Lect. 6 Differential forms ①

Covector = dual to vector : $L \simeq \mathbb{R}^n$ linear space

$L^* = \{ \xi : L \to \mathbb{R}$ linear functional $\}$

M manifold

$TM = \bigcup_{a \in M} T_a M$ Tangent space (tangent bundle – explained later)

$T^*M = \bigcup_{a \in M} (T_a M)^* = \bigcup_{a \in M} T_a^* M$ Cotangent bundle

Vector field: $X : M \to TM$ such that $X(a) \in T_a M$

Covector field $\xi : M \to T^*M$ — " — $\xi(a) \in T_a^* M$

Smoothness : $\forall X$ smooth $\in \mathcal{D}(M)$

$\langle \xi, X \rangle$ smooth function, $a \mapsto \langle \xi(a), X(a) \rangle \in \mathbb{R}$

Covector field = map $\xi : \mathcal{D}(M) \to C^\infty(M)$ || "Algebraic" description

additive + linear over $C^\infty(M)$: $\xi[fX] = f \cdot \xi[X]$ ||

Synonym : Differential 1-form. "Tensorial behavior"

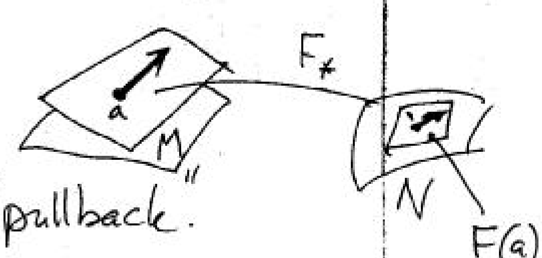
(Form = ancient name for "functionals", with empty slots to be filled by ~~some things~~ vector arguments) 1-form means 1 free argument.

0-forms = smooth functions) k-forms will be introduced in due turn.

Action by maps (= functoriality) Notation: $\Lambda^1(M)$

$F : M^m \to N^n$ smooth map

$F^* : \Lambda^1(N) \to \Lambda^1(M)$ "pullback"



$\langle (F^*\xi)(a), X(a) \rangle = \langle \xi(F(a)), (F_* X)(F(a)) \rangle$

This behaviour is much better than that of vector fields!

Example:

$f \in C^\infty(M) \Rightarrow df \in \Lambda^1(M)$

$\langle df(a), X(a) \rangle = Xf(a)$ directional derivative.

Exercise: $F : M \to N$, $g \in C^\infty(N)$

$$d(F^* g) = F^* dg.$$

◀ By definition ▶

Obvious observation: $\begin{cases} \xi_1, \dots, \xi_k \in \Lambda^1(M) \\ f_1 \dots f_k \in C^\infty(M) \end{cases}$

$$\Downarrow$$

$$\xi = f_1 \xi_1 + \dots + f_k \xi_k \in \Lambda^1(M)$$

[the structure of a module over the same algebra $\mathfrak{A} = C^\infty(M)$]

Example: Locally in a chart $\Leftrightarrow$ in a domain of $\mathbb{R}^n$

$x_1, \dots, x_n$ coordinates = functions on M

$dx_1 \dots dx_n$ their differentials

Claim: any 1-form can be represented as $\sum_1^n a_i(x)\, dx_i$

In particular, $df = \sum_1^n \left(\frac{\partial f}{\partial x_i}\right) dx_i$.

⚠ Warning, — the inverse is not true: not any 1-form is a differential.

Lie derivatives act on 1-forms:

$X \sim \{g^t\}_{t \in \mathbb{R}}$ automorphisms of M

$X \in \mathcal{D}(M)$

$$L_X \xi = \frac{d}{dt}\Big|_{t=0} (g^t)^* \xi.$$

linear operator; Leibnitz rule holds (the same proof as for vectors)

Rules of derivation

$$L_X \langle \xi, Y \rangle = \langle L_X \xi, Y \rangle + \langle \xi, L_X Y \rangle$$

(follows from definition) (exercise)

Hence $L_X(f\xi) = f \cdot L_X \xi + (L_X f) \cdot \xi = f \cdot L_X \xi + X(f) \cdot \xi$

