Exponential Rates for Error Probabilities in DMPSK Systems

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ABSTRACT Precise analytical asymptotic exponential rates of error, and bounds on those rates, for differential multiple-phase-shift keying (DMPSK) systems that include post-detection integration are provided. Easily computed bounds on these rates are provided, both in the case of floor bit error probability (i.e., with no additive noise) and in the case of weak additive noise. The derivation uses the theory of large deviations and illustrates its applicability to the analysis of communications systems.

1 Introduction

An important part of the design of communication systems involves the evaluation of their performance, both for a given design and on a comparative basis. Analytic expressions for the performance, when measured in terms of error probabilities, are often unavailable, and one resorts to numerical computations and, frequently, to simulations. While these methods are sometimes sharp and powerful, a comparative study of different designs and a study of the importance of various parameters in the system becomes impossible.

One of the possibilities for overcoming this difficulty is the use of asymptotics. Besides being

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interesting on their own merit, the asymptotics of the performance of a system when some parameter approaches zero (e.g., the noise-to-signal ratio) may serve as the basis for a comparative study and may help in gaining an understanding of the various parameters that influence the behavior of the system.

An often used method for obtaining asymptotics of the performance of communication systems is the Chebyshev bound. This time-honored technique yields upper bounds on probabilities of errors, but one is left to guess about their tightness. Thus, a comparative study of performance based on the Chebyshev bound may not lead to definitive conclusions nor to a reliable comparison between different designs.

In recent years, a mathematical method, called the *large deviations principle* (LDP) was developed in order to complement the available upper bounds obtained from Chebyshev's inequality. This method has found many applications in various aspects of mathematics, statistics and engineering (for representative examples and a gentle introduction to the theory, see [2].) We shall use [4] as our standard reference, for both notation and terminology. The bibliography of [4] should be consulted for references to this vast field. It is our goal in this paper to provide an example where this method can be successfully applied in communication theory, yielding analytic results that are otherwise not available. This example also illustrates the type of analytical machinery that is required in order to carry out a complete analysis of large deviations for a given system.

The vehicle that we have chosen for our presentation is the analysis of bit error probabilities in differential multiple phase-shift-keying (DMPSK) over an optical channel. DMPSK is an often used modulation technique for communication systems operating over a coherent optical channel. For background and a description of typical applications, see [8],[11],[12]. This paper deals with the asymptotic performance of DMPSK receivers that incorporate post-detection integration in the presence of both phase noise and additive noise. The evaluation of this performance is not a new question. For previous work, see [1],[6],[7],[9],[10]. We return below, in the concluding section, to a comparison of our results with theirs.

We model the optical communication channel as follows. Let $\{\gamma_k\}$ denote the sequence of modulating phases, with γ_k taking one of the values $2\pi l/M$, for $l = 0, \ldots, M-1$. During the k-th keying interval (time slot of length T), the information is contained in the difference $\{\gamma_k - \gamma_{k-1}\}$.

During the k-th time slot the transmitted signal is

$$s(t) = \cos\left[\omega_c t + \theta + \gamma_k + \sqrt{\epsilon} w(t)\right] \qquad \text{for } t \in [(k-1)T, kT). \tag{1.1}$$

Here θ is a fixed random phase, distributed uniformly over $[0, 2\pi)$, and ω_c denotes the carrier frequency. The phase-noise process, which arises from the spectral noise of the laser generating the signal in the optical channel, is modeled by a standard Brownian motion $w(t)$ (measured in units of radians). ϵ is a dimensionless small parameter related to "signal-to-noise ratio". (There is no commonly accepted definition of signal-to-noise ratio for this model; however as ϵ becomes smaller, the change in the phase noise during a keying interval becomes smaller compared with the signal).

The signal is observed in the presence of additive Gaussian narrow-band noise; i.e. the received signal (in units of volts) is

$$r(t) = s(t) + \sqrt{\epsilon N_0} n(t) \tag{1.2}$$

where $n(t)$ is a narrow-band Gaussian noise of bandwidth $2\pi W$ centered at ω_c, normalized to have total power of one square volt, and N_0 is a dimensionless parameter relating the strength of the phase noise to that of the additive noise. This additive noise models additional distortion at the input of the channel and in the electronic components of the front end of the receiver, and its influence on performance is typically weak when compared with that of the phase noise $w(t)$.

We comment here on the choice of the same asymptotic parameter ϵ in both equations (1.1) and (1.2). The reason is that, in situations where the additive noise is much weaker in the limit than the phase noise difference during one keying interval, one can show that its effect on the exponential decay of the probability of error is negligible, corresponding to the case $N_0 = 0$. On the other extreme, if the phase noise difference during one keying interval is weaker in the limit than the additive noise, the channel behaves like a channel with additive Gaussian noise alone, and the asymptotics of the bit error probability in such channels are well known. Thus, the interesting case of the optical channel is that in which the two noises are comparable.

Implicit in our model is the existence of an IF bandpass filter, which has the effect of both filtering the additive noise and of distorting the signal $s(t)$. Throughout the analysis here, we take the common road of neglecting this distortion, which is justified by the relatively large bandwidth of the IF filter. The techniques presented can also deal with the inclusion of the IF filter, albeit at the cost of a more involved analysis. For an explicit study of the influence of IF filtering on the floor bit error probability (i.e., the case with no additive noise, $N_0 = 0$), see [10].

