Limit theorems for one-dimensional transient random walks in Markov environments

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Abstract

We obtain non-Gaussian limit laws for one-dimensional random walk in a random environment assuming that the environment is a function of a stationary Markov process. This is an extension of the work of Kesten, M. Kozlov and Spitzer [14] for random walks in i.i.d. environments. The basic assumption is that the underlying Markov chain is irreducible and either with a finite state space or with the transition kernel dominated above and below by a probability measure.

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1 Introduction and Statement of Results

Let $\Omega = (0,1)^{\mathbb{Z}}$ and let $\mathcal{F}$ be the Borel $\sigma-$algebra on Ω. A *random environment* is an element $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ of Ω distributed according to a stationary and ergodic probability measure P on $(\Omega, \mathcal{F})$. The *random walk in the environment* ω is a time-homogeneous Markov chain $X = \{X_n\}_{n\in\mathbb{N}}$ on $\mathbb{Z}$ governed by the *quenched* law

$$P_\omega(X_0 = 0) = 1 \quad \text{and} \quad P_\omega(X_{n+1} = j | X_n = i) = \begin{cases} \omega_i & \text{if } j = i+1, \\ 1-\omega_i & \text{if } j = i-1. \end{cases}$$

Let $\left(\mathbb{Z}^{\mathbb{N}}, \mathcal{G}\right)$ be the canonical space for the paths of $\{X_n\}$, i.e. $\mathcal{G}$ is the cylinder $\sigma-$algebra. *The random walk in random environment* (RWRE) associated with P is the process (X, ω)

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on the measurable space $\left(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathcal{F} \otimes \mathcal{G}\right)$ having the *annealed* probability law $\mathbb{P} = P \otimes P_\omega$ defined by

$$\mathbb{P}(F \times G) = \int_F P_\omega(G) P(d\omega), \quad F \in \mathcal{F},\ G \in \mathcal{G}.$$

Since the process learns about the environment as time passes according to the Bayes rule, $\{X_n\}$ is in general not a Markov chain under the annealed measure $\mathbb{P}$. The model goes back to [16, 20] and, in physics, to [8, 24]. In this introduction we briefly discuss some basic results on the one-dimensional RWRE. We refer the reader to [22, 25] for recent comprehensive surveys of the field.

Recurrence criteria and possible speed regimes for the one-dimensional RWRE were established by Solomon [20] in the case where $\{\omega_n\}$ is an i.i.d. sequence and carried over to the general ergodic environments by Alili [1]. Let

$$\begin{aligned} \rho_n &= \frac{1-\omega_n}{\omega_n}, \\ R(\omega) &= 1 + \sum_{n=0}^{+\infty} \rho_0 \rho_{-1} \cdots \rho_{-n}, \end{aligned} \tag{1.1}$$

$T_0 = 0$, and for $n \in \mathbb{N}$,

$$T_n = \min\{k : X_k \geq n\} \quad \text{and} \quad \tau_n = T_n - T_{n-1}. \tag{1.2}$$

X_n is a.s. transient to the right (to the left) if $E_P(\log \rho_0) < 0$ (> 0) and is a.s. recurrent if $E_P(\log \rho_0) = 0$. If $E_P(\log \rho_0) < 0$ then (see [25, Sect 2.1]) $\lim_{n\to\infty} \mathbb{P}(X_n = +\infty) = 1$, T_n are a.s. finite, $\{\tau_n\}$ is a stationary and ergodic sequence, and we have the following law of large numbers:

$$\mathrm{v}_P := \lim_{n\to+\infty} \frac{X_n}{n} = \lim_{n\to+\infty} \frac{n}{T_n} = \frac{1}{\mathbb{E}(\tau_1)} = \frac{1}{2E_P(R) - 1}, \quad \mathbb{P} - \text{a.s.} \tag{1.3}$$

Thus, the transient walk X_n has a deterministic speed $\mathrm{v}_P = \lim_{n\to\infty} X_n/n$ which may be zero.

Solomon's law of large numbers for the transient walks in i.i.d. environment was completed by limit laws in the work of Kesten, M. Kozlov, and Spitzer [14]. The limit laws for the RWRE X_n are deduced in [14] from stable limit laws for the hitting times T_n, and the index κ of the stable distribution is determined by the condition

$$E_P(\rho_0^\kappa) = 1.$$

In particular, under certain conditions the central limit theorem holds with the standard normalization $\sqrt{n}$, and this case was extended to stationary and ergodic environments by Alili [1], Molchanov [17] and Zeitouni [25, Sect 2.2], see also Bremont [7].

In this paper we obtain limit laws for X_n for environments which are point-wise transformations of a stationary ergodic Markov process which satisfies Assumption 1.5 below. These laws are related to stable laws of index $\kappa \in (0,2]$, where, under the assumptions below, κ is determined by

$$\Lambda(\kappa) = 0\,, \quad \text{where } \Lambda(\beta) := \lim_{n\to\infty} \frac{1}{n} \log E_P \left(\prod_{i=0}^{n-1} \rho_i^\beta \right). \tag{1.4}$$

More precisely:

Basic setup: On a state space $\mathcal{S}$ equipped with a countably generated σ–field $\mathcal{T}$, let $\{x_n\}_{n\in\mathbb{Z}}$ be a stationary Markov chain, such that $\omega_{-n} = \omega(x_n)$ (and hence $\rho_{-n} = \rho(x_n)$) for measurable functions $\rho, \omega : \mathcal{S} \to \mathbb{R}$. We denote by $H(x, A)$ the transition probability measure of (x_n), by π its stationary probability measure, and use the notation $H(x, y)$ to denote $H(x, \{y\})$ for a single state $y \in \mathcal{S}$. With P_x denoting the law of the Markov chain with $x_0 = x$, the reader should not confuse P_x and P_ω.

Assumption 1.5.

(A1) Either

$$\mathcal{S} \text{ is a finite set and the Markov chain } (x_n) \text{ is irreducible,} \tag{1.6}$$

or, there exist a constant $c_r \geq 1$ and a probability measure ψ on $(\mathcal{S}, \mathcal{T})$ such that for some $m \in \mathbb{N}$,

$$c_r^{-1}\psi(A) < H^m(x, A) < c_r\psi(A) \quad \forall x \in \mathcal{S}, A \in \mathcal{T}, \tag{1.7}$$

where the kernel $H^n(x, A)$ is defined inductively by $H^0(x, A) = \mathbf{1}_A(x)$ for all $x \in \mathcal{S}, A \in \mathcal{T}$ and $H^n(x, A) = \int_{\mathcal{S}} H^{n-1}(x, dy)H(y, A)$, $n \geq 1$.

(A2) $P(\epsilon < \omega_0 < 1 - \epsilon) = 1$ for some $\epsilon \in (0, 1/2)$.

(A3) $P(\rho_0 > 1) > 0$ and $\limsup_{n\to\infty} \frac{1}{n} \log E_P \left(\prod_{i=0}^{n-1} \rho_i^\beta \right) < 0$ for some $\beta > 0$.

(A4) The stationary law of $\log \rho_0$ is non-arithmetic, that is its support is not contained in any proper sub-lattice of $\mathbb{R}$.

Note that condition *(A1)* refers to the underlying Markov chain (x_n), whereas conditions *(A2)–(A4)* refer to ω itself. Note that assumption (1.6) is not implied by assumption (1.7) since the Markov chain (x_n) may be periodic. Under Assumption *(A1)*, the environment ω is an ergodic sequence (see e.g. [9, p. 338] and [18, Theorem 6.15]). It follows from *(A3)* by Jensen's inequality that $E_P(\log \rho_0) < 0$, so that X_n is transient to the right. For future reference we denote

$$c_\rho = \frac{1 - \epsilon}{\epsilon}, \tag{1.8}$$

and note that by the ellipticity condition *(A2)*, $P(c_\rho^{-1} < \rho_0 < c_\rho) = 1$.

For $\kappa \in (0, 2]$ and $b > 0$ we denote by $\mathcal{L}_{\kappa,b}$ the stable law of index κ with the characteristic function

$$\log \widehat{\mathcal{L}}_{\kappa,b}(t) = -b|t|^\kappa \left(1 + i\frac{t}{|t|} f_\kappa(t) \right), \tag{1.9}$$

where $f_\kappa(t) = -\tan \frac{\pi}{2}\kappa$ if $\kappa \neq 1$, $f_1(t) = 2/\pi \log t$. With a slight abuse of notation we use the same symbol for the distribution function of this law. If $\kappa < 1$, $\mathcal{L}_{\kappa,b}$ is supported on the positive reals, and if $\kappa \in (1, 2]$, it has zero mean [19, Chapter 1]. Our main result is:

Theorem 1.10. *Let Assumption 1.5 hold. Then there is a unique $\kappa > 0$ such that* (1.4) *and the following hold for some $b > 0$:*

(i) If $\kappa \in (0,1)$, then $\lim_{n\to\infty} \mathbb{P}\left(n^{-\kappa} X_n \le \mathfrak{z}\right) = 1 - \mathcal{L}_{\kappa,b}(\mathfrak{z}^{-1/\kappa})$,

(ii) If $\kappa = 1$, then $\lim_{n\to\infty} \mathbb{P}\big(n^{-1}(\log n)^2(X_n - \delta(n)) \le \mathfrak{z}\big) = 1 - \mathcal{L}_{1,b}(-\mathfrak{z})$, *for suitable $A_1 > 0$ and $\delta(n) \sim (A_1 \log n)^{-1} n$,*

(iii) If $\kappa \in (1,2)$, then $\lim_{n\to\infty} \mathbb{P}\left(n^{-1/\kappa}\left(X_n - n\mathrm{v}_P\right) \le \mathfrak{z}\right) = 1 - \mathcal{L}_{\kappa,b}(-\mathfrak{z})$.

(iv) If $\kappa = 2$, then $\lim_{n\to\infty} \mathbb{P}\left((n \log n)^{-1/2}(X_n - n\mathrm{v}_P) \le \mathfrak{z}\right) = \mathcal{L}_{2,b}(\mathfrak{z})$.

In the setup of Theorem 1.10 it is not hard to check, and follows e.g. from [25], that the standard CLT holds if $\kappa > 2$.

As in [14], stable laws for X_n follow from stable laws for the hitting times T_n, and we direct our efforts to obtaining limit laws for the latter. We have:

Proposition 1.11. *Let Assumption 1.5 hold. Then there is a unique $\kappa > 0$ such that* (1.4) *and the following hold for some $\tilde{b} > 0$:*

(i) If $\kappa \in (0,1)$, then $\lim_{n\to\infty} \mathbb{P}\left(n^{-1/\kappa} T_n \le t\right) = \mathcal{L}_{\kappa,\tilde{b}}(t)$,

(ii) If $\kappa = 1$, then $\lim_{n\to\infty} \mathbb{P}\big(n^{-1}(T_n - nD(n)) \le t\big) = \mathcal{L}_{1,\tilde{b}}(t)$, *for suitable $c_0 > 0$ and $D(n) \sim c_0 \log n$,*

(iii) If $\kappa \in (1,2)$, then $\lim_{n\to\infty} \mathbb{P}\left(n^{-1/\kappa}\left(T_n - n\mathrm{v}_P^{-1}\right) \le t\right) = \mathcal{L}_{\kappa,\tilde{b}}(t)$.

(iv) If $\kappa = 2$, then $\lim_{n\to\infty} \mathbb{P}\left((n \log n)^{-1/2}(T_n - n\mathrm{v}_P^{-1}) \le t\right) = \mathcal{L}_{2,\tilde{b}}(t)$.

The proof that Theorem 1.10 follows from Proposition 1.11 is the same as in the i.i.d. case, and is based on the observation that for any positive integers η, ζ, n

$$\{T_\zeta \ge n\} \subset \{X_n \le \zeta\} \subset \{T_{\zeta+\eta} \ge n\} \bigcup \{\inf_{k \ge T_{\zeta+\eta}} X_k - (\zeta+\eta) \le -\eta\}. \tag{1.12}$$

Because the random variables $\inf_{k\ge T_{\zeta+\eta}} X_k - (\zeta+\eta)$ and $\inf_{k\ge 0} X_k$ have the same annealed distribution, the probability of the last event in (1.12) can be made arbitrary small uniformly in n and ζ by fixing η large (since the RWRE X_n is transient to the right). For $\kappa = 1$, the rest of the argument is detailed in [14, pp. 167–168], where no use of the i.i.d. assumption for ω is made at that stage, and a similar argument works for all $\kappa \in (0,2]$. All of our work in the sequel is directed toward the proof of Proposition 1.11.

Following [14], the analysis of T_n is best understood in terms of certain regeneration times ν_n, with excursion counts between regenerations forming a branching process Z_n with immigration in a random environment (see Section 2.2 for precise definitions). In the i.i.d. setup, the total population of the branching process between regenerations, denoted W_n, forms an i.i.d. sequence, and much of the work in [14] is to establish accurate enough tail estimates on them to allow for the application of the i.i.d. stable limit law for partial sums of W_n. The limit laws for T_n then easily follow from those for W_n.