◁ Proof: Take an arbitrary Y ... use the Leibniz rule for L_X on functions ▷

$L_{fX} \xi = ?$ Tensorial property? (Chances that it is a 2-form)

Hope: $L_X \xi$ again a 1-form:

$$\cancel{L_X \langle \xi, fY \rangle = f L_X \langle \xi, Y \rangle + X(f) \cdot \langle \xi, Y \rangle}$$

Verification:

$$\langle L_X \xi, fY \rangle = L_X \langle \xi, fY \rangle - \langle \xi, L_X(fY) \rangle$$

$$= L_X f \cdot \cancel{\langle \xi, Y \rangle} + f L_X \langle \xi, Y \rangle \quad (\text{Leibniz}_1)$$

$$- \langle \xi, f \cdot L_X Y \rangle - \cancel{\langle \xi, (L_X f) \cdot Y \rangle} \quad (\text{Leibniz}_2)$$

$$= f \langle L_X \xi, Y \rangle.$$

Second argument - wrong! How to correct?

Claim (answer) $L_X \xi - d\langle \xi, X \rangle$ - tensor: (one hidden argument Y, one explicit X)

$$\omega(X, \cdot) = L_X \xi - d\langle \xi, X \rangle$$

$$\omega(fX, \cdot) = f\omega(X, \cdot)$$

~~Computation~~

Computation of

$$\langle L_{fX}\xi, Y\rangle = L_{fX}\langle \xi, Y\rangle - \langle \xi, L_{fX} Y\rangle$$

$$= fL_X\langle \xi, Y\rangle + \langle \xi, L_Y(fX)\rangle \leftarrow \text{antisymmetry}$$

$$= \underbrace{f L_X\langle \xi, Y\rangle + f\langle \xi, L_Y X\rangle}_{\parallel} + \langle \xi, X\rangle \cdot L_Y f$$

$$= f L_X\langle \xi, Y\rangle + \underbrace{\langle \xi, X\rangle\langle df, Y\rangle}_{\text{"Wrong" term (non-tensorial)}}$$

"Correction:"

$$-d\langle \xi, fX\rangle = -f\, d\langle \xi, X\rangle - \langle \xi, X\rangle df \qquad (Lb)$$

Cumulatively,

$\omega(fX, Y) = f\,\omega(X,Y)$, $\Rightarrow$ ω is 1-form in each argument.

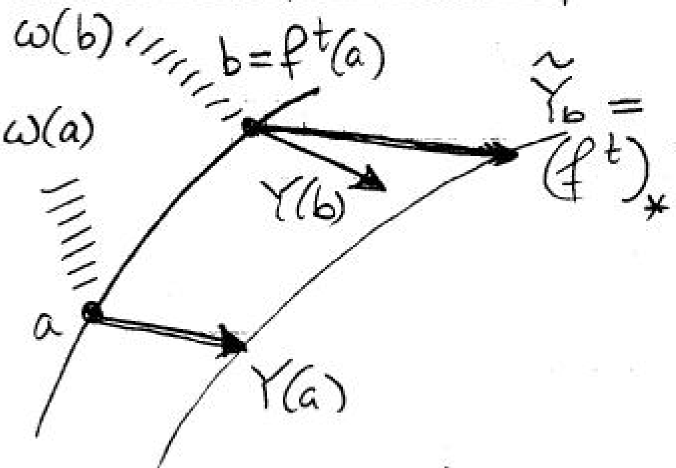
"Symmetry": $\omega(X,Y) = -\omega(Y,X)$

◁ Direct computation ... ▷ (related to antisymmetry of $L_X Y$)

Notation: $\omega = d\xi$. ← Exterior differential.

$$\xi \in \Lambda^1(M) \Rightarrow d\xi \in \Lambda^2(M) = \left\{ \begin{array}{l} \text{bilinear antisymmetric forms} \\ \omega : \mathcal{D}(M)\times\mathcal{D}(M) \to C^\infty(M) \end{array} \right\}$$