The receiver, at the IF level, is depicted in Figure 1. It forms the quadrature signals I_k, Q_k and

Figure 1.1: Receiver structure

the phase-difference estimate $\widehat{\gamma_k - \gamma_{k-1}} = \text{angle}\,[I_k - iQ_k]$. Let $n(t) = n_1(t)\cos\omega_c t + n_2(t)\sin\omega_c t$, where n_1, n_2 are the baseband noises of bandwidth πW, unit power, and covariances $R_{n_1}(\tau) = R_{n_2}(\tau)$. Then

$$\begin{aligned} 2I_k &= \int_{kT}^{(k+1)T} [\cos(\theta + \gamma_k + \sqrt{\epsilon}w(t)) + \sqrt{\epsilon N_0}n_1(t)][\cos(\theta + \gamma_{k-1} + \sqrt{\epsilon}w(t-T)) + \sqrt{\epsilon N_0}n_1(t-T)]\,dt \\ &+ \int_{kT}^{(k+1)T} [\sin(\theta + \gamma_k + \sqrt{\epsilon}w(t)) + \sqrt{\epsilon N_0}n_2(t)][\sin(\theta + \gamma_{k-1} + \sqrt{\epsilon}w(t-T)) + \sqrt{\epsilon N_0}n_2(t-T)]\,dt \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} 2Q_k &= \int_{kT}^{(k+1)T} [\cos(\theta + \gamma_k + \sqrt{\epsilon}w(t)) + \sqrt{\epsilon N_0}n_1(t)][\sin(\theta + \gamma_{k-1} + \sqrt{\epsilon}w(t-T)) + \sqrt{\epsilon N_0}n_2(t-T)]\,dt \\ &- \int_{kT}^{(k+1)T} [\sin(\theta + \gamma_k + \sqrt{\epsilon}w(t)) + \sqrt{\epsilon N_0}n_2(t)][\cos(\theta + \gamma_{k-1} + \sqrt{\epsilon}w(t-T)) + \sqrt{\epsilon N_0}n_1(t-T)]\,dt \end{aligned} \tag{1.4}$$

where, as is usual, we have filtered out the double frequency terms.

To compute the error probabilities, we shall assume that $\gamma_k = \gamma_{k-1} = 0$. When $N_0 = 0$, this loses no generality. In general, the assumption $\gamma_k = \gamma_{k-1} = 0$ does not influence the analysis much. Note that one may always assume that $\gamma_{k-1} = 0$.

The error event is the event

$$E \stackrel{\triangle}{=} \{I_k \sin\frac{\pi}{M} - Q_k \cos\frac{\pi}{M} \le 0\} \cup \{I_k \sin\frac{\pi}{M} + Q_k \cos\frac{\pi}{M} < 0\}$$

Note, by symmetry and the existence of densities, that

$$P(I_k \sin\frac{\pi}{M} - Q_k \cos\frac{\pi}{M} \le 0) = P(I_k \sin\frac{\pi}{M} + Q_k \cos\frac{\pi}{M} < 0)$$

Hence,

$$P(I_k \sin\frac{\pi}{M} - Q_k \cos\frac{\pi}{M} \le 0) \le P(E) \le 2\,P(I_k \sin\frac{\pi}{M} - Q_k \cos\frac{\pi}{M} \le 0)\,,$$

and, since we are interested in the asymptotics of the probability of error $P(E)$ (specifically, the limit of $\epsilon \log P(E)$ as $\epsilon \to 0$, which turns out to be finite and positive, see Theorems 3.1, 4.4, 4.17), it is enough to neglect the factor 2 and compute the asymptotics of

$$\tilde{P}_e \stackrel{\triangle}{=} P(I_k \sin\frac{\pi}{M} - Q_k \cos\frac{\pi}{M} \le 0)\,, \tag{1.5}$$

i.e. the limit

$$\lim_{\epsilon\to 0} \epsilon \log \tilde{P}_e = \lim_{\epsilon\to 0} \epsilon \log P(E)\,.$$

The organization of the paper is as follows. In section 2 we recall some preliminaries from the theory of large deviations. The reader is referred to [2],[4],[5] for a fuller account. In section 3, we derive the so-called floor limiting bit error probability. In section 4, we derive bounds on the bit error probability in the case of large signal-to-noise ratio in the presence of weak additive noise, and we illustrate their evaluation and tightness by means of a numerical example. Section 5 is devoted to a discussion and conclusions. Some of the proofs of the mathematical relations used are deferred to the appendix.

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2 Large Deviations – Preliminaries

We follow here the notations of [4]. We do not attempt to present here a full account of the theory but, rather, to present those elements that we shall use in the sequel.