In our case, the sequence W_n a-priori is not even stationary. However, using the regeneration property of the underlying Markov chain (x_n) (see Section 2.1), we introduce in Section 2.2 modified regeneration times $\bar{\nu}_n$ (a random subsequence of ν_n) such that the total population of the branching process between times $\bar{\nu}_n$ and $\bar{\nu}_{n+1}$, denoted by $\overline{W}_{n+1}$, is a one-dependent stationary sequence. This sequence is i.i.d. if either (1.7) with $m = 1$ or (1.6) hold. Again following the proof in [14], we obtain tails estimates for the random variables $\overline{W}_{n+1}$ yielding the stable limit laws for T_n stated in Proposition 1.11. Similarly to the i.i.d. case, the key to the proof is the derivation of tails estimates obtained in Section 2.3 for the random variable R defined in (1.1).

We conclude the introduction with a characterization of the speed v_P under Assumption 1.5, which will not be used in the sequel. Recall that $\rho_n = \rho(x_n)$ for a measurable function $\rho : \mathcal{S} \to \mathbb{R}$. If $\kappa \le 1$, then $\mathrm{v}_P = 0$, and if $\kappa > 1$, then $\mathrm{v}_P^{-1} = E_P\big(\rho(x_0)\xi(x_0)\big)$, where the function $\xi : \mathcal{S} \to (0, \infty)$ is the unique positive and bounded solution of the equation

$$\xi(x) = \int_{\mathcal{S}} H(x, dy)\rho(y)\xi(y) + 1 + 1/\rho(x). \tag{1.13}$$

This formula is essentially due to Takacs [23], who considered finite-state Markov environments. The proof in the general case is included at the end of Section 2.1.

The rest of the paper is organized as follows. Section 2, divided into three subsections, contains the proof of Theorem 1.10, except for the proofs of two propositions which are deferred to the Appendix. In Subsection 2.1 some basic properties of Markov chains that satisfy Assumption 1.5 are described. In particular, Condition B is introduced and shown to hold under Assumption 1.5. In Subsection 2.2, Condition C_κ is introduced and Proposition 1.11 is derived from it and Condition B, making use of the above mentioned branching process and a regeneration structure it possesses. Finally, Subsection 2.3 is devoted to the proof that Condition C_κ holds under Assumption 1.5.

2 Proofs

2.1 Some properties of the underlying Markov chain

We summarize here, using the framework of the Athreya-Ney and Nummelin theory of positive recurrent kernels (cf. [5, 6, 18]), some properties of the Markov chain (x_n) that follow from Assumption 1.5. The main objectives here are to introduce the regeneration times N_k and to obtain the Perron-Frobenius type Lemmas 2.6 and 2.9. One immediate consequence of these lemmas is that Condition B introduced subsequently is satisfied under Assumption 1.5.

First, we define a sequence of regeneration times for the Markov chain (x_n). If (1.6) holds, let $x^* \in \mathcal{S}$ be any (recurrent) state of the Markov chain (x_n) and pick any $r \in (0, 1)$. Let $(y_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. variables independent of (x_n) (in an enlarged probability space if needed) such that $P(y_0 = 1) = r$ and $P(y_0 = 0) = 1 - r$, and let

$$N_0 = 0, \quad N_{n+1} = \min\{k > N_n : x_n = x^*,\ y_n = 1\},\ n \ge 0.$$

Then, the blocks $\big(x_{N_n}, x_{N_n+1}, \ldots, x_{N_{n+1}-1}\big)$ are independent, and x_{N_n} are identically distributed for $n \ge 1$. Note that between two successive regeneration times, the chain evolves according to the sub-stochastic Markov kernel Θ defined by

$$H(x, y) = \Theta(x, y) + r\mathbf{1}_{\{y=x^*\}}H(x, y), \tag{2.1}$$

that is

$$P_x(x_1 = y, N_1 > 1) = \Theta(x, y). \tag{2.2}$$

If (1.7) holds, then the random variables N_k can be defined by the following procedure (see [5, 18] and [3]). Given an initial state x_0, generate x_m as follows: with probability $r < c_r^{-1}$ distribute x_m over $\mathcal{S}$ according to ψ and with probability $1-r$ according to $1/(1-r)\cdot\Theta(x_0, \cdot)$, where the kernel $\Theta(x, \cdot)$ is defined by

$$H^m(x, A) = \Theta(x, A) + r\psi(A), \quad x \in \mathcal{S}, A \in \mathcal{T}. \tag{2.3}$$

Then, (unless $m = 1$) sample the segment $\big(x_1, x_2, \ldots, x_{m-1}\big)$ according to the chain's conditional distribution, given x_0 and x_m. Generate x_{2m} and $x_{m+1}, x_{m+2}, \ldots, x_{2m-1}$ in a similar way, and so on. Since the "r-coin" is tossed each time independently, the event "the next move of the chain $(x_{mn})_{n\geq 0}$ is according to ψ" occurs i.o. Let $N_0 = 0$ and $\{N_k\}_{k\geq 1}$ be the successful times of its occurrence multiplied by m. By construction, the blocks $\big(x_{N_n}, x_{N_n+1}, \ldots, x_{N_{n+1}-1}\big)$ are one-dependent (if $m = 1$ they are actually independent), and for $n \geq 1$ they are identically distributed (x_{N_n} is distributed according to ψ).

Let us summarize the most important property of the regeneration times N_n as follows. For $n \geq 0$, let

$$D_n = \big(x_{N_n}, x_{N_n+1}, \ldots, x_{N_{n+1}-1}\big). \tag{2.4}$$

Then:

- The random blocks D_n are identically distributed for $n \geq 1$.
- If (1.6) or (1.7) with $m = 1$ hold, D_n are independent for $n \geq 0$.
- If (1.7) holds with $m > 1$, D_n are one-dependent for $n \geq 0$.

In both cases under consideration (either of (1.6) or of (1.7)), there exist constants $l, \delta > 0$, such that (cf. [5])

$$\inf_{x\in\mathcal{S}} P_x(N_1 \leq l) > \delta > 0. \tag{2.5}$$

The regeneration times N_n will be used in Section 2.2 for the construction of an auxiliary sequence $\overline{W}_n$ of stationary and one-dependent random variables playing a central role in the proof of Proposition 1.11. Throughout the paper we keep the notation $\Theta(x, dy)$ for the kernel introduced in (2.1) or (2.3).

We now turn to a Perron-Frobenius type theorem for positive finite kernels, having in mind applications to the kernels of the form $K(x, A) = E_x\left(\prod_{i=0}^n \rho_{-i}^\beta; x_n \in A\right)$. In the following two lemmas, we consider separately the cases of non-finite (assumption (1.7)) and finite (assumption (1.6)) state space $\mathcal{S}$. In particular, the properties of positive kernels described in these lemmas imply Condition B introduced below and are essential for the proof of the crucial Proposition 2.38.

Let B_b be the Banach space of bounded measurable real-valued functions on $(\mathcal{S},\mathcal{T})$ with the norm $\|f\| = \sup_{x\in\mathcal{S}} |f(x)|$. A positive and finite kernel $K(x,A)$ (a measurable function of x for all $A \in \mathcal{T}$ and a finite positive measure on $\mathcal{T}$ for all $x \in \mathcal{S}$) defines a bounded linear operator on B_b by setting $Kf(x) = \int_{\mathcal{S}} K(x,dy)f(y)$. We denote by $r_{_K}$ the spectral radius of the operator corresponding to the kernel K, that is

$$r_K = \lim_{n\to\infty} \sqrt[n]{\|K^n\mathbf{1}\|} = \lim_{n\to\infty} \sqrt[n]{\|K_n\|_{B_b\to B_b}},$$

where $\mathbf{1}(x) \equiv 1$.

Although the results stated in the following lemma are certainly well-known and appear elsewhere, their proofs are provided for the sake of completeness.

Lemma 2.6. *Let $K(x,A)$ be a positive kernel on $(\mathcal{S},\mathcal{T})$ such that for some constant $c \geq 1$ and probability measure ψ,*

$$c^{-1}\psi(A) \leq K(x,A) \leq c\psi(A), \quad \forall x \in \mathcal{S},\ A \in \mathcal{T}. \tag{2.7}$$

Further, let a function $q(x) \in B_b$ and a positive finite kernel $\widetilde{\Theta}(x,A)$ be such that

$$K(x,A) = \widetilde{\Theta}(x,A) + q(x)\psi(A), \quad x \in \mathcal{S}, A \in \mathcal{T}, \tag{2.8}$$

and $0 < q(x) < d < 1/c$ for some constant $d \in (0,1/c)$ and ψ-almost all $x \in \mathcal{S}$. Then,

(a) There exists a function $f \in B_b$ such that $\inf_x f(x) > 0$ and $Kf = r_K f$. There exists a constant $c_K \geq 1$ such that $c_K^{-1} r_K^n \leq K^n\mathbf{1} \leq c_K r_K^n$ for all $n \in \mathbb{N}$.

(b) There exists a function $g \in B_b$ such that $\inf_x g(x) > 0$ and $\widetilde{\Theta}g = r_{\widetilde{\Theta}}g$.

(c) $r_{\widetilde{\Theta}} \in (0, r_{_K})$.

(d) If $K = K_1^m$ for a positive finite kernel $K_1(x,A)$ and some $m \in \mathbb{N}$, then $r_{_{K_1}} = r_K^{1/m}$ and there exists a function $f_1 \in B_b$ such that $\inf_x f_1(x) > 0$ and $K_1 f_1 = r_{_{K_1}}^{1/m} f_1$.

Proof.

(a) The existence of a function $f : \mathcal{S} \to (0,\infty)$ and a constant $\lambda > 0$ such that $Kf = \lambda f$ follows from the Example in [18, p. 96]. It follows from (2.7) that $f(x)$ is bounded away from zero and infinity, i.e. $c_{_K}^{-1} \leq f(x) \leq c_{_K}$ for some $c_{_K} > 0$. Hence, for any $n > 0$, $K^n\mathbf{1} < c_{_K} K^n f = c_{_K}\lambda^n f < c_{_K}^2\lambda^n$. Similarly, $K^n\mathbf{1} > c_{_K}^{-2}\lambda^n$. That is $\lambda = r_{_K}$.

(b) The proof for the kernel $\widetilde{\Theta}$ is the same as for K, by using the following counterpart of (2.7): $(1/c - d)\psi(A) \leq \widetilde{\Theta}(x,A) \leq c\psi(A)$.

(c) Clearly $r_{\widetilde{\Theta}} \leq r_{_K}$. Since $r_K f \geq \widetilde{\Theta}f$, the equality $r_{\widetilde{\Theta}} = r_{_K}$ would imply [18, Theorem 5.1] that $f = g$ and $\widetilde{\Theta}f = r_K f = Kf$, that is impossible since $f > 0$ everywhere. Hence $r_{\widetilde{\Theta}} < r_{_K}$.

(d) Set $f_1 = \sum_{j=0}^{m-1} (1/r_K)^{j/m} K_1^j f$. $\square$

The finite-state counterpart of the previous lemma is stated as follows:

Lemma 2.9. *Let $\mathcal{S} = \{1, 2, \ldots, n\}$ and $K(i,j)$ be an irreducible $n \times n$ matrix with nonnegative entries. For some constants $r \in (0,1)$ and $j^* \in \{1, \ldots, n\}$ define the matrix $\widetilde{\Theta}(i,j)$ by*

$$K(i,j) = \widetilde{\Theta}(i,j) + r\mathbf{1}_{\{j=j^*\}} K(i,j), \quad 1 \le i, j \le n. \tag{2.10}$$

Then, and (a)–(c) of Lemma (2.6) *hold for the matrices K and $\widetilde{\Theta}$.*

Proof. Since $\widetilde{\Theta}$ and K have the same adjacency matrices ($K(i,j) = 0$ iff $\widetilde{\Theta}(i,j) = 0$), $\widetilde{\Theta}$ is irreducible as well. Assertions of *(a)* and *(b)* follow then from the Perron-Frobenius theorem, and *(c)* follows again from Theorem 5.1 in [18]. □

Since for any $\beta \ge 0$,

$$E_x \left(\prod_{k=0}^{n-1} (\rho_{-k})^\beta \right) = \rho(x)^\beta H_\beta^{n-1} \mathbf{1}, \tag{2.11}$$

where $H_\beta(x, dy) = H(x, dy)\rho(y)^\beta$, it follows from Lemmas 2.6 and 2.9 that for some constant $c_\beta \ge 1$ which depends on β only,

$$c_\beta^{-1} r_\beta^n \le E_x \left(\prod_{k=0}^{n-1} (\rho_{-k})^\beta \right) \le c_\beta r_\beta^n, \quad x \in \mathcal{S}, \ n \in \mathbb{N}, \tag{2.12}$$

where $r_\beta = r_{H_\beta}$. Therefore, the following Condition B is satisfied under Assumption 1.5. With future applications in mind, we make the formulation suitable for non-Markovian ergodic environments. Let

$$\mathcal{F}_0 = \sigma(\omega_n : n > 0) \tag{2.13}$$

be the $\sigma-$algebra generated by the "past" of the sequence $\{\omega_{-n}\}$.