How to differentiate in public



$$\tilde{Y}_b = (f^t)_* Y$$

$$L_X \underbrace{\langle \omega, Y\rangle}_{\text{function}} = ?$$

$$L_X \omega = ?$$

$$\boxed{\langle (f^t)^*\omega \cdot Y\rangle_a = \langle \omega_b \cdot (f^t)_* Y\rangle}$$

Def. of pull-back

$$L_X \langle \omega, Y\rangle \approx \frac{1}{t}\left\{\langle \omega_b \cdot Y_b\rangle - \langle \omega_a \cdot Y_a\rangle\right\} \quad (1)$$

$$\langle L_X\omega, Y\rangle \approx \frac{1}{t}\left\{\langle \omega_b \cdot \tilde{Y}_b\rangle - \langle \omega_a, Y_a\rangle\right\} \quad (2)$$

$$(L_X Y)_b \approx \frac{1}{t}\left\{ Y_b - \tilde{Y}_b \right\} \quad (3)$$

Taking

$\langle \omega_b \cdot (3)\rangle + (2)$, we obtain (1), therefore

$$L_X\langle \omega, Y\rangle = \langle L_X\omega, Y\rangle + \langle \omega, L_X Y\rangle$$

(Leibnitz rule for pairing $\langle \cdot, \cdot \rangle$)

Further obvious properties of the Lie derivation:

(1) $L_X(f\omega) = f(L_X\omega) + (L_X f)\cdot\omega$ (easy)

(2) $L_{fX}\omega = fL_X\omega + (L_X f)\cdot\omega$ (via Leibnitz-dual)

(3) $L_X df = dL_X f$

acts on Y: $L_X\langle df, Y\rangle - \langle df, L_X Y\rangle = XYf - L_{[X,Y]}f$

$= (XY - XY + YX)f = YXf = \langle d(Xf), Y\rangle = \langle dL_X f, Y\rangle$

Let Ω be a 2-form, def. d by the equation

$$\Omega(X,\cdot) = L_X\omega - d\langle\omega, X\rangle$$

Direct computation: bilinearity in each vector argument separately.
More precisely, by Leibnitz-dual,

$$\Omega(X,Y) = L_X\langle\omega, Y\rangle - \langle\omega, [X,Y]\rangle - L_Y\langle\omega, X\rangle$$

– immediately antisymmetric ...

This independently proves also linearity in X.

Notation: $\Omega = d\omega$ (the same symbol d, different meaning).

To distinguish sometimes denote $d_0 f$, differential of a 0-form f

$d_1\omega$, ...

Properties: • $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ (trivial)

• $d_1 d_0 f = 0$: By definition,

$$ddf = L_X df - dL_X f = 0.$$

What about $d_1(f\omega)$, $f \in C^\infty(M)$?

- New operation, exterior (wedge) product of 1-forms

$\alpha, \beta \in \Lambda^1(M)$ $\quad X, Y \in \mathcal{D}(M)$

$\alpha \wedge \beta \in \Lambda^2(M)$; $\quad (\alpha\wedge\beta)(X,Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$

– automatically antisymmetric & bilinear.

- $d_1(f\omega) = d_0 f \wedge \omega + f \cdot d_1\omega$

◀ $$\begin{aligned} \text{L.h.s.} &= L_X\langle f\omega, Y\rangle - L_Y\langle f\omega, X\rangle - \langle f\omega, [X,Y]\rangle \\ &= f\cdot d_1\omega + L_X f\cdot\langle\omega, Y\rangle - L_Y f\cdot\langle\omega, X\rangle \\ &= f\cdot d_1\omega + \langle df, X\rangle\langle\omega, Y\rangle - \langle df, Y\rangle\langle\omega, X\rangle \end{aligned}$$ ▶

- (Exercise) $\quad L_X(\alpha\wedge\beta) = (L_X\alpha)\wedge\beta + \alpha\wedge(L_X\beta)$

Reference table

Forms: Operations $\omega_1 \pm \omega_2$ $\quad \omega_{1,2} \in \Lambda^k(M)$

$f\omega$, $\quad f \in C^\infty(M)$

Forms constitute a superalgebra over the ring of smooth functions.