We say that a functional F on the space of continuous functions on $[0, 2T]$ is *continuous* if

$$\max_{t\in[0,2T]} |\phi_n(t) - \phi(t)| \underset{n\to\infty}{\longrightarrow} 0 \text{ implies } F(\phi_n) \underset{n\to\infty}{\longrightarrow} F(\phi)\,. \tag{2.1}$$

For every continuous functional F (we have in mind the particular functional F in Lemma 2.6), define the function $I_F(x)$ as

$$I_F(x) = \frac{1}{2} \inf_{\{\phi : F(\phi)=x, \phi(0)=0\}} \int_0^{2T} \dot{\phi}^2(t)\, dt\,, \tag{2.2}$$

where $\dot{\phi}(t) \equiv d\phi(t)/dt$ and we have implicitly assumed that $\dot{\phi}(t)$ is square integrable on $[0, 2T]$. That is, $I_F(x)$ is T times the greatest lower bound on the mean square value of the derivative of ϕ, over all ϕ such that $\phi(0) = 0$ and $F(\phi) = x$. The following theorem combines Schilder's theorem and the contraction principle of large deviations. For a proof, see Theorems 4.2.1 and 5.2.3 in [4] or Theorem 1.3.27 and Lemma 2.1.4 in [5].

Theorem 2.3 *Let F be a continuous functional. Then, for any constant a,*

$$\begin{aligned} -\inf_{x<a} I_F(x) &\leq \liminf_{\epsilon\to 0} \epsilon \log P(F(\sqrt{\epsilon} w) < a) \\ &\leq \limsup_{\epsilon\to 0} \epsilon \log P(F(\sqrt{\epsilon} w) \leq a) \\ &\leq -\inf_{x\leq a} I_F(x)\,, \end{aligned} \tag{2.4}$$

where w denotes, as before, a Brownian motion. In particular, if $\inf_{x\leq a} I_F(x) = \inf_{x<a} I_F(x)$, then

$$\lim_{\epsilon\to 0} \epsilon \log P(F(\sqrt{\epsilon} w) \leq a) = -\inf_{x\leq a} I_F(x)\,. \tag{2.5}$$

The following information concerning the use of Theorem 2.3 will be particularly useful to us.

Lemma 2.6 *Let $F(\phi) = \int_T^{2T} \sin[\phi(t) - \phi(t-T) + \pi/M]\, dt$. Then F, being the integral of a continuous function, is continuous and, moreover, for $a \in (-T, T)$,*

$$\inf_{x\leq a} I_F(x) = \inf_{x<a} I_F(x)\,. \tag{2.7}$$

Note that (2.7) holds since $F(\phi) = a$ implies that $dF(\phi + \theta\psi)/d\theta|_{\theta=0} < 0$ for $\psi(t) = -(t - T)\cos[\phi(t) - \phi(t-T) + \pi/M]$ for $t \in [T, 2T]$ and $\psi(t) = 0$ for all other values of t.

3 DMPSK Performance – floor bit error probability

In this section, we analyze $\tilde{P}_e$ in the case $N_0 = 0$. Our main result is

Theorem 3.1 *For $N_0 \equiv 0$,*

$$-\overline{I}_0 \stackrel{\triangle}{=} \lim_{\epsilon\to 0} \epsilon \log \tilde{P}_e = -\frac{1}{2} \inf_{\{\phi: \int_T^{2T} \sin\left[\phi(t)-\phi(t-T)+\frac{\pi}{M}\right] dt \le 0, \phi(0)=0\}} \int_0^{2T} \dot{\phi}^2(t)\, dt \tag{3.2}$$

Moreover,

$$\frac{\pi^2}{2M^2T} \le \overline{I}_0 \le \frac{\pi^2}{M^2T}. \tag{3.3}$$

Proof: Recall that we are interested in the case $\gamma_k = \gamma_{k-1} = 0$. To avoid repetition in the next section, we do not yet assume that $N_0 = 0$. By simple algebraic manipulations, one can separate the contributions of the phase and additive noises to the total error in the following way:

$$I_k \sin\frac{\pi}{M} - Q_k \cos\frac{\pi}{M} = A_k + B_k + N$$

where

$$A_k = \int_{kT}^{(k+1)T} \sin\left(\sqrt{\epsilon}(w(t) - w(t-T)) + \frac{\pi}{M}\right) dt$$

$$\begin{aligned} B_k &= \sqrt{\epsilon N_0} \int_{(k-1)T}^{kT} \left[n_1(t)\sin\left(\frac{\pi}{M} + \theta + \sqrt{\epsilon}w((k-1)T)\right) - n_2(t)\cos\left(\frac{\pi}{M} + \theta + \sqrt{\epsilon}w((k-1)T)\right)\right] dt \\ &+ \sqrt{\epsilon N_0} \int_{kT}^{(k+1)T} \left[n_1(t)\sin\left(\frac{\pi}{M} - \theta - \sqrt{\epsilon}w((k-1)T)\right) + n_2(t)\cos\left(\frac{\pi}{M} - \theta - \sqrt{\epsilon}w((k-1)T)\right)\right] dt \end{aligned}$$