Condition B. $\{\omega_{-n}\}$ *is a stationary and ergodic sequence such that*

(B1) Ellipticity condition: $P(\epsilon < \omega_0 < 1 - \epsilon) = 1$ for some $\epsilon \in (0, 1/2)$.

(B2) For any $\beta > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \log E_P \left(\prod_{k=0}^{n-1} \rho_{-k}^\beta \,\middle|\, \mathcal{F}_0 \right) = \Lambda(\beta), \ a.s., \tag{2.14}$$

with uniform (in ω) rate of convergence, with $\Lambda(\beta)$ as in (1.4)*. Further, there exists a unique $\kappa > 0$ such that $\Lambda(\kappa) = 0$, and $\Lambda(\beta)(\beta - \kappa) \ge 0$ for all $\beta > 0$.*

The last statement follows since $\Lambda(\beta)$ is a convex function of β in $[0, \infty)$, taking both negative and positive values by Assumption *(A3)*, with $\Lambda(0) = 0$.

We conclude this subsection with the proof of (1.13). It follows from (1.3), (1.1) and (2.12) that $\mathrm{v}_P = 0$ for $\kappa \le 1$. Assume that $\kappa > 1$ and consider the following decomposition for the hitting time τ_1 defined in (1.2)):

$$\tau_1 = \mathbf{1}_{\{X_1 = 1\}} + \mathbf{1}_{\{X_1 = -1\}}(1 + \tau_0'' + \tau_1'),$$

where $1+\tau_0''$ is the first hitting time of 0 after time 1, and $1+\tau_0''+\tau_1'$ is the first hitting time of 1 after time $1+\tau_0''$. Taking expectations in both sides of the equation (first for a fixed environment and then integrating over the set of environments) gives

$$\mathbb{E}(\tau_1|x_0=x)=1+\rho(x)\left(1+\mathbb{E}(\tau_0''|x_0=x)\right).$$

Since $\mathbb{E}(\tau_0''|x_0=x)=\mathbb{E}(\tau_1|x_1=x)=\int_{\mathcal{S}}\mathbb{E}(\tau_1|x_0=y)H(x,dy)$, we obtain that the function $\xi(x):=\mathbb{E}(\tau_1|x_1=x)/\rho(x)$ solves equation (1.13). Let $H_1:f(x)\to\int_{\mathcal{S}}H(x,dy)\rho(y)f(y)$ be the linear operator acting on the space of bounded and measurable functions on $(\mathcal{S},\mathcal{T})$. It follows from identity (2.12) and Condition B, that its spectral radius is strictly less than one, and a simple truncation argument (by (1.13), $\xi_M\le H_1\xi_M+1+1/\rho$, where $\xi_M(x):=\mathbb{E}\left(\min\{\tau_1,M\}|x_1=x\right)/\rho(x)$ for a constant $M>0$) shows that $\xi(x)$ is a bounded function of x, yielding that $\mathbb{E}(\tau_1)=E_P\big(\rho(x_0)\xi(x_0)\big)$. This implies (1.13) by (1.3) (Lemmas 2.1.11 and 2.1.17 in [25]).

2.2 The branching model and its regeneration structure

We consider here a branching process $\{Z_n\}$ in random environment with immigration closely related to the RWRE (see e.g., [1, 14, 25]). The random variables T_n are associated by (2.15) to the partial sums of the branching process Z_n. This leads us naturally to the variables $\overline{W}_n$, defined in (2.22), which are random partial sums of Z_n. The aim in introducing the branching process is to transform the limit problem of T_n into a limit problem for the partial sums of the sequence $\overline{W}_n$, which turns out to be a stationary and one-dependent sequence in a stable domain of attraction.

Let

$$U_i^n=\#\{k<T_n:X_k=i,\ X_{k+1}=i-1\},\quad i,n\in\mathbb{Z},$$

the number of moves to the left from site i up to time T_n. Then

$$T_n=n+2\sum_{i=-\infty}^{n}U_i^n. \tag{2.15}$$

When $U_n^n=0,U_{n-1}^n,\dots,U_{n-i+1}^n$ and $\omega_n,\omega_{n-1}\dots,\omega_{n-i}$ are given, U_{n-i}^n is the sum of U_{n-i+1}^n+1 i.i.d. geometric random variables that take the value k with probability $\omega_{n-i}(1-\omega_{n-i})^k$, $k=0,1,\dots$ Assuming that the RWRE is transient to the right we have:

$$\sum_{i\le 0}U_i^n\le\ \text{total time spent by }\{X_t\}\text{ in }(-\infty;0]<\infty\text{ a.s.} \tag{2.16}$$

Therefore, in order to prove the limit laws for T_n it is sufficient to prove the corresponding result for the sums $\sum_{i=1}^n U_i^n$. These sums have the same distribution as

$$\sum_{k=0}^{n-1}Z_k, \tag{2.17}$$

where $Z_0=0,Z_1,Z_2,\dots$ forms a branching process in random environment with one immigrant at each unit of time.

Without loss of generality, we shall extend the underlying sample space $\left(\Omega \times \mathbb{Z}^{\mathbb{N}}\right)$ to $(\Omega \times \Upsilon)$, where Υ is large enough to fit not only the random walk but also the branching process, and assume that P_ω (and hence $\mathbb{P}$) is suitably extended.

Thus, when ω and $Z_0, \ldots, Z_n$ are given, Z_{n+1} is the sum of $Z_n + 1$ independent variables $V_{n,0}, V_{n,1}, \ldots, V_{n,Z_n}$ each having the geometric distribution

$$P_\omega\{V_{n,j} = k\} = \omega_{-n}(1 - \omega_{-n})^k, \;\; k = 0, 1, 2, \ldots \tag{2.18}$$

Extending (2.13), let for $n \in \mathbb{N}$,

$$\mathcal{F}_n = \sigma\left(Z_0, Z_1, Z_2, \ldots, Z_{n-1}, Z_n\right) \vee \sigma\left(\omega_j : j > -n\right), \tag{2.19}$$

that is, the σ-algebra generated by the branching process $\{Z_i\}_{i=0}^n$ and the environment $\{\omega_i\}_{i=-n+1}^\infty$ before time n.

As in [14], the random variables

$$\nu_0 = 0, \quad \nu_n = \min\{k > \nu_n - 1 : Z_k = 0\}$$

are the successive stopping times at which the population becomes extinct, and the variables

$$W_n = \sum_{k=\nu_{n-1}}^{\nu_n - 1} Z_k$$

measure the total number of individuals born between two such extinction times.

Recall the definition of the $\sigma-$algebra $\mathcal{F}_0$ given in (2.13). The proof of the following proposition, which is a modification of Lemma 2 in [14] adapted to non-i.i.d. environments, is included in Appendix A.

Proposition 2.20. *Assume that Condition B holds. Then, there exist $C_1,\ C_2 > 0$ such that $P-$ a.s., $\mathbb{P}(\nu_1 > n|\mathcal{F}_0) \le C_1 e^{-C_2 n}$, for any $n > 0$.*

The following corollary is immediate since C_1, C_2 above are deterministic.

Corollary 2.21. *Assume that Condition B holds. Then, there exist $C_1,\ C_2 > 0$ such that $P-$ a.s., $\mathbb{P}(\nu_{j+1} - \nu_j > n|\mathcal{F}_{\nu_j}) \le C_1 e^{-C_2 n}$, for any $j \ge 0$ and $n > 0$.*

Let $\{N_k\}_{k=0}^\infty$ be the sequence of successive regeneration times for the chain (x_n) defined in Section 2.1, let $\bar{\nu}_0 = 0$, and for $n \ge 0$ define the stopping times:

$$\bar{\nu}_{n+1} = \inf\{k > \bar{\nu}_n : k = \nu_i = N_j \text{ for some } i, j > 0\},$$

$$\overline{W}_{n+1} = \sum_{k=\bar{\nu}_n}^{\bar{\nu}_{n+1}-1} Z_k. \tag{2.22}$$

By construction of the random times N_n, the segments of the environment between $\bar{\nu}_n$ and $\bar{\nu}_{n+1} - 1$ are one-dependent (see (2.4) and the subsequent summary), and hence the variables $\{\overline{W}_n\}_{n\ge1}$ form a one-dependent sequence, which is even independent if either (1.6) or (1.7) with $m = 1$ hold.

Lemma 2.23. *Let Assumption 1.5 hold. Then,*

(a) The distribution of $\bar{\nu}_1$, conditioned on the "past" has exponential tails: there exist K_1, $K_2 > 0$ such that $P-a.s.$,

$$\mathbb{P}(\bar{\nu}_1 > n|\mathcal{F}_0) \leq K_1 e^{-K_2 n}, \quad \forall n > 0, \tag{2.24}$$

and, more generally,

$$\mathbb{P}(\bar{\nu}_{j+1} - \bar{\nu}_j > n|\mathcal{F}_{\bar{\nu}_j}) \leq K_1 e^{-K_2 n} \tag{2.25}$$

for any $j \geq 0$.

(b) The law of large numbers holds for $\bar{\nu}_n$: $\mathbb{P}\left(\lim_{n\to\infty} \frac{\bar{\nu}_n}{n} = \mu\right) = 1$, where $\mu = \mathbb{E}(\bar{\nu}_2 - \bar{\nu}_1) > 0$.

(c) The central limit theorem holds for $\bar{\nu}_n$: there exists a constant $b > 0$ such that the law of $(\bar{\nu}_n - n\mu)/\sqrt{n}$ converges to $\mathcal{L}_{2,b}$.

Proof.

(a) Clearly, it is sufficient to prove (2.24), since the constant K_1 and K_2 are deterministic. Let $F_1 = \{Z_1 = 0\}$, and for $2 \leq j \leq l$, where l is defined in (2.5),

$$F_j = \{Z_1 = Z_2 = \ldots = Z_{j-1} = 1, Z_j = 0\},$$

and

$$\mathcal{S}_j = \{x \in \mathcal{S} : P_x(N_1 = j) > \delta/l\}.$$

Then $\bigcup_{j=1}^l \mathcal{S}_j = \mathcal{S}$, and we have for $x \in \mathcal{S}_j$:

$$\begin{aligned}\mathbb{P}\big(\nu_1 = N_1 \leq l|x_0 = x\big) &\geq \mathbb{P}\big(F_j \cap \{N_1 = j\}|x_0 = x\big) = \\ &= P_x(N_1 = j)\mathbb{P}\big(F_j|x_0 = x, N_1 = j\big) \geq \frac{\delta}{l}\mathbb{P}\big(F_j|x_0 = x, N_1 = j\big).\end{aligned}$$

Using the ellipticity condition *(A2)*, we obtain that $P-$ a.s., $P_\omega(F_1) = \omega_0 \geq \epsilon$, and for $2 \leq j \leq l$,

$$P_\omega(F_j) = \omega_0(1-\omega_0)\prod_{k=1}^{j-2}\big(2\omega_{-k}^2(1-\omega_{-k})\big)\omega_{-j+1}^2 \geq 2^{j-2}\epsilon^{2j-1}(1-\epsilon)^{j-1} \geq \epsilon^{2l}(1-\epsilon)^{l-1}.$$

implying that $\mathbb{P}\big(\nu_1 = N_1 \leq l|x_0 = x\big) \geq \delta/l \cdot \epsilon^{2l}(1-\epsilon)^{l-1} > 0$ for $P-$almost every $x \in \mathcal{S}$. Thus, in view of Corollary 2.21, $\bar{\nu}_1$ is stochastically dominated by a sum of a geometric random number of i.i.d. variables with exponential tails, yielding (2.24). We note in passing that, in view of the uniform bounds in the proof above, the same argument yields uniform exponential tails for the distribution of $\bar{\nu}_{i+1} - \bar{\nu}_i$ conditioned on $\sigma\{\omega_j, j > -\bar{\nu}_i\}$.

(b) Follows from (2.24) and the ergodic theorem, since $\bar{\nu}_{n+1} - \bar{\nu}_n$, $n \geq 1$, are one-dependent identically distributed variables.

(c) Follows e.g. from the CLT for stationary and uniformly mixing sequences [9, p. 427]. □

Recall the function $R(\omega)$ defined in (1.1). We shall prove in Subsection 2.3 that under Assumption 1.5 the following condition holds for some $\kappa > 0$.

Condition C_κ. *There exists a strictly positive random variable function $K(\omega)$ such that for some positive constants K_3, K_4, t_c the following hold $P-a.s.$:*

$$t^\kappa P(R > t|\mathcal{F}_0) > K_3 \ \ \forall t > t_c \quad \textit{and} \quad t^\kappa P(R > t|\mathcal{F}_0) < K_4 \ \ \forall t > 0, \tag{2.26}$$

$$\lim_{t\to\infty} t^\kappa P(R > t|\mathcal{F}_0) = K(\omega). \tag{2.27}$$

It follows from (2.26) and (1.3) that the case $\kappa \le 1$ corresponds to zero speed, and the case $\kappa > 1$ to a positive speed. Note that if Condition $\mathrm{C}_{\bar\kappa}$ and Condition B hold simultaneously, then $\bar\kappa = \kappa$.