Wedge product $\left.\begin{matrix} \\ \alpha \in \Lambda^k,\ \beta \in \Lambda^l \end{matrix}\right\}$ $\alpha \wedge \beta \in \Lambda^{k+l}(M)$ – anticommutative.

Defined on 1-forms:

$$(\alpha_1 \wedge \dots \wedge \alpha_k)(X_1, \dots, X_k) = \det \begin{vmatrix} \alpha_1(X_1) & \cdots & \alpha_k(X_1) \\ \vdots & \ddots & \vdots \\ \alpha_1(X_k) & \cdots & \alpha_k(X_k) \end{vmatrix}$$

Properties: associative, multilinear, antisymmetric

In general: $\alpha \wedge \beta = (-1)^{\deg\alpha \cdot \deg\beta} \cdot \beta \wedge \alpha$

(via transpositions of factors of degree 1)

Operators on forms: L_X Lie derivative

i_X Antiderivative: $(i_X\omega)(\dots) = \omega(X, \dots)$

$$i_X : \Lambda^k \to \Lambda^{k-1}$$

$$d : \Lambda^k \to \Lambda^{k+1}, \quad k = 0, 1, \dots, n-1$$

Table of identities:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg\alpha}\, \alpha \wedge d\beta$$

$d(f) =$ differential df

Th: $d^2 = 0$

$[L_X, L_Y] = L_{[X,Y]}$ (Jacobi)

$[L_X, i_Y] = i_{[X,Y]}$ (Cartan)

Homotopy formula: (can be used to define d inductively)

$$L_X = i_X d + d i_X$$

Thm: $\exists !$ operation $d : \Lambda^k \to \Lambda^{k+1}$ such that

$$\begin{cases} d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (d\beta) & \text{(Leibnitz)} \\ df = \text{differential} \end{cases}$$

$\triangle$ (a) If two forms coincide on an open set $U \subseteq M$, then their differentials there coincide.

Locality

Let $a \in U$ be an arbitrary pt and $f = \begin{cases} \equiv 1 & \text{near } a \\ 0 & \text{outside } U \end{cases}$

$\omega|_U \equiv 0 \Rightarrow f\omega \equiv 0$, and by Leibnitz

$$0 = d(f\omega) = \underbrace{f \cdot d\omega}_{= d\omega|_{\text{near } a}} + \underbrace{\cancel{df \wedge \omega}}_{df = 0 \text{ near } a} \Rightarrow d\omega|_{\text{near } a} = 0.$$

(b) Computation in charts: (uniquely)

$$(*) \qquad \alpha = \sum f \, dg_1 \wedge \ldots \wedge dg_k \Rightarrow d\alpha = \sum df \wedge dg_1 \wedge \ldots \wedge dg_k$$

(c) Existence: Define d by $(*)$. One has to verify the Leibnitz rule, which amounts to checking that

$$d(fg \; dx_1 \wedge \ldots \wedge dx_k \wedge dy_1 \wedge \ldots \wedge dy_l) =$$

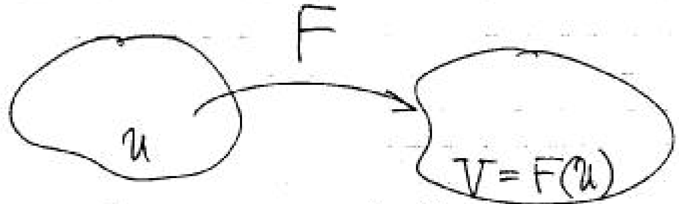
$$= (f\,dg + g\,df) \wedge (\quad \text{—}''\text{—} \quad)$$

$$= (\text{the correct answer after the re-arrangement of terms})$$

Integration of forms.

