V)$ of the form ε/p^a, where ε is a root of unity;

(3) the action of φ^a on $D(V)_\mu$ is semisimple.

Here p^a is the number of elements in $\widetilde{K}$, $D(V) = (V \otimes_{\mathbf{Q}_p} B_{\text{st}})^G$, and $D(V)_\mu$ is the maximal φ^a-invariant K_0-vector subspace of $D(V)$, where the eigenvalues of φ^a are roots of unity. For example, as we saw above, the representations of G on the étale cohomology groups $H^m(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_p)$, where $\mathcal{Y}$ is a strictly semistable projective scheme over K°, possess the properties (1)–(3).

Let $\mathcal{R}(K)$ denote the abelian category of p-adic representations of G whose restriction to the Galois group of a finite extension of K is semistable and possesses the properties (1)–(3). If V is such a representation, then its restriction to the Galois group of *any* finite extension of K, over which V is semistable, possesses the properties (1)–(3)

and, in particular, for any finite extension K' of K, there is a functor $\mathcal{R}(K) \to \mathcal{R}(K')$. We also remark that, since the functor $V \mapsto D(V)$ on the category of p-adic semistable representations is exact, any p-adic representation of G, which is a subquotient of $V \in \mathrm{Ob}(\mathcal{R}(K))$, is an object of $\mathcal{R}(K)$.

Lemma 6.1. The functor $V \mapsto V^G$ is exact on the category $\mathcal{R}(K)$. □

Proof. If V is a semistable p-adic representation of G, then $V^{sm} = (D(V) \otimes_{K_0} K_0^{nr})^{N=0,\,\varphi=1} \cap \mathrm{Fil}^0(D(V) \otimes_{K_0} K^a)$. Assume that V satisfies the properties (1)–(3). Then (1) implies that $V^{sm} = (D(V) \otimes_{K_0} K_0^{nr})^{N=0,\,\varphi=1}$ and, therefore, $V^G = D(V)^{N=0,\,\varphi=1}$. By the property (2), the latter coincides with $D(V)^{\varphi=1}$. Indeed, if $\varphi(x) = x$, then $\varphi^a(Nx) = (1/p^a)Nx$ and, therefore, $Nx = 0$. Now let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence in $\mathcal{R}(K)$. We can replace K by a finite extension so that each of V_i's possesses the properties (1)–(3). By the property (3), there exists $n \geq 1$ such that the action of φ^{na} on each $D(V_i)_\mu$ is trivial and, therefore, the action of φ on each $D(V_i)_\mu$, considered as a $\mathbf{Q}_p$-vector space, is semisimple. Since the sequence $0 \to D(V_1)_\mu \to D(V_2)_\mu \to D(V_3)_\mu \to 0$ is exact, it follows that the sequence $0 \to D(V_1)^{\varphi=1} \to D(V_2)^{\varphi=1} \to D(V_3)^{\varphi=1} \to 0$ is exact. ■

Now let $\mathcal{X} = \mathcal{Y} \backslash \mathcal{Z}$, where $\mathcal{Y}$ is a projective scheme over K° and $\mathcal{Z}$ is a divisor in $\mathcal{Y}$ such that the pair $(\mathcal{Y}, \mathcal{Z})$ is strictly semistable. By [deJ, §6.4], $\mathcal{Y}$ is locally isomorphic in the étale topology to $\mathrm{Spec}(A)$, where

$$A = K^\circ[T_1, \dots, T_m, S_1, \dots, S_n, V_1, \dots, V_q]/(T_1 \cdot \cdots \cdot T_m - \pi),$$

π is a generator of the maximal ideal in K°, and the irreducible components of $\mathcal{Z}$ (in $\mathrm{Spec}(A)$) are defined by the equations $T_i = 0$ and $S_j = 0$. Thus, the results of G. Faltings from [Fal] (see also [Kis]) can be applied to $\mathcal{X}$, and it follows that the p-adic representations of G on $H^i(\overline{\mathcal{X}}, \mathbf{Q}_p)$ and $H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_p)$ are semistable. *We claim that* $H^i(\overline{\mathcal{X}}, \mathbf{Q}_p)$ *possess the properties (1)–(3) and that* $H^i(\overline{\mathcal{Y}}_\eta, \mathbf{Q}_p)^{sm} \xrightarrow{\sim} H^i(\overline{\mathcal{X}}, \mathbf{Q}_p)^{sm}$. Indeed, let $Z_1, \dots, Z_n$ be the irreducible components of $\mathcal{Z}$ that are flat over K°, and, for $0 \leq j \leq n$, let $\mathcal{X}_j = \mathcal{Y}_\eta \backslash (Z_{1,\eta} \cup \cdots \cup Z_{j,\eta})$. We apply the functor $V \mapsto D(V)$ to the Gysin exact sequence $(*)$ from §5. We know that $\mathrm{Fil}^0(D(V) \otimes_{K_0} K) = D(V) \otimes_{K_0} K$ and that the eigenvalues of φ^a on $D(V)$ for $V = H^i(\overline{\mathcal{X}}_0, \mathbf{Q}_p)$ and $V = H^i(\overline{Z}_{1,\eta}, \mathbf{Q}_p(-1))$ are Weil numbers of weights greater than or equal to zero and greater than or equal to 2, respectively. Since $Z_{j+1,\eta} \cap \mathcal{X}_j = (Z_{j+1} \backslash (Z_1 \cup \cdots \cup Z_j))_\eta$, the induction on j shows that the same is true for $V = H^i(\overline{\mathcal{X}}_j, \mathbf{Q}_p)$ and $V = H^i(\overline{Z}_{j+1,\eta} \cap \overline{\mathcal{X}}_j, \mathbf{Q}_p(-1))$, respectively. The claim follows. Since $H^i(|\mathcal{Y}_\eta^{an}|, \mathbf{Q}_p) \xrightarrow{\sim} H^i(|\mathcal{X}_\eta^{an}|, \mathbf{Q}_p)$, it follows that the canonical map $H^i(|\mathcal{X}_\eta^{an}|, \mathbf{Q}_p) \to H^i(\overline{\mathcal{X}}, \mathbf{Q}_p)^G \xrightarrow{\sim} D(H^i(\overline{\mathcal{X}}, \mathbf{Q}_p))^{\varphi=1}$ is bijective. Furthermore, *we claim that* $H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_p)$ *also* *possess the properties (1)–(3) and that* $H^i_c(|\mathcal{X}^{an}_\eta|, \mathbf{Q}_p) \tilde{\to} H^i_c(\overline{\mathcal{X}}, \mathbf{Q}_p)^G$. Indeed, consider the commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^i_c(|\overline{\mathcal{X}}^{an}_{j+1}|, \mathbf{Q}_p) & \longrightarrow & H^i_c(|\overline{\mathcal{X}}^{an}_j|, \mathbf{Q}_p) & \longrightarrow & H^i_c(|(\overline{Z}_{j+1,\eta} \cap \overline{\mathcal{X}}_j)^{an}|, \mathbf{Q}_p) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^i_c(\overline{\mathcal{X}}_{j+1}, \mathbf{Q}_p) & \longrightarrow & H^i_c(\overline{\mathcal{X}}_j, \mathbf{Q}_p) & \longrightarrow & H^i_c(\overline{Z}_{j+1,\eta} \cap \overline{\mathcal{X}}_j, \mathbf{Q}_p) & \longrightarrow \cdots \end{array}$$

Using the previous claim for cohomology without support, the second row, and induction on j, we see that the groups $H^i_c(\overline{\mathcal{X}}_j, \mathbf{Q}_p)$ possess the properties (1) and (2). To establish the other facts, we can replace K by a finite extension so that all groups in the first row are defined over K and that, for each group V from the second row, the eigenvalues of φ^a on $D(V)_\mu$ are equal to 1. In this case, using the induction on j again and the five-lemma, we see that the canonical map $H^i_c(|\mathcal{X}^{an}_j|, \mathbf{Q}_p) \to D(H^i_c(\overline{\mathcal{X}}_j, \mathbf{Q}_p))_\mu$ is an isomorphism. This gives the required facts.

Finally, let $\mathcal{X}$ be an arbitrary separated scheme of finite type over K. Take an open immersion $j : \mathcal{X} \hookrightarrow \mathcal{Y}_\eta$, where $\mathcal{Y}$ is a proper scheme over K°. We may assume that $\mathcal{Z} = \mathcal{Y} \backslash \mathcal{X}$ is nowhere dense in $\mathcal{Y}$. By the construction from [Del3, §6.2] and [deJ, Theorem 6.5], there exists a simplicial scheme $\mathcal{Y}_\bullet$ and an augmentation $\mathcal{Y}_\bullet \to \mathcal{Y}$ such that $\mathcal{Y}_{\bullet,\eta} \to \mathcal{Y}_\eta$ is a proper hypercovering and each $\mathcal{Y}_n$ is a finite disjoint union of integral schemes $\mathcal{Y}'$. In turn, each $\mathcal{Y}'$ is projective over K'°, where K' is a finite extension of K, such that, if $\mathcal{Z}'$ is the reduction of the preimage of $\mathcal{Z}$ in $\mathcal{Y}'$, then either $\mathcal{Z}' = \mathcal{Y}'$ or the pair $(\mathcal{Y}', \mathcal{Z}')$ is strictly semistable over K'°. Thus, if $\mathcal{X}_n$ is the preimage of $\mathcal{X}$ in $\mathcal{Y}_n$, we get a proper hypercovering $\mathcal{X}_\bullet \to \mathcal{X}$ such that the statement of Theorem 1.1(a″) is true for each $\mathcal{X}_n$ and the groups $H^i(\overline{\mathcal{X}}_n, \mathbf{Q}_p)$ and $H^i_c(\overline{\mathcal{X}}_n, \mathbf{Q}_p)$ are objects of the category $\mathcal{R}(K)$. By [SGA4, Exp. V bis] applied to the constant sheaf $\mathbf{Q}_p$ (resp., the sheaf $j_!(\mathbf{Q}_p)$), there are spectral sequences

$$'E_1^{m,n} = H^n(|\mathcal{X}^{an}_m|, \mathbf{Q}_p) \Longrightarrow H^{m+n}(|\mathcal{X}^{an}|, \mathbf{Q}_p)$$

and

$$''E_1^{m,n} = H^n(\overline{\mathcal{X}}_m, \mathbf{Q}_p) \Longrightarrow H^{m+n}(\overline{\mathcal{X}}, \mathbf{Q}_p)$$
$$(\text{resp., } 'E_1^{m,n} = H^n_c(|\mathcal{X}^{an}_m|, \mathbf{Q}_p) \Longrightarrow H^{m+n}_c(|\mathcal{X}^{an}|, \mathbf{Q}_p)$$

and

$$''E_1^{m,n} = H^n_c(\overline{\mathcal{X}}_m, \mathbf{Q}_p) \Longrightarrow H^{m+n}_c(\overline{\mathcal{X}}, \mathbf{Q}_p)),$$

and a canonical homomorphism of spectral sequences $'E \to'' E$. We know that the latter induces isomorphisms $'E_1^{m,n} \xrightarrow{\sim} (''E_1^{m,n})^G$ and that each $''E_1^{m,n}$ is an object of the category $\mathcal{R}(K)$. From Lemma 6.1, it follows that $'E_r^{m,n} \xrightarrow{\sim} (''E_r^{m,n})^G$ for all $r \geq 1$ and, in particular, $'E_\infty^{m,n} \xrightarrow{\sim} (''E_\infty^{m,n})^G$. But $'E_\infty^{i,m-i} \xrightarrow{\sim} F^i\,'H^m/F^{i+1}\,'H^m$ and $''E_\infty^{i,m-i} \xrightarrow{\sim} F^i\,''H^m/F^{i+1}\,''H^m$, where

$$F^0\,'H^m = {}'H^m \supset F^1\,'H^m \supset \cdots \supset F^{m+1}\,'H^m = 0$$

and

$$F^0\,''H^m = {}''H^m \supset F^1\,''H^m \supset \cdots \supset F^{m+1}\,''H^m = 0$$

are the filtrations of $'H^m = H^m(|\mathcal{X}^{an}|, \mathbf{Q}_p)$ and $''H^m = H^m(\overline{\mathcal{X}}, \mathbf{Q}_p)$ (resp., $'H^m = H_c^m(|\mathcal{X}^{an}|, \mathbf{Q}_p)$ and $''H^m = H_c^m(\overline{\mathcal{X}}, \mathbf{Q}_p)$) induced by the spectral sequences. Since the image of $F^i\,'H^m$ in $F^i\,''H^m$ is contained in $(F^i\,''H^m)^G$, it follows that

$$\left(F^i\,''H^m\right)^G / \left(F^{i+1}\,''H^m\right)^G \xrightarrow{\sim} \left(F^i\,''H^m/F^{i+1}\,''H^m\right)^G$$

and, therefore,

$$H^m\left(|\mathcal{X}^{an}|, \mathbf{Q}_p\right) \xrightarrow{\sim} H^m\left(\overline{\mathcal{X}}, \mathbf{Q}_p\right)^G \left(\text{resp., } H_c^m\left(|\mathcal{X}^{an}|, \mathbf{Q}_p\right) \xrightarrow{\sim} H_c^m\left(\overline{\mathcal{X}}, \mathbf{Q}_p\right)^G\right). \qquad \blacksquare$$

Remark 6.2. Using the results of G. Faltings [Fal] and J. de Jong [deJ], M. Kisin [Kis] recently proved potential semistability of the cohomology groups $H^m(\overline{\mathcal{X}}, \mathbf{Q}_p)$ and $H_c^m(\overline{\mathcal{X}}, \mathbf{Q}_p)$ for an arbitrary separated scheme $\mathcal{X}$ of finite type over K. (This was earlier proven by T. Tsuji [Tsu2] for proper $\mathcal{X}$.) Having this result, the proof of Theorem 1.1(a″) shows that the above groups are always objects of the category $\mathcal{R}(K)$. It follows also that, if one of the above groups V is semistable, then the eigenvalues of φ^a on $D(V)$ are Weil numbers of weights greater than or equal to zero, and each eigenvalue that is a Weil number of weight zero is a root of unity.

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