and

$$\begin{aligned} N &= \sqrt{\epsilon N_0} \int_{kT}^{(k+1)T} n_1(t) \left[\sin\left(\frac{\pi}{M} - \theta - \sqrt{\epsilon}w(t-T)\right) - \sin\left(\frac{\pi}{M} - \theta - \sqrt{\epsilon}w((k-1)T)\right)\right] dt \\ &+ \sqrt{\epsilon N_0} \int_{kT}^{(k+1)T} n_1(t-T) \left[\sin\left(\frac{\pi}{M} + \theta + \sqrt{\epsilon}w(t)\right) - \sin\left(\frac{\pi}{M} + \theta + \sqrt{\epsilon}w((k-1)T)\right)\right] dt \\ &+ \sqrt{\epsilon N_0} \int_{kT}^{(k+1)T} n_2(t) \left[\cos\left(\frac{\pi}{M} - \theta - \sqrt{\epsilon}w(t-T)\right) - \cos\left(\frac{\pi}{M} - \theta - \sqrt{\epsilon}w((k-1)T)\right)\right] dt \\ &- \sqrt{\epsilon N_0} \int_{kT}^{(k+1)T} n_2(t-T) \left[\cos\left(\frac{\pi}{M} + \theta + \sqrt{\epsilon}w(t)\right) - \cos\left(\frac{\pi}{M} + \theta + \sqrt{\epsilon}w((k-1)T)\right)\right] dt \\ &+ \epsilon N_0 \int_{kT}^{(k+1)T} \left[n_1(t)n_1(t-T) + n_2(t)n_2(t-T)\right] \sin\left(\frac{\pi}{M}\right) dt \\ &- \epsilon N_0 \int_{kT}^{(k+1)T} \left[n_1(t)n_2(t-T) - n_2(t)n_1(t-T)\right] \cos\left(\frac{\pi}{M}\right) dt \end{aligned} \tag{3.4}$$

We return now to the assumption that $N_0 \equiv 0$, which makes $B_k = N \equiv 0$. Then (3.2) is a direct consequence of Theorem 2.3 and Lemma 2.6. Unfortunately, it does not seem that the constrained optimization problem in (3.2) can be solved analytically, and thus bounds on it are of importance. The bound (3.3) is a consequence of the following slightly more general lemma, which will serve us also in the case of additive noise, and whose proof is presented in the appendix.

Lemma 3.5 *Let $\alpha \in [-T\sin\pi/M,\ T\sin\pi/M]$. Define*

$$\overline{I}_\alpha \stackrel{\triangle}{=} \frac{1}{2} \inf_{\{\phi: \int_T^{2T} \sin\left[\phi(t)-\phi(t-T)+\frac{\pi}{M}\right] dt \le \alpha, \phi(0)=0\}} \int_0^{2T} \dot{\phi}^2(t)\, dt \tag{3.6}$$

and $\alpha = T\sin\psi$. Then

$$\frac{1}{2T}\left(\frac{\pi}{M}-\psi\right)^2 \le \overline{I}_\alpha \le \frac{1}{T}\left(\frac{\pi}{M}-\psi\right)^2; \tag{3.7}$$

(3.3) is just Lemma 3.5 in the case $\alpha = \psi = 0$. □

Remark It can be shown (for details, see [4], Section 5.4) that $\lim_{M\to\infty} TM^2\overline{I}_0 = \frac{3}{4}\pi^2$. Moreover, a numerical evaluation of the solution to the optimization problem (3.2) reveals that, actually, $\overline{I}_0 \sim 3\pi^2/4M^2T$ as $M \to \infty$.

4 DMPSK Performance in the presence of weak additive noise

We return now to the case $N_0 \neq 0$. An inspection of the expression (1.5) for $\tilde{P}_e$ reveals that, if one could get rid of the cross term N (see (3.4)), one would have only to deal with the sum of two independent random variables, and the evaluation of the exponential rate of decay would be much simplified. That is, if it holds that, for some $\delta > 0$,

$$\liminf_{\epsilon\to 0}\ \epsilon \log P(A_k + B_k \le -\delta) > \limsup_{\epsilon\to 0}\ \epsilon\log P(|N| \ge \delta)\,, \tag{4.1}$$

then it follows that

$$\liminf_{\epsilon\to 0}\ \epsilon\log P(A_k+B_k \le -\delta) \le \liminf_{\epsilon\to 0}\ \epsilon\log\tilde{P}_e \le \limsup_{\epsilon\to 0}\ \epsilon\log\tilde{P}_e \le \limsup_{\epsilon\to 0}\ \epsilon\log P(A_k+B_k\le\delta)\,. \tag{4.2}$$

While any δ will do, one would like to have δ as small as possible in order to get tight bounds in (4.2). We show below (see Theorem 4.17 and (4.22)) that, for weak additive noise (i.e., N_0 small enough), condition (4.1) is satisfied, and that the smaller N_0 is, the smaller δ can be chosen, and tighter bounds can be obtained. Before presenting the analysis, we summarize the steps one has to take in order to choose an appropriate δ and compute the analytic bounds of this section. A numerical example is provided at the end of this section.

1) Given T, M, N_0 and the covariance of the narrow-band noise $n_1(t)$, check that N_0 is small enough for (4.23) to be satisfied. This is the regime where the cross term N appearing in (3.4) is not dominant on an exponential scale.

2) Evaluate $\lambda_{\rm cr}(N_0)$ by using (4.24). This measures the size (on an exponential scale) of the cross term N.

3) Compute $\delta_{\min}/T$, the solution to (4.21). This parameter will influence the tightness of the bounds obtained, by giving an absolute exponential bound on the contribution of the cross term N.

4) Evaluate $\underline{\psi}_{(-\delta_{\min})}$ and $\overline{\psi}_{\delta_{\min}}$, the solutions of (4.8) and (4.9), and use these and (4.10) to derive an upper bound on $J_{(-\delta_{\min})}$ and a lower bound on $J_{\delta_{\min}}$.

5) The asymptotic bounds on $\tilde{P}_e$ are given by (4.19) in terms of the quantities computed in step 4.

For small N_0 (such that $N_0 << M^2T^2\sin(\pi/M)/160\pi^2\lambda_{n_1}^2$, where λ_{n_1} is defined below (4.22)), we have $\delta_{\min} \ll T$, and then steps 4 and 5 above simply yield the bounds

$$-\frac{1}{T}\left(\frac{\pi}{M}+\frac{\delta_{\min}}{T}\right)^2 \le \lim_{\epsilon\to 0}\epsilon\log\tilde{P}_e \le -\frac{1}{2T}\left(\frac{\pi}{M}-\frac{\delta_{\min}}{T}\right)^2 . \tag{4.3}$$

We turn now to the analysis. We have for the RHS and LHS of (4.2) (the proof is deferred to the appendix):

Theorem 4.4 *For any η with $|\eta| < T\sin(\pi/M)$,*

$$\epsilon\log P(A_k+B_k\le\eta) \underset{\epsilon\to 0}{\longrightarrow} -J_\eta\,, \tag{4.5}$$

where

$$J_\eta \stackrel{\triangle}{=} \inf_{\alpha\in(-T\sin\frac{\pi}{M},-\eta]}\left(\frac{(\alpha+\eta)^2}{2N_0\sigma^2}+\overline{I}_{-\alpha}\right) \tag{4.6}$$

(see (3.6)), and

$$\sigma^2 = 2\int_0^T \int_0^T R_{n_1}(s-t)\,ds\,dt \;-2\cos\left(\frac{2\pi}{M}\right)\int_{-T}^0 \int_0^T R_{n_1}(s-t)\,ds\,dt\,. \tag{4.7}$$

Explicit bounds on J_η can be obtained by replacing $\overline{I}_{-\alpha}$ by the bounds from (3.7) and then minimizing over α. Indeed, let $(\overline{\psi}_\eta, \underline{\psi}_\eta)$ solve the transcendental equations

$$\frac{T^2 \sin\underline{\psi}_\eta \cos\underline{\psi}_\eta}{N_0\sigma^2} - \frac{\eta T\cos\underline{\psi}_\eta}{N_0\sigma^2} = \frac{2}{T}\left(\frac{\pi}{M} - \underline{\psi}_\eta\right), \quad \text{for } \underline{\psi}_\eta \in \left[-\frac{\pi}{M}, \frac{\pi}{M}\right] \tag{4.8}$$

$$\frac{T^2 \sin\overline{\psi}_\eta \cos\overline{\psi}_\eta}{N_0\sigma^2} - \frac{\eta T\cos\overline{\psi}_\eta}{N_0\sigma^2} = \frac{1}{T}\left(\frac{\pi}{M} - \overline{\psi}_\eta\right), \quad \text{for } \overline{\psi}_\eta \in \left[-\frac{\pi}{M}, \frac{\pi}{M}\right] \tag{4.9}$$

(Take η small enough to allow a solution, and note that for $\eta = 0$ a solution exists since the LHS increases with $\underline{\psi}_\eta(\overline{\psi}_\eta)$, while the RHS decreases with the same). Then

$$\frac{1}{2T}\left(\frac{\pi}{M} - \overline{\psi}_\eta\right)^2 + \frac{(\eta - T\sin\overline{\psi}_\eta)^2}{2N_0\sigma^2} \le J_\eta \le \frac{1}{T}\left(\frac{\pi}{M} - \underline{\psi}_\eta\right)^2 + \frac{(\eta - T\sin\underline{\psi}_\eta)^2}{2N_0\sigma^2}\,. \tag{4.10}$$

The last ingredient required for the evaluation of the DMPSK performance in the presence of additive noise is the derivation of explicit conditions for (4.1) to hold true. Note that, by the independence of A_k and B_k,

$$P(A_k + B_k \le -\delta) \ge P(B_k \le 0)P(A_k \le -\delta) = \frac{1}{2}\,P(A_k \le -\delta) \tag{4.11}$$

Hence,

$$\liminf_{\epsilon\to 0}\ \epsilon \log P(A_k + B_k \le -\delta) \ge -\overline{I}_{-\delta}\,. \tag{4.12}$$

On the other hand, for all $\alpha > 0$, considering $k = 1$ without loss of generality, we have

$$\frac{|N|}{\epsilon} \le \int_0^{2T} \frac{w(t)^2}{\alpha} dt + (\alpha+1)N_0 \int_0^{2T} n_1(t)^2 dt + (\alpha+1)N_0 \int_0^{2T} n_2(t)^2 dt\,. \tag{4.13}$$

Since all three terms in (4.13) are quadratic forms in Gaussian processes independent of ϵ, and since w, n_1, n_2 are independent, it follows that, for some $\lambda_{\rm cr}(N_0) > 0$,

$$\sup_{\epsilon>0} E\left(e^{\pm\lambda_{\rm cr}(N_0)N/\epsilon}\right) < \infty\,, \tag{4.14}$$

where E denotes the expectation over the distribution of N. Therefore, by Chebyshev's inequality,

$$\begin{aligned} P(|N| > \delta) &= P(N > \delta) + P(N < -\delta) \\ &\leq E\left(e^{\lambda_{\rm cr}(N_0)(N-\delta)/\epsilon}\right) + E\left(e^{-\lambda_{\rm cr}(N_0)(N+\delta)/\epsilon}\right) \\ &\leq E\left(e^{\frac{\lambda_{\rm cr}(N_0)N}{\epsilon}}\right) \cdot e^{-\delta\lambda_{\rm cr}(N_0)/\epsilon} + E\left(e^{-\lambda_{\rm cr}(N_0)\frac{N}{\epsilon}}\right) e^{-\delta\lambda_{\rm cr}(N_0)/\epsilon} \end{aligned} \tag{4.15}$$

Therefore,

$$\limsup_{\epsilon\to 0}\ \epsilon \log P(|N| > \delta) \leq -\delta\lambda_{\rm cr}(N_0)\,. \tag{4.16}$$

For explicit bounds on $\lambda_{\rm cr}$, see the remark following Theorem 4.17. Note that one can choose $\lambda_{\rm cr}$ so that $\lambda_{\rm cr}(N_0) \underset{N_0\to 0}{\longrightarrow} \infty$. In particular, for every positive δ one can find an N_0 small enough such that $\delta\lambda_{\rm cr}(N_0) > \bar{I}_{-\delta}$. We can now combine the results of this section to obtain

Theorem 4.17 *Let $T\sin(\pi/M) \geq \delta > 0$ and N_0 be small enough that*

$$\lambda_{\rm cr}(N_0) > \frac{\bar{I}_{-\delta}}{\delta} \tag{4.18}$$

where $\lambda_{\rm cr}(N_0)$ is defined in (4.14). Then

$$-J_{(-\delta)} \leq \liminf_{\epsilon\to 0}\ \epsilon \log \tilde{P}_e \leq \limsup_{\epsilon\to 0}\ \epsilon\log\tilde{P}_e \leq -J_\delta\,. \tag{4.19}$$

In particular, if

$$\lambda_{\rm cr}(N_0) > \frac{4\pi^2}{M^2T^2\sin(\pi/M)}\,, \tag{4.20}$$

then $\delta = \delta_{\rm min}$ suffices, where $\delta_{\rm min}$ is the unique solution of the equation

$$T\lambda_{\rm cr}(N_0)\delta_{\rm min} = [\pi/M + \sin^{-1}(\delta_{\rm min}/T)]^2. \tag{4.21}$$

Explicit bounds on J_δ and $J_{(-\delta)}$ are provided in (4.10).

Remark: An inspection of (4.13) reveals that a sufficient condition for (4.14) is that

$$E(e^{N_0\lambda_{\rm cr}(N_0)(\alpha+1)\int_0^{2T} n_1^2(t)dt}) < \infty \quad \text{and} \quad E(e^{\alpha^{-1}\lambda_{\rm cr}(N_0)\int_0^{2T} w_t^2 dt}) < \infty\,.$$

Recall that (see [3], Chapter 6), for any Gaussian process $x(t)$,

$$\text{for all } \beta < (2\lambda_1^2)^{-1}, \qquad E(e^{\beta\int_0^{2T} x_s^2 ds}) = \prod_{l=1}^{\infty}(1-2\beta\lambda_l^2)^{-1/2}\,,$$

where λ_l^2 are the eigenvalues of the Karhunen-Loève expansion of $x(t)$, and λ_1^2 denotes the largest eigenvalue of this expansion. It follows that

$$\lambda_{\rm cr}(N_0) \geq \min(\frac{1}{2(\alpha+1)N_0\lambda_{n_1}^2}, \frac{\alpha}{2\lambda_w^2}), \tag{4.22}$$

where $\lambda_{n_1}^2$ is the maximal eigenvalue of the Karhunen–Loève expansion of $n_1(t)$ on $[0, 2T]$, and $\lambda_w = 2\sqrt{2}T/\pi$ is the square-root of maximal eigenvalue of the Karhunen–Loève expansion of the Brownian motion on [0,2T]. Thus, to satisfy (4.20) (and hence (4.18)) it is enough to take $\alpha > 16$.

It follows from the computations above that a sufficient condition for (4.18) is that

$$N_0 \leq \frac{M^2T^2\sin(\pi/M)}{160\pi^2\lambda_{n_1}^2}, \tag{4.23}$$

and then (4.22) is transformed to

$$\lambda_{\rm cr}(N_0) \geq \frac{\pi^2}{32T^2}\left(\sqrt{1+\frac{32T^2}{\pi^2N_0\lambda_{n_1}^2}}-1\right). \tag{4.24}$$

Finally, note that $\lambda_{n_1} \sim k(WT)T/WT$, where $k(\cdot)$ depends on the precise shape of the covariance $R_{n_1}(\tau)$. In the case of ideal bandpass noise, $k(\cdot)$ is related to the prolate-spheroidal wave functions described in [13] and tabulated in table I there, with $k(1) = 0.57$ and $k(WT) = 1$ for $WT > 8$.

We conclude this section with a numerical example for the evaluation of the bounds. For simplicity, we take $M = 2$. Since the bounds depend on the precise shape of $R_{n_1}(s-t)$, we assume that $\sigma^2 = c\lambda_{n_1}^2$, where σ^2 is defined in (4.7) and c is a numerical factor, for which we check the cases of $c = 1$ and $c = 10$ (the case $c = 1$ corresponding roughly to ideal bandpass noise with $WT = 1$). As we shall see, the value of the bound depends only weakly on the precise value of c, especially for small N_0.

We now follow the steps described in the beginning of this section.

Steps 1, 2 and 3 The condition (4.23) implies the bound $N_0 \leq T^2/(40\pi^2\lambda_{n_1}^2)$. We thus consider three case:

a) $N_0\lambda_{n_1}^2 = T^2/40\pi^2$, which implies $\lambda_{\rm cr} = 11/T^2$ and $\delta_{\rm min}/T = 0.33$.

b) $N_0\lambda_{n_1}^2 = T^2/400\pi^2$, which implies $\lambda_{\rm cr} = 35/T^2$ and $\delta_{\rm min}/T = 0.075$.

c) $N_0\lambda_{n_1}^2 = T^2/4000\pi^2$, which implies $\lambda_{cr} = 110/T^2$ and $\delta_{\min}/T = 0.02$.

From here, the evaluation of the bounds differs for $c = 1$ and $c = 10$. Whereas for $c = 1$ $\underline{\psi}_{\delta_{\min}} \sim \overline{\psi}_{\delta_{\min}} \sim \delta_{\min}/T$, for $c = 10$ one actually needs to solve the transcendental equation (4.8) and (4.9). The values of the bounds are shown in Tables 1 and 2.

$N_0\lambda_{n_1}^2$	Upper bound	Lower bound	Upper-to-lower bound ratio
$T^2/40\pi^2$	0.77/T	3.61/T	4.69
$T^2/400\pi^2$	1.12/T	2.70/T	2.41
$T^2/4000\pi^2$	1.2/T	2.53/T	2.10
0	1.23/T	2.46/T	2

Table 1: Bounds for $\sigma^2/(\lambda_{n_1}^2) = c = 1$

$N_0\lambda_{n_1}^2$	Upper bound	Lower bound	Upper-to-lower bound ratio
$T^2/40\pi^2$	0.73/T	3.3/T	4.5
$T^2/400\pi^2$	1.07/T	2.45/T	2.29
$T^2/4000\pi^2$	1.14/T	2.29/T	2.01
0	1.23/T	2.46/T	2

Table 2: Bounds for $\sigma^2/(\lambda_{n_1}^2) = c = 10$

5 Discussion and Conclusions

We have presented an analysis, based on large deviations, of the performance of a DMPSK system in the presence of both phase noise and weak additive noise. As with any asymptotic study, the question of for which values of ϵ the asymptotic analysis is close to reality is of interest. Unfortunately this question cannot be answered analytically, and thus the method presented is no substitute for a numerical evaluation of the performance for a *specific* value of the signal-to-noise ratio. Its main purpose, as explained in the introduction, is to clarify the roles of various systems parameters and to allow for a comparison of different designs and values of the system's parameters.

In the particular case treated in this paper, some numerical results are available in the literature for differential two-phase keying (i.e., DPSK, with $M = 2$). The analysis presented in this paper leads to the computation of the limiting slope of $\log P(E)$, the logarithm of the bit error probability, as a function of the phase noise and additive noise. An evaluation of $P(E)$ is presented, e.g., in [9] (where sampling, and not post-detection integration, is used) and in [6],[7],[10]. In all cases, one sees that the system operates within the linear decay of $\log P(E)$ as predicted by this paper for values of $P(E)$ smaller than $10^{-6} - 10^{-9}$, depending on the precise values of the system's parameters.

Another analytic method available for the evaluation of $P(E)$ is based on the evaluation of moments of the input signal (see [1] and [9]). This method yields sharp results for small signal-to-noise ratios but becomes numerically unstable for high signal-to-noise ratio, hence, is unsuitable for computations in the region of practical interest. Moreover, because of the summation over a large number of terms, the effect of the system's parameters on the performance is hard to evaluate.

Some authors have analyzed the influence of the IF filter on the performance ([6],[7],[10]). Although we have not attempted to do so here, such an analysis via large deviations is possible and mainly involves considering the minimizing path in the constrained-optimization problem described in Section 3.

We comment here on the effect of post-detection integration in the DMPSK model. Many of the references dealing with the DMPSK compute the exponential rate of decay of the floor bit error probability. In the notations used in most references, $\epsilon = 2\pi\Delta\nu$, where $\Delta\nu$ denotes the spectral impurity of the laser, and the computed probability-of-error exponent $\overline{I}_0$ ranges from $\pi^2/8T$ (when there is neither IF filtering nor post-detection integration, see [6]) to $\pi^2/4T$ or $\pi^2/6T$ or $\pi^2/5.8T$ (when IF filtering is present but not post-detection integration, see [7],[10]). The inclusion of post-detection integration considerably complicates the standard analysis, and an attempt to use a Chebyshev upper bound does not change the value of the computed $\overline{I}_0$ from the value for the case of no post-detection integration. The analysis of this paper, on the other hand, predicts an improvement in the floor bit-error probability of 1.7 dB (when one uses the remark below Lemma 3.5) over the case with no integration.

A Appendix

Proof of Lemma 3.5: Let $\tilde{\phi}(t) = -t(\pi/M - \psi)/T$. Note that $\int_T^{2T} \sin\left[\tilde{\phi}(t) - \tilde{\phi}(t-T) + \pi/M\right] = \alpha$ and $\overline{I}_\alpha \le \frac{1}{2}\int_0^{2T} \dot{\tilde{\phi}}^2\, dt = (\frac{\pi}{M} - \psi)^2/T$. To see the reverse inequality, note that, for any $s \in [T, 2T]$, it follows from the Cauchy–Schwartz inequality that

$$\frac{1}{2}\int_0^{2T} \dot{\phi}^2 dt \ge \frac{1}{2}\int_{s-T}^{s} \dot{\phi}^2 dt \ge \frac{1}{2T}\left(\int_{s-T}^{s} \dot{\phi}\, dt\right)^2 = \frac{1}{2T}\left|\phi_s - \phi_{s-T}\right|^2 . \tag{A.1}$$

Now define

$$\tau = \inf\{s \ge T : \quad |\phi_s - \phi_{s-T}| \ge \frac{\pi}{M} - \psi\}.$$

Then one can verify that, for $\alpha \in [-T\sin(\pi/M), T\sin(\pi/M)]$,

$$\int_T^{2T} \sin\left(\phi(t) - \phi(t-T) + \frac{\pi}{M}\right) dt \le \alpha \ \text{ implies } \ \tau \le 2T . \tag{A.2}$$

Combining (A.1) and (A.2), we conclude that

$$\overline{I}_\alpha \ge \frac{1}{2T}\left|\frac{\pi}{M} - \psi\right|^2 \tag{A.3}$$

and the proof of Lemma 3.5 is complete. ☐

Proof of Theorem 4.4 : Note that A_k and B_k (defined above (3.4)) are independent, B_k (conditioned on $\{w(s),\ 0 \le s \le (k-1)T\}$) is a zero-mean normal random variable with variance $\epsilon N_0\sigma^2$, and $|A_k| \le T$. Thus,

$$\begin{aligned} P(A_k + B_k \le \eta) &= \frac{1}{\sqrt{2\pi\epsilon N_0\sigma^2}}\int_{-\infty}^{\infty} e^{-x^2/2\epsilon N_0\sigma^2}\ P(A_k \le \eta - x)\, dx \\ &= \frac{1}{\sqrt{2\pi\epsilon N_0\sigma^2}}\int_{-\infty}^{-T} e^{-(x+\eta)^2/2\epsilon N_0\sigma^2}\, dx \\ &+ \frac{1}{\sqrt{2\pi\epsilon N_0\sigma^2}}\int_{-T}^{T} e^{-(x+\eta)^2/2\epsilon N_0\sigma^2} P(A_k \le -x)\, dx . \end{aligned} \tag{A.4}$$

From Theorem 2.3 and Lemma 2.6, it follows that

$$\epsilon \log P(A_k \le -x) \underset{\epsilon\to 0}{\longrightarrow} -\overline{I}_{-x} \tag{A.5}$$

where $\overline{I}_{-x}$ is as defined in Lemma 3.5. Thus, using the monotonicity of $\overline{I}_{-x}$ in x, one obtains

$$\epsilon \log P(A_k + B_k \le \eta) \underset{\epsilon\to 0}{\longrightarrow} -\min\left(\frac{(T-\eta)^2}{2N_0\sigma^2}\ ,\ \inf_{\alpha\in[-T,T]}\left[\frac{(\alpha+\eta)^2}{2N_0\sigma^2} + \overline{I}_{-\alpha}\right]\right) \tag{A.6}$$

Note that, by substituting $\phi \equiv 0$, it follows that $\overline{I}_\alpha = 0$ for $\alpha \geq T\sin(\pi/M)$. This and the monotonicity (in α) of $\overline{I}_{-\alpha}$ imply that

$$\epsilon \log P(A_k + B_k \leq \eta) \underset{\epsilon\to 0}{\longrightarrow} -\inf_{\alpha\in(-T\sin\frac{\pi}{M},-\eta]} \left(\frac{(\alpha+\eta)^2}{2N_0\sigma^2} + \overline{I}_{-\alpha}\right) = -J_\eta\,. \tag{A.7}$$

□

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FIGURE CAPTIONS

Fig. 1.1: Receiver structure

TABLES CAPTIONS

Table 1: Bounds for $\sigma^2/(\lambda_{n_1}^2) = c = 1$

Table 2: Bounds for $\sigma^2/(\lambda_{n_1}^2) = c = 10$