(a) Integration over domains in $\mathbb{R}^n$

$$\omega = f\, dx_1 \wedge \ldots \wedge dx_n$$



$$\int_U \omega = \text{Riemann integral of } f$$
against the measure
$$d\mu = dx_1 \ldots dx_n$$

Theorem: F diffeo

$$\int_{F(U)} \omega = \pm \int_U F^* \omega$$

◁ $F = (f_1, \ldots, f_n)$

$y_i = f_i(x)$

$dy_i = df_i$

$df_1 \wedge \ldots \wedge df_n =$

$= \det \left|\left(\frac{\partial F}{\partial x}\right)\right| dx_1 \wedge \ldots \wedge dx_n$

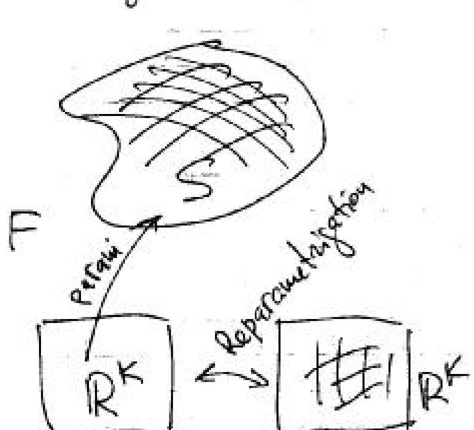
Formula for the change of variables in integrals

Sign $\pm$ depends on the sign of $\det \left(\frac{\partial F}{\partial x}\right)$

= "orientation" of the domain."

= choice of a "positive" coordinate system.

(b) Integration over ~~domains~~ submanifolds in $\mathbb{R}^n$



$$\int_S \omega := \int_{U = F^{-1}(S)} F^* \omega$$

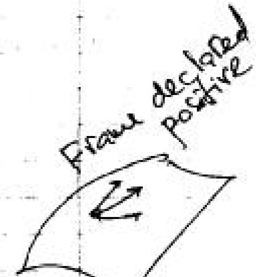
Thm: "Doesn't depend on the parametrization" (modulo sign $\pm$).

S = k-dim submanifold parameterized by

$$F : U \hookrightarrow M$$
$$U \subset \mathbb{R}^k$$

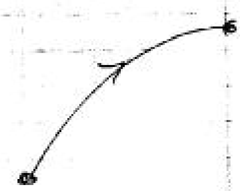
Proper definition:

Oriented submanifold.

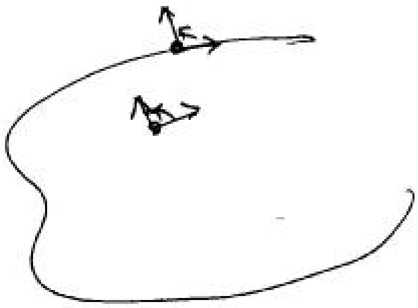


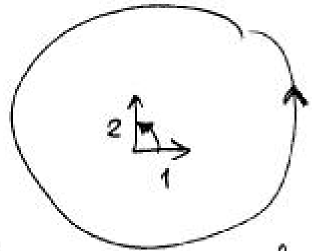
Examples:

Oriented curves.



Orientation of a boundary

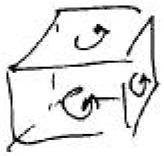




"Extra first" rule
$\left.\begin{array}{l}\text{(exterior normal} \\ \text{+ positive frame of bnd}\end{array}\right\} =$
= positive frame of the body itself.

Addition of pieces: formal operation.
$$\int_{S_1+S_2} \omega := \int_{S_1} \omega + \int_{S_2} \omega$$

Boundary of the cube (example)



$$\partial(\text{Cube}) = \sum_{\text{six}} \text{faces}$$

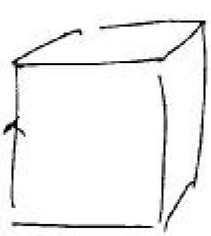
$$\partial\partial(\text{Cube}) = \sum_{\text{six faces}} \sum_{\substack{\text{four}\\ \text{edges}}} \text{edges};$$

Each edge enters twice, with two opposite signs, $\Rightarrow$ $\partial\partial(\text{cube}) = 0$.

Stokes theorem:

$$\int_{\partial S} \omega = \int_{S} d\omega.$$

Proof: (a) $S =$ ~~image of~~ a cube



$$\omega = f \, dx_2 \wedge \dots \wedge dx_n$$

$$d\omega = \frac{\partial f}{\partial x_1} \, dx_1 \wedge \dots \wedge dx_n$$

(b) S = image of a cube: