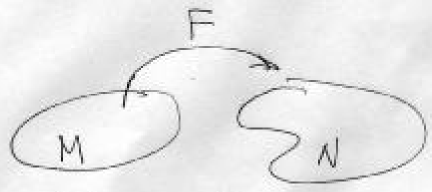


F = smooth map

$$C^\infty(N) \xrightarrow{F^*} C^\infty(M)$$

Pullback operator:

$$(F^* g)(\underset{\Uparrow \atop M}{x}) = g(\underset{\Uparrow \atop N}{\underbrace{F(x)}})$$

Obvious properties:

- F^* is a ring (algebra) homomorphism

$$F^*(g_1 g_2) = (F^* g_1)\cdot(F^* g_2) \quad + \text{linearity}$$

$$F^*(\text{const}) = \text{const}$$

Functoriality:

$$(F_1 \circ F_2)^* = F_2^* F_1^*$$

How F can be restored from F^*?

Passing to charts:

$$M \subseteq \mathbb{R}^m, \quad N \subseteq \mathbb{R}^n$$

$$y_1 \dots y_n \in C^\infty(N)$$

$$F^* y_i = f_i(x_1 \dots x_m) \text{ - smooth functions.}$$

Claim: $F = (f_1, \dots, f_n)$. On functions y_i by construction

On polynomials – by homomorphy

On all smooth functions – "by continuity"!

Problematic!

Vector fields:

$M \subseteq \mathbb{R}^m$ a domain: $\begin{cases} v_1(x_1 \dots x_m) \\ \vdots \\ v_m(x_1 \dots x_m) \end{cases}$ Coordinate functions of the field

X acts on functions

$$Xf = \sum v_j \frac{\partial f}{\partial x_j}$$

Obvious properties:

X linear + satisfies the Leibnitz rule: ~~(Alg)~~ Derivation of the algebra

$$X(fg) = f\cdot(Xg) + g\cdot(Xf)$$

Conversely: ~~(Alg)~~ X is a derivation $\Rightarrow$ X is a vector field.

(on $M \subseteq \mathbb{R}^n$)

$\triangleleft$ $\forall a \in M \subseteq \mathbb{R}^n$ $\quad \forall f \in C^\infty(M)$

$(*)$ $f = f(a) + \sum_1^n (x_i - a_i) f_i(x)$, $\quad f_i \in C^\infty(M)$, $f_i(a) = \frac{\partial f}{\partial x_i}(a)$

We claim that

$$Xf = \sum v_i \frac{\partial f}{\partial x_i}, \quad v_i = X(x_i)$$

Indeed, $X(1\cdot 1) = X(1) + X(1) \Rightarrow X(\text{const}) = 0$

$$(*) \Rightarrow (Xf)(a) = \left(\sum (x_i - a_i) X f_i\right)(a) + \sum_1^n f_i(a)\cdot X(x_i - a_i) = \sum \frac{\partial f}{\partial x_i}(a)\cdot v_i$$

– Indeed, nothing new.

How F acts on X?

$$M \xrightarrow{F} N$$

$X \qquad Y$

$X \in \mathcal{D}(M) \qquad \mathcal{D}(N)$?

= Construct a derivation of $C^\infty(N)$

$$g \xrightarrow{F^*} F^*g \xrightarrow{X} X(F^*g)$$

$g \in C^\infty(N)$, $\quad F^*g \in C^\infty(M)$, $\quad X(F^*g) \downarrow ?$

$$Y = (F^*)^{-1} \circ X \circ (F^*) \leftarrow \text{"conjugacy"}$$

$$(F^*)^{-1} X F^*(g)$$

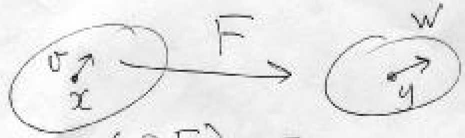
~~For vectors in charts:~~

Formulas in charts $\mathbb{R}^m \supseteq M \longrightarrow N \subseteq \mathbb{R}^n$

$$g(y) \longrightarrow g(f_1(x)\dots f_n(x)) \longrightarrow \sum v_i(x)\frac{\partial}{\partial x_i}(g\circ F)$$

$$= \sum_{i,j} v_i(x)\frac{\partial g}{\partial y_j}\frac{\partial f_j}{\partial x_i} = \sum_j w_j \frac{\partial g}{\partial y_j}, \quad w_j = \sum_i \frac{\partial f_j}{\partial x_i} v_i$$

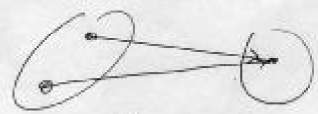
considered as a function of y



$w(F(x)) = \left(\frac{\partial F}{\partial x}\right)\cdot v$

this explains why invertibility is required!

$\left(\frac{\partial F}{\partial x}\right) = J$



two points bring two different values.

What can one do with Vector fields? – "Integrate"!

(restore trajectories)

$(X^0 = id)$

X a derivation $\in \mathcal{D}(M)$

Formal identity: ~~Postpone!~~ $\exp tX = \sum_{k=0}^{\infty} \frac{t^k X^k}{k!}$

" F^t : automorphism of $C^\infty(M)$ onto itself

" Formal property:

F^t is an automorphism for all t!"

Ignoring the convergence

" $F^{t+s} = F^t F^s = F^s F^t$ "

One-Parametric Group: F^t differentiably dep. on t

$Xf := \lim_{t\to 0} \frac{F^t f - f}{t}$ = exists.

Claim: X is a derivation

$$X(fg) = \lim_{t\to 0} \frac{F^t(fg) - fg}{t}$$

$$= \frac{F^t f \cdot F^t g - fg}{t} = \frac{F^t f(F^t g - g)}{t} + F \ldots$$