For $n \ge 1$ let

$$\widetilde{W}_n = \sum_{j=1}^n \overline{W}_j.$$

The next proposition is an analogue of [14, Lemma 6] for non-i.i.d environments and is applicable for non-Markov environments too. Once Conditions B and C_κ are assumed, the proof of the tail estimates for W_1 in the i.i.d case [14] will work for the partial sums of the variables $\overline{W}_n$ as well, and yield (2.30) and the uniform estimate (2.29). The verification of this claim is not difficult but not short, so we defer it to Appendix B.

Proposition 2.28. *Assume Conditions B and C_κ. Then, for any $n \ge 1$ there exist constants $t_n, L_n, J_n > 0$ and a strictly positive random variable $\widetilde{K}_n(\omega)$ such that the following hold $P-a.s.$:*

$$J_n < t^\kappa \mathbb{P}\left(\widetilde{W}_n > t|\mathcal{F}_0\right), \ \forall t > t_n \quad \textit{and} \quad t^\kappa \mathbb{P}\left(\widetilde{W}_n > t|\mathcal{F}_0\right) < L_n, \ \forall t > 0, \tag{2.29}$$

and

$$\lim_{t\to\infty} t^\kappa \mathbb{P}\left(\widetilde{W}_n > t|\mathcal{F}_0\right) = \widetilde{K}_n(\omega). \tag{2.30}$$

Remark 2.31.

(i) The proof in [14] works nearly verbatim with Conditions B and C_κ compensating for the lack of independence of ω. Nevertheless, since the proof is rather long and technical, its detailed modification is included in Appendix B.

(ii) The proposition remains valid with the random variables $\widetilde{W}_n$ replaced by the variables $\widehat{W}_n = \sum_{j=1}^n W_n$. The proof is essentially the same, the only (obvious) difference being that Proposition 2.20 can be applied directly instead of (2.24).

(iii) Just as with Corollary 2.21 and Lemma 2.23 (i), Proposition 2.28 implies the corresponding uniform estimates for the tails $\mathbb{P}\left(\widetilde{W}_{m+n} - \widetilde{W}_m > t|\mathcal{F}_{\bar\nu_m}\right)$ as well, for every $m \ge 1$.

By the bounded convergence theorem, (2.29) and (2.30) yield

$$\lim_{t\to\infty} t^\kappa \mathbb{P}\big(\widetilde{W}_n > t\big) = E_P\big(\widetilde{K}_n\big) \in (0,\infty). \tag{2.32}$$

Note that if either (1.6) or (1.7) holds with $m = 1$, the random variables $\overline{W}_n$ are independent, and the limit laws for their partial sums follow from the standard i.i.d. limit laws [10, 19]. More generally, we have:

Proposition 2.33. *Let Assumption 1.5 hold.*

(a) Assume that $\kappa \neq 1$. Let $B_n = n^{1/\kappa}$ if $\kappa \in (0,2)$, $B_n = (n \log n)^{1/2}$ if $\kappa = 2$, and $A_n = 0$ if $\kappa \in (0,1)$, $A_n = n\mathbb{E}(\overline{W}_2)$ if $\kappa \in (1,2]$. Then, $\Big(\widetilde{W}_n - A_n\Big)/B_n$ converges in distribution to a stable law of the form (1.9)*.*

(b) Assume that $\kappa = 1$. Then, there exist a sequence $\widetilde{D}(n) \sim \log n$ and a positive constant $\tilde{c}_0$ such that the law of $\frac{1}{n}\Big(\widetilde{W}_n - \tilde{c}_0 n\widetilde{D}(n)\Big)$ converges to a stable law of the form (1.9)*.*

Proof. The random variables $\overline{W}_n$ are identically distributed and one-dependent for $n \geq 2$ (see the summary after (2.4), and note that we start from $n = 2$ because the slightly different law of $\overline{W}_1$). Clearly, it is sufficient to show that the appropriately normalized and centered sums $S_n = \sum_{j=2}^n \overline{W}_j$ converge to a stable law of the form (1.9). For $\kappa < 2$, apply [15, Corollary 5.7], noting that the uniform estimates of Proposition 2.28 imply that

$$\forall\, \varepsilon > 0,\ \ \forall\, j \geq 3, \geq 2 \qquad n\mathbb{P}\left(\overline{W}_2 \geq \varepsilon n^{1/\kappa}, \overline{W}_j \geq \varepsilon n^{1/\kappa}\right) \ \to_{n\to\infty} 0,$$

which is the tail condition needed to apply Corollary 5.7 of Kobus [15].

In the case $\kappa = 2$, we note first that $\overline{W}_2$ and $\overline{W}_2 + \overline{W}_3$ both belong by Proposition 2.28 to the domain of attraction of a normal distribution. We seek to apply the limit theorem in [21, p. 328], for which we need to check that $S_2 = \overline{W}_2$ and $S_3 = \overline{W}_2 + \overline{W}_3$ have different parameters $b_i = \lim_{n\to\infty} t^\kappa \mathbb{P}\left(S_i > t\right), i = 2, 3$. But,

$$\begin{aligned} b_3 &= \lim_{t\to\infty} t^\kappa \mathbb{P}\left(\overline{W}_2 + \overline{W}_3 > t\right) \geq \lim_{t\to\infty} t^\kappa \mathbb{P}\left(\overline{W}_2 < t, \overline{W}_3 > t\right) + \lim_{t\to\infty} t^\kappa \mathbb{P}\left(\overline{W}_3 < t, \overline{W}_2 > t\right) \\ &= \lim_{t\to\infty} t^\kappa \mathbb{P}\left(\overline{W}_3 > t | \overline{W}_2 < t\right) \mathbb{P}\left(\overline{W}_2 < t\right) + \lim_{t\to\infty} t^\kappa \mathbb{P}\left(\overline{W}_2 > t\right) \mathbb{P}\left(\overline{W}_3 < t | \overline{W}_2 > t\right) \\ &\geq J_1 + b_2 > b_2, \end{aligned} \tag{2.34}$$

where J_1 is the constant appearing in (2.29), and we used the uniform exponential estimates of Proposition 2.28 and the fact that $\mathbb{P}\left(\overline{W}_3 < t | \overline{W}_2 > t\right) \to_{t\to\infty} 1$ which is also implied by these estimates, as can be seen by conditioning on the environment to the right of $-\bar{\nu}_2$. Here and in the remainder of the proof, any reference to Proposition 2.28 actually includes Remark 2.31 *(iii)*. We have

$$\lim_{t\to\infty} \mathbb{P}\left(\overline{W}_3 < t | \overline{W}_2 > t\right) = \lim_{t\to\infty} \mathbb{E}\left(\mathbb{P}\left(\overline{W}_3 < t | \mathcal{F}_{\bar{\nu}_2}\right) | \overline{W}_2 > t\right). \tag{2.35}$$

By Proposition 2.28,

$$\mathbb{P}\left(\overline{W}_3 < t | \mathcal{F}_{\bar{\nu}_2}\right) \geq 1 - L_1 t^{-\kappa}, \quad P-\text{a.s.},$$

implying that the limit in (2.35) exists and is equal to 1. Therefore, by (2.34) and since we know a-priori from (2.30) that $b_3 = \lim_{t\to\infty} t^\kappa \mathbb{P}(\overline{W}_2 + \overline{W}_3 > t)$ is well-defined, the following limit exists and can be bounded below by using (2.29):

$$\lim_{n\to\infty} t^\kappa \mathbb{P}\left(\overline{W}_3 > t | \overline{W}_2 < t\right) = \lim_{t\to\infty} t^\kappa \mathbb{E}\left(\mathbb{P}\left(\overline{W}_3 > t|\mathcal{F}_{\bar{\nu}_2}\right) |\overline{W}_2 < t\right) \geq J_1.$$

This completes the proof of the proposition. □

Completion of the proof of Proposition 1.11. The limit laws for T_n announced in Proposition 1.11 are obtained from stable laws for partial sums of $\overline{W}_n$ in the same way as in [14], by a standard argument using Lemma 2.23. To illustrate the argument we consider here the case $\kappa = 2$, omitting the proof for $\kappa \in (0,2)$. Let $\zeta(n) = \max\{i : \bar{\nu}_i < n\}$ and $\varsigma(n) = [n/\mu - C\sqrt{n}]$ for a constant $C > 0$. Using part *(c)* of Lemma 2.23, we obtain, with $\mu = \mathbb{E}(\bar{\nu}_2 - \bar{\nu}_1)$,

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{P}\left(\zeta(n) \geq n/\mu - C\sqrt{n}\right) &\geq \lim_{n\to\infty} \mathbb{P}\left(\bar{\nu}_{\varsigma(n)} \leq n\right) \\ &= \lim_{n\to\infty} \mathbb{P}\left(\frac{\bar{\nu}_{\varsigma(n)} - \varsigma(n)\mu}{\sigma\sqrt{\varsigma(n)}} \leq \frac{n - \varsigma(n)\mu}{\sigma\sqrt{\varsigma(n)}}\right) = \mathcal{L}_{2,\frac{\sigma}{\sqrt{2}}}\left(C\mu^{3/2}\right).\end{aligned}$$

Hence, for all $\epsilon > 0$ and some $C = C(\varepsilon) > 0$ and all $n > N_2(\varepsilon)$, $\mathbb{P}\left(\zeta(n) \leq n/\mu - C\sqrt{n}\right) \leq \varepsilon$. It follows, letting $a = \mathbb{E}(\overline{W}_2)$, that for any n large enough,

$$\begin{aligned}&\mathbb{P}\left(\frac{\sum_{i=1}^n Z_i - na/\mu}{\sqrt{n\log n}} \leq x\right) \leq \mathbb{P}\left(\sum_{i=1}^{\zeta(n)} \overline{W}_i \leq x\sqrt{n\log n} + na/\mu\right) \\ &\leq \mathbb{P}\left(\sum_{i=1}^{[n/\mu - C\sqrt{n})]} \overline{W}_i \leq x\sqrt{n\log n} + na/\mu\right) + \varepsilon \to \mathcal{L}_{2,\tilde{b}}\left(x\sqrt{\mu}\right) + \varepsilon,\end{aligned}$$

where $\mathcal{L}_{2,\tilde{b}}$ is the limiting law for sums of $\overline{W}_n$. Similarly,

$$\begin{aligned}&\mathbb{P}\left(\frac{\sum_{i=1}^n Z_i - na/\mu}{\sqrt{n\log n}} \leq x\right) \geq \mathbb{P}\left(\sum_{k=1}^{\zeta(n)+1} \overline{W}_k \leq x\sqrt{n\log n} + na/\mu\right) \\ &\geq \mathbb{P}\left(\sum_{k<n/\mu + C\sqrt{n}} \overline{W}_k \leq x\sqrt{n\log n} + an/\mu\right) - \varepsilon \to \mathcal{L}_{2,\tilde{b}}\left(x\sqrt{\mu}\right) - \varepsilon.\end{aligned}$$

Since ε was arbitrary, Proposition 1.11 now follows from the limit laws for partial sums of Z_n by (2.15)–(2.17). Since the law defined by (1.9) has expectation zero, $\mathrm{v}_P = a/\mu = \mathbb{E}(\tau_1)$, where τ_1 is defined by (1.2). □

As shown in the Introduction this completes the proof of Theorem 1.10.

2.3 Tails of distribution of the random variable R

The aim of this subsection is to prove that Condition C_κ holds for some $\kappa > 0$. Proposition 2.38 below extends the following theorem, valid in the i.i.d. setup, to some Markov-dependent variables.

Theorem 2.36 (Kesten). *[12, Theorem 5] Let (Q_n, M_n), $n \in \mathbb{N}$, be independent copies of a $\mathbb{R}^2$-valued random vector (Q, M), satisfying the following conditions:*

(i) $P(M > 0) = 1$ *and* $P(Q > 0) = 1$.

(ii) For some $\kappa > 0$, $E(M^\kappa) = 1$, $E\left(M^\kappa \log^+ M\right) < \infty$, *and* $E(Q^\kappa) < \infty$.

(iii) The law of $\log M$ *is non-arithmetic and* $P(Q = (1-M)c) < 1$, $\forall c \in \mathbb{R}$.

Then there exists a constant $\widehat{K} > 0$ *such that*

$$\lim_{t\to\infty} t^\kappa P(\widehat{R} \geq t) = \widehat{K}, \tag{2.37}$$

where $\widehat{R} := Q_1 + M_1(Q_2 + M_2(Q_3 \ldots))$.

We have:

Proposition 2.38. *Let Assumption 1.5 hold. Then Condition C_κ is satisfied for the $\kappa > 0$ defined by* (1.4). *Further, if* (1.7) *holds, then there exists*

$$\widehat{K} := \lim_{t\to\infty} t^\kappa P_\psi(R > t) \in (0, \infty). \tag{2.39}$$

Proof. If either (1.6) or (1.7) with $m = 1$ hold, this proposition can be deduced rather directly from Kesten's theorem. It will be convenient to give a separate proof for the case where the state space $\mathcal{S}$ is finite, i.e. under assumption (1.6).

Assume first that (1.6) **holds**. Then, it is sufficient to show that

$$K_x := \lim_{t\to\infty} t^\kappa P_x(R > t) \in (0, \infty)$$

exists for all $x \in \mathcal{S}$. For $n \geq 0$, let

$$Q_n = 1 + \mathbf{1}_{\{N_{n+1} \geq N_n + 2\}} \sum_{i=N_n}^{N_{n+1}-2} \prod_{j=N_n}^{i} \rho_{-j} \quad \text{and} \quad M_n = \prod_{i=N_n}^{N_{n+1}-1} \rho_{-i}. \tag{2.40}$$

Then, $(M_n, Q_n)_{n\geq 1}$ is an i.i.d. sequence, and $R = Q_0 + M_0(Q_1 + M_1(Q_2 + \ldots))$. First, we will show that Kesten's theorem is applicable to this sequence, that is the following limit

$$\widehat{K} := \lim_{t\to\infty} t^\kappa P_x(\widehat{R} > t) \in (0, \infty) \tag{2.41}$$

exists, where

$$\widehat{R} = Q_1 + M_1(Q_2 + M_2(Q_3 \ldots)), R = Q_0 + M_0 \widehat{R}. \tag{2.42}$$

Let f_κ be a strictly positive Perron-Frobenius eigenvector of the matrix $H_\kappa(x,y) := H(x,y)\rho(y)^\kappa$. By virtue of (2.11) and Condition B, it corresponds to the eigenvalue 1. Recall now the definitions of the state x^* and the matrix Θ from (2.1). By Lemma 2.9, the Perron-Frobenius eigenvalue (the spectral radius) of the matrix $\Theta_\kappa(x,y) = \Theta(x,y)\rho(y)^\kappa$ is strictly less than one. So, the vector f_κ normalized by the condition $f_\kappa(x^*)\rho^\kappa(x^*) = 1$ is the unique positive vector in $\mathbb{R}^{|\mathcal{S}|}$ solving the equation $(I - \Theta_\kappa)f = s$, where $s(x) := H(x,x^*)$. Hence (this is a very particular case of the results of [6] and [18, Theorem 5.1])

$$f_\kappa(x) = \rho(x)^{-\kappa} E_x\left(\prod_{i=0}^{N_1-1} \rho_{-i}^\kappa\right) = \sum_{n=0}^\infty \Theta_\kappa^n s(x)\,, \tag{2.43}$$

and

$$E_{x^*}\left(\prod_{i=0}^{N_1-1} \rho_{-i}^\kappa\right) = E_P(M_1^\kappa) = 1. \tag{2.44}$$

The second equality in (2.43) follows since the chain (x_i) evolves according to the kernel Θ until N_1 (see (2.2)), while (2.44) follows from the normalization condition $f_\kappa(x^*)\rho^\kappa(x^*) = 1$.

The distribution of $\log M_1$ is non-arithmetic and $P\big(Q_1 = (1-M_1)c\big) < 1$ for any $c \in \mathbb{R}$, by the assumption *(A4)*. In order to prove (2.41), it remains to show that $E_P(Q_1^\kappa) < \infty$ and $E_P\left(M_1^\kappa \log^+ M_1\right) < \infty$. Thus, it is sufficient to prove that there exists $\beta > \kappa$ such that

$$E_x\big(Q_0^\beta\big) \text{ is a bounded function of } x. \tag{2.45}$$

Since for any $n \in \mathbb{N}$ and positive numbers $\{a_i\}_{i=1}^n$ we have

$$(a_1 + a_2 + \dots a_n)^\beta \le n^\beta(a_1^\beta + a_2^\beta + \dots a_n^\beta),$$

we obtain for any $\beta > 0$ and $x \in \mathcal{S}$:

$$\begin{aligned} E_x\big((Q_0-1)^\beta\big) &= E_x\left(\sum_{n=2}^\infty \sum_{i=1}^{n-1}\prod_{j=0}^{i-1} \rho_{-j}\mathbf{1}_{\{N_1=n\}}\right)^\beta = \sum_{n=2}^\infty E_x\left(\sum_{i=1}^{n-1}\prod_{j=0}^{i-1} \rho_{-j}\mathbf{1}_{\{N_1=n\}}\right)^\beta \\ &\le \sum_{n=2}^\infty (n-1)^\beta \sum_{i=1}^{n-1} E_x\left(\prod_{j=0}^{i-1} \rho_{-j}^\beta \mathbf{1}_{\{N_1\ge n\}}\right). \end{aligned} \tag{2.46}$$

But $E_x\left(\prod_{j=0}^{i-1}\rho_{-j}^\beta \mathbf{1}_{\{N_1\ge n\}}\right) = \rho(x)^\beta \Theta^{n-i}\Theta_\beta^{i-1}\mathbf{1}$, where $\Theta_\beta(x,y) := \Theta(x,y)\rho(y)^\beta$. Since the spectral radius of the matrices Θ_κ and Θ are strictly less than one, it follows from (2.46) that (2.45) holds for some $\beta > \kappa$. This yields (2.41).

By (2.41) and the bounded convergence theorem, and since the random variables M_0 and $\widehat{R}$ are independent under the measure P_x, the following limit exists:

$$K_x := \lim_{t\to\infty} t^\kappa P_x(M_0\widehat{R} > t) = \widehat{K} E_x(M_0^\kappa) \in (0,\infty).$$

Fix any $\alpha \in \left(\frac{\kappa}{\beta}, 1\right)$. It follows from (2.39) and (2.45) that for all $t > 1$,

$$\begin{aligned} t^\kappa P_x(R > t) &\leq t^\kappa P_x\big(Q_0 + M_0\widehat{R} > t,\ Q_0 < t^\alpha\big) + t^\kappa P_x(Q_0 \geq t^\alpha) \\ &\leq t^\kappa P_x\big(M_0\widehat{R} > t - t^\alpha\big) + \frac{t^\kappa}{t^{\alpha\beta}} E_x(Q_0^\beta), \end{aligned}$$

and

$$t^\kappa P_x(R > t) = t^\kappa P_x\big(Q_0 + M_0\widehat{R} > t\big) \geq t^\kappa P_x\big(M_0\widehat{R} > t\big).$$

We conclude, by taking the limit in the above inequalities as $t \to \infty$, that

$$\lim_{t\to\infty} t^\kappa P_x(R > t) = \lim_{t\to\infty} t^\kappa P_x(M_0\widehat{R} > t) = K_x,$$

completing the proof of the proposition in the case (1.6).

Assume now that (1.7) **holds**. First, we will prove that (2.27) and (2.39) hold for some function $K(\omega)$ and constant $\widehat{K}$. We follow Goldie's proof [11] of Kesten's Theorem 2.36. Let

$$\eta(x) := \log \rho(x),$$

$$\Pi_0 = 1,\ \Pi_n = \prod_{k=0}^{n-1} \rho_{-k}, \qquad n \geq 1,$$

$$\eta_n = \log \rho_{-n},\ V_n = \log \Pi_n \quad n \geq 0,$$

$$R = R^0 = \sum_{n=0}^{\infty} \Pi_n, \quad R_0 = 0,\ R_n = \sum_{k=0}^{n-1} \Pi_k, \quad R^n = (R - R_n)/\Pi_n, \quad n \geq 1. \tag{2.47}$$

Following Goldie [11], we write for any numbers $n \in \mathbb{N},\ t \in \mathbb{R}$ and any initial distribution λ of x_0 (the cases of interest for us will be $\lambda = \psi$ and $\lambda = \lambda_z$, the probability distribution concentrated at a point $z \in \mathcal{S}$)

$$P_\lambda(R > e^t) = \sum_{k=1}^{n} [P_\lambda(e^{V_{k-1}} R^{k-1} > e^t) - P_\lambda(e^{V_k} R^k > e^t)] + P_\lambda(e^{V_n} R^n > e^t).$$

We have, by using the identity $R^{k-1} = 1 + \rho_{-k+1} R^k$,

$$\begin{aligned} &P_\lambda(e^{V_{k-1}} R^{k-1} > e^t) - P_\lambda(e^{V_k} R^k > e^t) = \\ &\int_{\mathbb{R}} \int_{\mathcal{S}} [P(R^{k-1} > e^{t-u} | x_{k-1} = x) - P(\rho_{-k+1} R^k > e^{t-u} | x_{k-1} = x)] P_\lambda(V_{k-1} \in du, x_{k-1} \in dx) \\ &= \int_{\mathbb{R}} \int_{\mathcal{S}} [P_x(R > e^{t-u}) - P_x(R - 1 > e^{t-u})] P_\lambda(V_{k-1} \in du, x_{k-1} \in dx). \end{aligned}$$

Thus, letting $\delta_n(\lambda, t) = e^{\kappa t} P_\lambda(e^{V_n} R^n > e^t)$ and $f(x,t) = e^{\kappa t}[P_x(R > e^t) - P_x(R-1 > e^t)]$,

$$r_\lambda(t) := e^{\kappa t} P_\lambda(R > e^t) = \sum_{k=0}^{n-1} \int_{\mathbb{R}} \int_{\mathcal{S}} f(x, t-u) e^{\kappa u} P_\lambda(V_k \in du, x_k \in dx) + \delta_n(\lambda, t).$$

By Lemma 2.6 and (2.11), there exists a positive measurable function $h(x) : \mathcal{S} \to \mathbb{R}$ bounded away from zero and infinity such that:

$$h(x) = \int_{\mathcal{S}} H(x, dy) \rho^\kappa(y) h(y).$$

This implies, by [18, Theorem 5.2], that there is a probability measure π_κ invariant for the kernel $H_\kappa(x, dy) := H(x, dy)\rho^\kappa(y)$, that is, since $r_{H_\kappa} = 1$ by (1.4) and (2.11),

$$\int_{\mathcal{S}} H_\kappa(x, A) \pi_\kappa(dx) = \pi_\kappa(A), \quad \forall\ A \in \mathcal{T}. \tag{2.48}$$

The measure $\pi_h(dx) = h(x)\pi_\kappa(dx)$ is a finite invariant measure for the kernel

$$\widetilde{H}(x, dy) := \frac{1}{h(x)} H_\kappa(x, dy) h(y).$$

The measure π_κ and hence π_h are equivalent to the original stationary distribution π. Indeed, by (2.48),

$$\int_{\mathcal{S}} H_\kappa^m(x, A) \pi_\kappa(dx) = \pi_\kappa(A), \quad \forall\ A \in \mathcal{T}.$$

Hence, by (1.7) and the ellipticity condition *(A2)*, $c_r^{-1} c_\rho^{-m} \pi_\kappa(A) \le \pi(A) \le c_r c_\rho^m \pi(A)$, where the constant c_ρ is defined in (1.8).

Let $\widetilde{P}$ be the stationary law of the Markov chain $(x_k)_{k \ge 0}$ governed by the transition probability measure $\widetilde{H}(x, A)$. Then,

$$r_\lambda(t) = \sum_{k=0}^{n-1} \int_{\mathcal{S}} \left[\int_{\mathbb{R}} \int_{\mathcal{S}} f(x, t-u) \frac{\rho^\kappa(z) h(z)}{\rho^\kappa(x) h(x)} \widetilde{P}_y(V_k \in du, x_k \in dx) \right] \lambda(dz) + \delta_n(\lambda, t).$$

Since P – a.s., $\Pi_n R^n \to 0$ as n goes to infinity, $P\big(\lim_{n\to\infty} \delta_n(z,t) = 0\big) = 1$, for any fixed $t > 0$ and $z \in \mathcal{S}$, where $\delta_n(z,t)$ denotes $\delta_n(\lambda_z, t)$. Further, since the renewal measure ψ is equivalent to the invariant probability measure π_h, $\lim_{n\to\infty} \delta_n(\psi, t) = 0$, for any fixed $t > 0$. Therefore, P – a.s.,

$$r_z(t) := e^{\kappa t} P_z(R > e^t) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \int_{\mathcal{S}} f(x, t-u) \frac{\rho^\kappa(z) h(z)}{\rho^\kappa(x) h(x)} \widetilde{P}_z(V_k \in du, x_k \in dx),$$

and

$$r_\psi(t) = \int_{\mathcal{S}} r_z(t) \psi(dz) = \sum_{k=0}^{\infty} \int_{\mathcal{S}} \left[\int_{\mathbb{R}} \int_{\mathcal{S}} f(x, t-u) \frac{\rho^\kappa(z) h(z)}{\rho^\kappa(x) h(x)} \widetilde{P}_\psi(V_k \in du, x_k \in dx) \right] \psi(dz).$$

We will use the following Tauberian lemma :

Lemma 2.49. *[11, Lemma 9.3] Let R be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that for some constants $\kappa, K \in (0,\infty)$, $\int_0^t u^\kappa P(R > u)du \sim Kt$ as $t \to \infty$. Then $t^\kappa P(R > t) \sim K$.*

It follows from Lemma 2.49 that in order to prove (2.27) and (2.39), it is sufficient to show that $P-$ a.s. there exist

$$\lim_{t\to\infty} \check{r}_z(t) \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \check{r}_\psi(t) \in (0,\infty),$$

where the smoothing transform $\check{q}$ is defined, for a measurable function $q : \mathbb{R} \to \mathbb{R}$ bounded on $(-\infty, t]$ for all t, by

$$\check{q}(t) := \int_{-\infty}^t e^{-(t-u)} q(u)du.$$

Let

$$\begin{aligned} g(x,t) &:= \frac{1}{e^{\kappa\eta(x)}h(x)} \int_{-\infty}^t e^{-(t-u)} f(x,u)du \\ &= \frac{1}{e^{\kappa\eta(x)}h(x)} \int_{-\infty}^t e^{-(t-u)} e^{\kappa u}[P_x(R > e^u) - P_x(R-1 > e^u)]du \\ &= \frac{e^{-t}}{e^{\kappa\eta(x)}h(x)} \int_0^{e^t} v^\kappa [P_x(R > v) - P_x(R-1 > v)]dv. \end{aligned} \tag{2.50}$$

Then, since $\check{r}_\lambda(t) = \sum_{k=0}^\infty \widetilde{E}_\lambda(g(x_k, t - V_k))$ for either $\lambda = \psi$ or $\lambda = \lambda_z$, $z \in \mathcal{S}$, it is sufficient to show that for any $z \in \mathcal{S}$,

$$\lim_{t\to\infty} \sum_{k=0}^\infty \widetilde{E}_z(g(x_k, t - V_k)) = \lim_{t\to\infty} \sum_{k=0}^\infty \widetilde{E}_\psi(g(x_k, t - V_k)) \in (0,\infty) \tag{2.51}$$

exists. So, our goal now is to prove (2.51).

Toward this end, note first that the kernel $\widetilde{H}$ satisfies condition (1.7) and hence the chain (x_n) is ergodic under the measure $\widetilde{P}$. Further, the random walk $V_n = \sum_{j=0}^{n-1} \eta_j$ has a positive drift under the measure $\widetilde{P}_x$. Indeed, similarly to [11] and [12], we obtain for some $c > 0$ and any $\gamma > 0$,

$$\begin{aligned} \widetilde{P}_x\left(e^{V_n} \le e^{-\gamma n^{1/4}}\right) &= \frac{e^{-\kappa\eta(x)}}{h(x)} E_x\left(e^{kV_n} h(x_{n-1}); e^{V_n} \le e^{-\gamma n^{1/4}}\right) \le cE_x\left(e^{kV_n}; e^{V_n} \le e^{-\gamma n^{1/4}}\right) \\ &\le ce^{-\kappa\gamma n^{1/4}}. \end{aligned}$$

Thus, $\lim_{n\to\infty} \tilde{P}_x\left(V_n \le -\gamma n^{1/4}\right) = 0$, implying by the central limit theorem for bounded additive functionals of Doeblin recurrent Markov chains (see e.g. [18, p. 134]), $\widetilde{E}_{\pi_h}(\eta_0) > 0$.

The limits and the equality in (2.51) follow from the version of the Markov renewal theorem as given in [2, Theorem 1] (see also [4, 13]), provided that we are able to show that the following holds:

$$g(x,\cdot) \text{ is a continuous function for } \pi_h - \text{almost all } x \in \mathcal{S}, \tag{2.52}$$

and

$$\int_{\mathcal{S}} \sum_{n\in\mathbb{Z}} \sup_{n\delta\le t<(n+1)\delta} \{|g(x,t)|\}\pi_h(dx) < \infty \quad \text{for some } \delta > 0. \tag{2.53}$$

Note that Theorem 1 in [2] gives the condition for the existence of the first limit in (2.51), whereas the equality in (2.51) is a by-product of the proof of this theorem and holds by [2, (5.3) and Lemma A.5].

The assertion (2.52) follows from the continuity of $\int_0^{e^t} v^\kappa [P_x(R > v) - P_x(R-1 > v)]dv$ in t for every $x \in \mathcal{S}$. For some $M > 0$ and any $\varepsilon \in (0,1)$, we get from (2.50):

$$\begin{aligned} g(x,t) &\le Me^{-t}\int_0^{e^t} v^\kappa [P_x(R>v) - P_x(R-1>v)]dv \\ &\le Me^{-\varepsilon t}\int_0^{e^t} v^{\kappa-1+\varepsilon}[P_x(R>v)-P_x(R-1>v)]dv \le Me^{-\varepsilon t}E_x[R^{\kappa+\varepsilon}-(R-1)^{\kappa+\varepsilon}], \end{aligned}$$

where the last inequality follows from [11, Lemma 9.4]. Since for any $\gamma > 0$ and $R > 1$, $R^\gamma - (R-1)^\gamma \le \max\{1,\gamma\}R^{\max\{1,\gamma\}-1}$, we obtain by Condition B that

$$E_x[R^{\kappa+\varepsilon} - (R-1)^{\kappa+\varepsilon}] \le L,$$

for some constant $L > 0$ independent of x, yielding (2.53) and consequently (2.27) and (2.39).

We now turn to the proof of (2.26). Recall the random variables (Q_n, M_n) defined in (2.40) and $\widehat{R}$ defined in (2.42). First, we will show, using (2.46), that (2.45) holds for some $\beta > \kappa$ under assumption (1.7). Let

$$\widetilde{\Theta}_\beta(x,dy) := \Theta(x,dy)E_P(\rho_0^\beta\rho_{-1}^\beta\rho_{-2}^\beta\ldots\rho_{-m}^\beta|x_0 = x, x_m = y),$$

where the kernel $\Theta(x,dy)$ is defined by (2.1), and

$$K_\beta(x,y) := H^m(x,dy)E_P(\rho_0^\beta\rho_{-1}^\beta\rho_{-2}^\beta\ldots\rho_{-m}^\beta|x_0 = x, x_m = y).$$

Then, $\widetilde{\Theta}_\beta\mathbf{1} = E_x\left(\prod_{j=0}^m \rho_{-j}^\beta \mathbf{1}_{\{N_1\ge m+1\}}\right)$ and $K_\beta\mathbf{1}(x) = E_x\left(\prod_{j=0}^m \rho_{-j}^\beta\right) = \rho(x)^\beta H_\beta^m\mathbf{1}(x)$, where $H_\beta(x,dy) = H(x,dy)\rho(y)^\beta$. By Lemma 2.6 and Condition B, the spectral radius of H_κ and hence K_κ is equal to 1. On the other hand, by Assumptions *(A1)* and *(A2)*, K and $\widetilde{\Theta}$ satisfy (2.7) and (2.8) respectively with some $c \ge 1$ and $q(x) \in (0, c^{-1})$. Thus, by Lemma 2.6, the spectral radius of $\widetilde{\Theta}_\kappa$ is strictly less than one. Since $r_{\widetilde{\Theta}_\beta}$ is a continuous function of β, we have for some $\beta > \kappa$:

$$r_{\widetilde{\Theta}_\beta} < 1. \tag{2.54}$$

Denote $\hat{l} = \min\{m[l/m], 1\}$ for $l \in \mathbb{N}$. Using the ellipticity condition *(A2)*, we obtain from (2.54) that for any $l \in \mathbb{N}, k \ge l$ and for suitable constants $A_\beta > 0, \widetilde{\Lambda}_\beta < 0$:

$$E_x\left(\prod_{j=0}^{l-1}\rho_{-j}^\beta\mathbf{1}_{\{N_1\ge k\}}\right) \le c_\rho^{m-1}E_x\left(\prod_{j=0}^{\hat{l}-1}\rho_{-j}^\beta\mathbf{1}_{\{N_1\ge\hat{k}\}}\right) \le c_\rho^m\widetilde{\Theta}_\beta^{\hat{l}-1}\Theta^{\hat{k}-\hat{l}}\mathbf{1} \le A_\beta e^{k\widetilde{\Lambda}_\beta}, \tag{2.55}$$

where c_ρ is defined in (1.8). This yields (2.45) by virtue of (2.46).

The end of the proof is similar to that for the case where (1.6) holds. Fix some $\beta > \kappa$ which satisfies (2.54) and $\alpha \in \big(\frac{\kappa}{\beta}, 1\big)$. By (2.46) and the Chebyshev inequality, $\lim_{t\to\infty} t^\kappa P_x(Q_0 \geq t^\alpha) = 0$ uniformly in x. Let

$$M_{0,1} = \prod_{i=0}^{N_1-m} \rho_{-i} \quad \text{and} \quad M_{0,2} = \mathbf{1}_{\{m=1\}} + \mathbf{1}_{\{m>1\}} \cdot \prod_{i=N_1-m+1}^{N_1-1} \rho_{-i}.$$

Then, $M_0 = M_{0,1} \cdot M_{0,2}$ and

$$c_\rho^{-m} M_0 \leq M_{0,1} \leq c_\rho^m M_0,$$

where c_ρ is defined in (1.8). The random variables $M_{0,1}$ and $\widehat{R}$ are independent under the measure P_x because only $m-1$ last variables in the block D_0 defined in (2.4) are dependent on x_{N_1}. It follows from (2.39) that for some $L > 0$ and all $t > 1$,

$$\begin{aligned}
&t^\kappa P_x(R > t) = t^\kappa P_x\big(Q_0 + M_0 \widehat{R} > t,\ Q_0 < t^\alpha\big) + t^\kappa P_x(Q_0 \geq t^\alpha) \\
&\leq t^\kappa P_x\big(M_0 \widehat{R} > t - t^\alpha\big) + \frac{t^\kappa}{t^{\alpha\beta}} E_x(Q_0^\beta) \leq t^\kappa P_x\big(M_{0,1} \cdot c_\rho^m \cdot \widehat{R} > t - t^\alpha\big) + E_x(Q_0^\beta) \\
&\leq L \left(\frac{t}{t - t^\alpha}\right)^\kappa E_x(M_{0,1}^\kappa) + E_x(Q_0^\beta),
\end{aligned}$$

yielding the upper bound in (2.26). To get the lower bound, write for $\zeta > 0$,

$$\begin{aligned}
&t^\kappa P_x(R > t) = t^\kappa P_x\big(Q_0 + M_0 \widehat{R} > t\big) \geq t^\kappa P_x\big(M_0 \widehat{R} > t\big) \\
&\geq t^\kappa P_x\big(M_{0,1} \cdot c_\rho^{-m} \cdot \widehat{R} > t, M_{0,1} \geq \zeta\big) \geq t^\kappa P_x\big(\zeta \cdot c_\rho^{-m} \cdot \widehat{R} > t\big) \cdot P_x(M_{0,1} \geq \zeta) \\
&\geq J \zeta^\kappa \cdot P_x(M_{0,1} \geq \zeta),
\end{aligned}$$

for some constant $J > 0$ and all t large enough. To complete the proof it remains to show that for some $\zeta > 0$ there exists a number $\eta > 0$ such that

$$P_x\big(M_{0,1} \geq \zeta\big) > \eta \quad \text{a.s.}$$

Toward this end observe that due to the ellipticity condition *(A2)*, with $l \in \mathbb{N}$ defined in (2.5) and c_ρ defined in (1.8),

$$P_x\left(M_{0,1} \geq c_\rho^{-(l-m)}\right) \geq P_x\left(M_{0,1} \geq \min_{m \leq i \leq l} c_\rho^{-(i-m)}; N_1 \leq l\right) = P_x\big(N_1 \leq l\big) \geq \delta,$$

where $\delta > 0$ is defined in (2.5). □

It should be mentioned that essentially the same proof leads to similar tail estimates for random variables of the form $R = \sum_{n=0}^\infty Q_n \prod_{j=0}^{n-1} M_j$ with a more general type of Markov-dependent coefficients (Q_n, M_n) (e.g. Q_n need not be a constant and M_n need not be a.s. positive). Although this result is apparently interesting in its own right, we restrict ourselves to the particular case arising in the present context.

3 Summary

We have dealt with the random walk $(X_n)_{n\geq 0}$ in a random environment $\omega \in [0,1]^{\mathbb{Z}}$, associating with it an auxiliary Galton–Watson process $(Z_k)_{k\geq 0}$ with one immigrant at each instant and random branching mechanism $\mathrm{Geom}(\omega_{-k})$.

Without stating it explicitly the following has in fact been proved.

Theorem 3.1. *Assume the environment ω satisfies Conditions B and C_κ (for the $\kappa > 0$ involved in Condition B) introduced in Section 2. Furthermore, assume the existence of an increasing sequence of stopping times η_m, with $\eta_0 = 0$, with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ (defined in* (2.13) *and* (2.19)*) for which*

i) the LLN and CLT hold: there exist $\mu > 0$ and $\sigma \in \mathbb{R}$ such that

$$\frac{\eta_m}{m} \longrightarrow \mu \quad \text{a.s.} \qquad \text{and} \qquad \frac{\eta_m - m\mu}{\sqrt{m}} \quad \xrightarrow{\mathcal{D}} \quad N(0,\sigma^2)$$

ii) for some $b > 0$, $$\frac{1}{B_m}\left(\sum_{i=1}^{\eta_m} Z_i \; - \; A_m\right) \quad \xrightarrow{\mathcal{D}} \quad \mathcal{L}_{\kappa,b} \qquad \text{(defined in (1.9))}$$

$$\text{where } A_m \begin{cases} = 0 & \kappa \in (0,1) \\ \sim c_1 m \log m & \kappa = 1 \\ = c_2 m & \kappa \in (1,2] \end{cases} \qquad \text{and} \qquad B_m = \begin{cases} m^{\frac{1}{\kappa}} & \kappa \in (0,2) \\ (m\log m)^{\frac{1}{2}} & \kappa = 2 \end{cases}$$

for suitable positive constants c_1, c_2.

Then the random walk X_n satisfies a stable limit law in the sense that the conclusion (*i*)–(*iv*) *of Theorem 1.10 hold.*

In the Markov setup of this paper, and under Assumption 1.5, we have shown (see Lemma 2.23 and Proposition 2.33) that the environment ω indeed satisfies the conditions of Theorem 3.1 (with respect to the stopping times $\eta_n = \bar{\nu}_n$), thus obtaining the stable limit laws in this case.

Appendix

Recall $\mathcal{F}_0 = \sigma(\omega_k : k > 0)$. For brevity, we denote the conditional probabilities $P(\cdot|\mathcal{F}_0)$ and $\mathbb{P}(\cdot|\mathcal{F}_0)$ by P_+ and $\mathbb{P}_+ = P_+ \otimes P_\omega$ respectively. We usually do not indicate the argument ω of these functions meaning that the inequalities below hold $P-$ a.s. We denote by θ the shift on Ω, given by $(\theta\omega)_i = \omega_{i+1}$. For an event A, the notation $I(A)$ is used to denote the indicator function of A.

A Proof of Proposition 2.20

The key to the proof is

Lemma A.1. *[14, (2.12)] Suppose that the environment ω is stationary and ergodic, and $a_P := E_P(\log \rho_0) < 0$. Choose any $\gamma \in (a_P, 0)$ and define*

$$\begin{aligned} U_n &= \sum_{i=0}^{n-1} \{\log \rho_{-i} - \gamma\} \quad (U_0 = 0), \\ \zeta_0 &= 0, \quad \zeta_{k+1} = \inf\{n > \zeta_k : U_n \le U_{\zeta_k}\}. \end{aligned}$$

Then there exist constants $K_5, K_6 > 0$ such that $P - a.s.$,

$$P_\omega(\nu_1 > \zeta_k) \le K_5 e^{-K_6 k}, \qquad k > 0.$$

Remark A.2. *This lemma is proved in [14] for the special case $\gamma = a_P/2$, but an inspection of the proof reveals that $a_P/2$ can be replaced by any constant between a_P and zero in the definition of the random walk U_n.*

By virtue of Lemma A.1, it is sufficient to find $\gamma <\in (a_P, 0)$ such that for some constants $b > 0$ and $K_7, K_8 > 0$

$$P_+(\zeta_k > bk) < K_7 e^{-K_8 k}, \qquad k \ge 0.$$

Let $\eta(n) = \max\{j : \zeta_j \le n\}$ and recall $c_\rho = (1-\epsilon)/\epsilon$. Since for any $n > 0$,

$$U_n \ge \sum_{j=1}^{\eta(n)} (U_{\zeta_j} - U_{\zeta_{j-1}}) \ge -\eta(n)(\gamma + \log c_\rho),$$

for any $k > 0$, the event $\{\zeta_{k+1} > n\} = \{\eta(n) \le k\}$ is included in $\{U_n \ge -k \log c_\rho - k\gamma\}$. Therefore, for any $\gamma \in (a_P, 0)$ and $b \in \mathbb{N}$ we have

$$P_+(\zeta_{k+1} > bk) \le P_+\left(\sum_{i=0}^{bk-1} \log \rho_{-i} \ge -k \log c_\rho + k(b-1)\gamma \right).$$

Let $\gamma = \frac{1}{2} \cdot \lim_{\beta \to +0} \frac{\Lambda(\beta)}{\beta}$, where $\Lambda(\beta)$ is as in (1.4), noting that since $\Lambda(\beta)$ is convex, γ is negative by Condition B and is greater than a_P by Jensen's inequality. Hence, by Chebyshev's inequality and Condition B, we obtain for any fixed $b > 0$ and $\beta > 0$ small enough,

$$\limsup_k \frac{1}{k} \log \left[P_+(\zeta_{k+1} > kb)\right] \le \beta \log c_\rho - (b-1)\gamma\beta + \frac{3b\gamma\beta}{2} = \beta \left(\log c_\rho + \gamma + \frac{b\gamma}{2} \right).$$

Taking $b > -4 \log c_\rho / \gamma$ in the last inequality gives

$$\limsup_k \frac{1}{k} \log P_+\big(\zeta_{k+1} > kb\big) < \beta(-\log c_\rho + \gamma) < 0.$$

This completes the proof of Proposition 2.20.

B Proof of Proposition 2.28

As mentioned in Remark 2.31 *(i)*, this proof will follow the one of [14, Lemma 6] very closely, at times word by word, with the necessary changes made in annealed arguments to take the dependence of the environment into account. Quenched arguments, where no changes are needed, will be skipped.

Throughout we fix a number $\tilde{n} \in \mathbb{N}$ and denote $\widetilde{W} := \widetilde{W}_{\tilde{n}} = \sum_{j=1}^{\tilde{n}} \overline{W}_j$, $\tilde{\nu} := \bar{\nu}_{\tilde{n}}$. Recall the filtration $(\mathcal{F}_n)_{n\geq 0}$ introduced in (2.13) and (2.19), and for all $A > 0$ define its stopping time $\varsigma_A = \inf\{n : Z_n > A\}$. The random variable $\widetilde{W}$ can be represented on the event $\{\varsigma_A < \tilde{\nu}\}$ in the following form:

$$\widetilde{W} = \sum_{n=0}^{\varsigma_A - 1} Z_n + S_{\varsigma_A} + \sum_{\varsigma_A \leq n < \tilde{\nu}} Y_n, \tag{B.1}$$

where

$$\begin{aligned}
Z_{n,k} &= \text{number of progeny alive at time } k \text{ of the immigrant who entered at time } n < k, \\
Y_n &= \sum_{k>n} Z_{n,k} = \#\{\text{progeny of the immigrant at time } n, \text{ not including the immigrant}\} \\
S_n &= Z_n + \text{total progeny of the } Z_n \text{ particles present at } t.
\end{aligned}$$

It will turn out that for a large A, the main contribution to $\widetilde{W}$ in (B.1) comes from the second term and $\mathbb{P}_+\big(\widetilde{W} \geq t\big) \approx \mathbb{P}_+\big(S_{\varsigma_A} \geq t, \varsigma_A < \tilde{\nu}\big)$. If an environment ω is fixed, then $S_{\varsigma_A} - Z_{\varsigma_A}$ counts the progeny of Z_{ς_A} independent particles, and thus with a large probability S_{ς_A} is not very different from $Z_{\varsigma_A}\big(1 + E_\omega(Y_{\varsigma_A})\big) = Z_{\varsigma_A} R\big(\theta^{-\varsigma_A}\omega\big)$, where the random variable R is defined by (1.1). We will obtain

$$\lim_{t\to\infty} t^\kappa \mathbb{P}_+\big(\widetilde{W} \geq t\big) = \lim_{A\to\infty} \lim_{t\to\infty} t^\kappa \mathbb{P}_+\big(S_{\varsigma_A} \geq t, \varsigma_A < \tilde{\nu}\big) = \lim_{A\to\infty} \mathbb{E}_+\left(Z_{\varsigma_A}^\kappa K(\theta^{-\varsigma_A}\omega); \varsigma_A < \tilde{\nu}\right),$$

where the random variable $K(\omega)$ is defined by (2.27).

We shall then end the proof by showing that for all t and A large enough, $\mathbb{E}_+\left(Z_{\varsigma_A}^\kappa; \varsigma_A < \tilde{\nu}\right)$ and therefore $t^\kappa \mathbb{P}_+\big(\widetilde{W} \geq t\big) \approx \mathbb{E}_+\left(Z_{\varsigma_A}^\kappa K(\theta^{-\varsigma_A}\omega); \varsigma_A < \tilde{\nu}\right)$ is uniformly bounded away from zero and infinity by constants independent of ω.

To carry out this outline, the three terms in the right-hand side of (B.1) are evaluated in the following series of lemmas, which are versions of the corresponding statements (Lemmas 2–5) in [14], and their proofs are deferred to the end of this Appendix.

We start with the following corollary to Proposition 2.20.

Lemma B.2. *Assume that Condition B is satisfied. Then,*

(a) There exist $C_3,\ C_4 > 0$ such that $P - a.s.$, $\mathbb{P}(\tilde{\nu} > n|\mathcal{F}_0) \leq C_3 e^{-C_4 n}$, for any $n > 0$.

(b) There exists a deterministic function $\eta_t > 0,\ t \geq 0$ such that $\lim_{t\to\infty} \eta_t = 0$ and $\mathbb{P}_+\big(\varsigma_A < \tilde{\nu}\big) \leq \eta_A$.

Fix now any $\delta > 0$. It follows from part *(a)* of Lemma B.2 that for any $A > 0$,

$$\mathbb{P}_+\left(\sum_{n=0}^{\min\{\varsigma_A,\tilde{\nu}\}-1} Z_n \geq \delta t\right) \leq \mathbb{P}_+(A\tilde{\nu} \geq t) \leq C_3 e^{-C_4\delta t/A} = o(t^{-\kappa}), \quad t \to \infty,$$

and thus

$$\mathbb{P}_+(\widetilde{W} \geq \delta t, \varsigma_A \geq \tilde{\nu}) \leq \mathbb{P}_+(A\tilde{\nu} \geq t) \leq C_3 e^{-C_4\delta t/A} = o(t^{-\kappa}), \quad t \to \infty, \tag{B.3}$$

$$\mathbb{P}_+\left(\sum_{n=0}^{\varsigma_A-1} Z_n \geq \delta t, \varsigma_A < \tilde{\nu}\right) \leq \mathbb{P}_+(A\tilde{\nu} \geq \delta t) \leq C_3 e^{-C_4\delta t/A} = o(t^{-\kappa}), \quad t \to \infty. \tag{B.4}$$

Lemma B.5.

(i) There exists a constant $K_9 > 0$ such that $\mathbb{P}_+(Y_0 \geq t) \leq K_9 t^{-\kappa}$ for all $t > 0$.

(ii) For all $\delta > 0$ there exists an $A_0 = A_0(\delta) < \infty$ such that

$$\mathbb{P}_+\left(\sum_{\varsigma_A \leq n < \tilde{\nu}} Y_n \geq \delta t\right) \leq \delta t^{-\kappa} \qquad \textit{for all } A \geq A_0. \tag{B.6}$$

It follows from (B.1), taking estimates (B.3), (B.4) and (B.6) into account, that for any $A > A_0(\delta)$ (where A_0 is given by (B.6)) there exists $t_A > 0$ such that

$$\mathbb{P}_+(\varsigma_A < \tilde{\nu}, S_{\varsigma_A} \geq t) \leq \mathbb{P}_+(\widetilde{W} \geq t) \leq \mathbb{P}_+(\varsigma_A < \tilde{\nu}, S_{\varsigma_A} \geq t(1-2\delta)) + 3\delta t^{-\kappa}, \tag{B.7}$$

for all $t > t_A$. Thus, $\widetilde{W}$ can be approximated by S_{ς_A}.

Recall the random variable R defined by (1.1). Note that $R(\omega) = E_\omega(Y_0)$, and, denote (as in (2.47)) $R^{\varsigma_A} = R(\theta^{-\varsigma_A}\omega)$. We have the following law of large numbers with random normalizing constant Z_{ς_A}.

Lemma B.8.

(i) There exist functions $K_{10} = K_{10}(A) > 0$ and $K_{11} = K_{11}(A) > 0$ independent of ω such that

$$K_{10}(A) < \mathbb{E}_+\left(Z_{\varsigma_A}^\kappa; \varsigma_A < \tilde{\nu}\right) \leq K_{11}(A). \tag{B.9}$$

(ii) For all $\delta > 0$ there exists an $A_1 = A_1(\delta)$ such that

$$\mathbb{P}_+\left(\left|S_{\varsigma_A} - Z_{\varsigma_A} R^{\varsigma_A}\right| \geq \delta t, \varsigma_A < \tilde{\nu}\right) \leq \delta t^{-\kappa} \mathbb{E}_+\left(Z_{\varsigma_A}^\kappa; \varsigma_A < \tilde{\nu}\right) \tag{B.10}$$

for $A \geq A_1$.

It follows from (B.7) and (B.10) that for A and t sufficiently large,

$$\begin{aligned}\mathbb{P}_+\big(\varsigma_A < \tilde{\nu}, Z_{\varsigma_A} R^{\varsigma_A} \geq (1+\delta)t\big) - \delta t^{-\kappa}\mathbb{E}_+\left(Z_{\varsigma_A}^\kappa; \varsigma_A < \tilde{\nu}\right) \leq \mathbb{P}_+\big(\widetilde{W} \geq t\big) \\ \leq \mathbb{P}_+\big(\varsigma_A < \tilde{\nu}, Z_{\varsigma_A} R^{\varsigma_A} \geq (1-3\delta)t\big) + \delta t^{-\kappa}\left(3 + \mathbb{E}_+\left(Z_{\varsigma_A}^\kappa; \varsigma_A < \tilde{\nu}\right)\right).\end{aligned} \tag{B.11}$$

For a fixed $A > 0$, we obtain from Condition C_κ and the dominated convergence theorem that

$$\lim_{t\to\infty} t^\kappa \mathbb{P}_+\big(\varsigma_A < \tilde{\nu}, Z_{\varsigma_A} R^{\varsigma_A} \geq t\big) = \lim_{t\to\infty} t^\kappa \mathbb{E}_+ \left(I(\varsigma_A < \tilde{\nu}) \cdot \mathbb{P}_+\big(Z_{\varsigma_A} R^{\varsigma_A} \geq t \big| \mathcal{F}_{\varsigma_A}\big)\right)$$
$$= \mathbb{E}_+ \left(I(\varsigma_A < \tilde{\nu}) \cdot Z^\kappa_{\varsigma_A} \cdot K\big(\theta^{-\varsigma_A}\omega\big)\right) = \mathbb{E}_+ \left(Z^\kappa_{\varsigma_A} \cdot K\big(\theta^{-\varsigma_A}\omega\big); \varsigma_A < \tilde{\nu}\right), \tag{B.12}$$

and, with constants K_3 and K_4 defined in (2.26),

$$K_3 \mathbb{E}_+ \left(Z^\kappa_{\varsigma_A}; \varsigma_A < \tilde{\nu}\right) \leq t^\kappa \mathbb{P}_+\big(\varsigma_A < \tilde{\nu}, Z_{\varsigma_A} R^{\varsigma_A} \geq t\big) \leq K_4 \mathbb{E}_+ \left(Z^\kappa_{\varsigma_A}; \varsigma_A < \tilde{\nu}\right)$$

for all t sufficiently large.

It follows from (B.11) and (B.12) that

$$\lim_{t\to\infty} t^\kappa \mathbb{P}_+\big(\widetilde{W} \geq t\big) = \lim_{A\to\infty} \mathbb{E}_+ \left(Z^\kappa_{\varsigma_A} \cdot K\big(\theta^{-\varsigma_A}\omega\big); \varsigma_A < \tilde{\nu}\right),$$

where the last limit is finite by (2.26) and (B.9). The limit in the right-hand side exists since the limit in the left-hand side does not depend of A.

Furthermore, it follows from (B.11) and (2.26) that for some $\delta_0 > 0, A_2 > 0$,

$$\begin{aligned} 0 < \left(\frac{K_3}{(1+\delta_0)^\kappa} - \delta_0\right) \cdot \mathbb{E}_+ \left(Z^\kappa_{\varsigma_A}; \varsigma_A < \tilde{\nu}\right) &\leq t^\kappa \mathbb{P}_+\big(\widetilde{W} \geq t\big) \\ &\leq \left(\frac{K_4}{(1-3\delta_0)^\kappa} + \delta_0\right) \cdot \mathbb{E}_+ \left(Z^\kappa_{\varsigma_A}; \varsigma_A < \tilde{\nu}\right) + 3\delta_0, \end{aligned}$$

for all $t > t_0$. Therefore, by (B.9),

$$0 < K_{10}(A_2) \left(\frac{K_3}{(1+\delta_0)^\kappa} - \delta_0\right) \leq t^\kappa \mathbb{P}_+\big(\widetilde{W} \geq t\big) \leq K_{11}(A_2) \left(\frac{K_4}{(1-3\delta_0)^\kappa} + \delta_0\right) + 3\delta_0,$$

completing the proof of Proposition 2.28.

Proof of Lemma B.2

(a) Follows from part *(a)* of Lemma 2.23 (which itself is a corollary to Proposition 2.20).

(b) It is enough to consider $A \in \mathbb{N}$. For any $n > 0$ we have

$$\begin{aligned} \mathbb{P}_+(\varsigma_A < \tilde{\nu}) &= \mathbb{P}_+(\varsigma_A < \tilde{\nu}, \tilde{\nu} > n) + \mathbb{P}_+(\varsigma_A < \tilde{\nu}, \tilde{\nu} \leq n) \leq \mathbb{P}_+(\tilde{\nu} > n) + \mathbb{P}_+(\varsigma_A < n) \\ &\leq C_3 e^{-C_4 n} + \mathbb{P}_+(\varsigma_A < n). \end{aligned} \tag{B.13}$$

For any $n \in \mathbb{N}$ let $b_n = (1-1/n)^{1/n}$ and define a sequence of natural numbers $\{a_{i,n}\}_{i=0}^n$ by the following rule: $a_{0,n} = 0$ and

$$a_{i+1,n} = \min\left\{ j \in \mathbb{N} : j > \max\left\{ a_{n-1,n-1}; \frac{(a_{k,n}+1)(1-\epsilon)}{(1-b_n)\epsilon}\right\}\right\}.$$

Then,

$$\mathbb{P}_+\big(Z_i > a_{i,n} | Z_j \le a_{j,n},\ j = 0, 1, \ldots, i-1\big) \le \frac{1}{a_{i,n}} \mathbb{E}_+\big(Z_i | Z_{i-1} = a_{i-1,n}\big)$$
$$= \frac{(a_{i-1,n}+1)(1-\omega_{-i+1})}{a_{i,n}\cdot\omega_{-i+1}} \le \frac{(a_{i-1,n}+1)(1-\epsilon)}{a_{i,n}\cdot\epsilon} \le 1 - b_n.$$

We conclude that

$$\mathbb{P}_+\big(Z_i \le a_{i,n} | Z_j \le a_{j,n},\ j = 0, 1, \ldots, i-1\big) \ge b_n,$$

and hence $\mathbb{P}_+\big(\varsigma_A(a_{n,n}) > n\big) \ge \mathbb{P}_+\big(Z_i \le a_{i,n},\ i = 1, 2, \ldots, n\big) \ge 1 - 1/n$. By construction, $a_{n,n}$ is a strictly increasing sequence and it follows from (B.13) that for any $A > a_{n,n}$,

$$\mathbb{P}_+\big(\varsigma_A(A) < \tilde{\nu}\big) \le \mathbb{P}_+\big(\varsigma_A(a_{n,n}) < \tilde{\nu}\big) \le C_3 e^{-C_4 n} + 1/n,$$

completing the proof. □

Proof of Lemma B.5

(i) Recall $R^n = 1 + \sum\limits_{i=1}^{\infty} \prod\limits_{j=n}^{n+i-1} \rho_{-j}$ and let $A_n = Z_{0,n} - Z_{0,n-1}\rho_{-(n-1)}$. Then, $Y_0 = \sum\limits_{n=1}^{\infty} A_n R^n$, and using the identity $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6 < 2$, we obtain from Condition C_κ that

$$\begin{aligned}\mathbb{P}_+(Y_0 \ge t) &= \mathbb{P}_+\left(\sum_{n=1}^{\infty} A_n R^n \ge 6\pi^{-2} t \sum_{n=1}^{\infty} n^{-2}\right) \le \sum_{n=1}^{\infty} \mathbb{P}_+\left(|A_n| R^n \ge \frac{t}{2n^2}\right) \\ &\le 2^\kappa t^{-\kappa} K_4 \sum_{n=1}^{\infty} n^{2\kappa} \mathbb{E}_+\big(|A_n|^\kappa\big).\end{aligned}$$

Since [14, pp. 158–159] $\mathbb{E}_+\big(|A_n|^\kappa\big) \le K_{12} E_P \left(\prod_{i=0}^{n-2} \rho_{-i}^{\kappa/2} | \mathcal{F}_0\right)$ for some constant $K_{12} > 0$, it follows from Condition B that $\mathbb{P}_+(Y_0 \ge t) \le K_9 t^{-\kappa}$, for some $K_9 > 0$.

(ii) Recall the σ-algebra $\mathcal{F}_n$ defined in (2.19). Using the first part of the proposition, we obtain:

$$\mathbb{P}_+\left(\sum_{\varsigma_A \le n < \tilde{\nu}} Y_n \ge \delta t\right) = \mathbb{P}_+\left(\sum_{n=1}^{\infty} Y_n I(\varsigma_A \le n < \tilde{\nu}) \ge 6\delta t \pi^{-2} \sum_{n=1}^{\infty} n^{-2}\right)$$
$$\le \sum_{n=1}^{\infty} \mathbb{E}_+\Big(I(\varsigma_A \le n < \tilde{\nu}) \cdot \mathbb{P}\big(Y_n \ge 1/2 \cdot \delta t n^{-2} | \mathcal{F}_n\big)\Big)$$
$$\le K_9 2^\kappa t^{-\kappa} \delta^{-\kappa} \mathbb{E}_+\left(\tilde{\nu}^{2\kappa+1}; \varsigma_A < \tilde{\nu}\right) \le K_9 2^\kappa t^{-\kappa} \delta^{-\kappa} \sqrt{\mathbb{E}_+\left(\tilde{\nu}^{4\kappa+2}\right)} \cdot \sqrt{\mathbb{P}_+(\varsigma_A < \tilde{\nu})}.$$

The claim follows now from Lemma B.2, the first square root being bounded and the second one going to zero as $A \to \infty$, both uniformly in ω. □

Proof of Lemma B.8

(i) For the lower bound:

$$\begin{aligned}\mathbb{E}_+\left(Z_{\varsigma_A}^\kappa;\varsigma_A<\tilde\nu\right) &\ge A^\kappa\mathbb{P}_+\big(\varsigma_A<\tilde\nu\big)\ge A^\kappa\mathbb{P}_+\big(Z_1=A+1\big)=A^\kappa\omega_0(1-\omega_0)^{1+A}\\ &\ge A^\kappa\epsilon^{A+2}:=K_5(A)>0.\end{aligned}$$

We now turn to the upper bound. For a fixed environment ω, we obtain by the ellipticity condition *(B1)*, that

$$\begin{aligned}E_\omega\left(Z_{\varsigma_A}^\kappa\right) &= \sum_{n\ge1}\sum_{a=0}^{A}E_\omega\left(Z_n^\kappa\mid\varsigma_A=n,Z_{n-1}=a\right)P_\omega\left(\varsigma_A=n,Z_{n-1}=a\right)\\ &\le \sup_{\omega,n\in\mathbb{N},a\le A}E_\omega\left(Z_n^\kappa\mid Z_n>A,Z_{n-1}=a\right)\le\sup_{\omega,n\in\mathbb{N},a\le A}\frac{E_\omega\left(Z_n^\kappa\mid Z_{n-1}=a\right)}{P_\omega\left(Z_n>A\mid Z_{n-1}=a\right)}\\ &\le \sup_\omega\frac{E_\omega\left(Z_1^\kappa\mid Z_0=A\right)}{P_\omega\left(Z_1>A\mid Z_0=0\right)}\le A^{\kappa+1}\epsilon^{-A-2}\sup_\omega E_\omega\left[(V_{0,0})^\kappa\right]<\infty,\end{aligned}$$

where the random variables $V_{n,j}$ are defined in (2.18). This completes the proof of part *(i)* of the Lemma.

(ii) The proof is similar to that of Lemma B.5. If $\varsigma_A<\tilde\nu$, let

$S_{\varsigma_A,j}=$ number of progeny alive at time t of the Z_{ς_A} particles present at time ς_A,

and $B_j=S_{\varsigma_A,j}-S_{\varsigma_A,j-1}\cdot\rho_{-(j-1)}$. We have $\sum_{j=\varsigma_A}^\infty S_{\varsigma_A,j}-Z_{\varsigma_A}R^{\varsigma_A}=\sum_{j=\varsigma_A}^\infty B_jR^j$, and obtain from Condition C_κ that on the set $\{\varsigma_A<\tilde\nu\}$,

$$\begin{aligned}&\mathbb{P}_+\left(\Big|\sum_{j=\varsigma_A}^\infty S_{\varsigma_A,j}-Z_{\varsigma_A}R^{\varsigma_A}\Big|\ge\delta t\Big|\mathcal{F}_{\varsigma_A}\right)\le\sum_{j=\varsigma_A}^\infty\mathbb{E}_+\left(\mathbb{P}_+\left(|B_j|R^j\ge\frac{\delta t}{2(j-\varsigma_A+1)^2}\Big|B_j,\mathcal{F}_{\varsigma_A}\right)\right)\\ &\le K_4\left(\frac{2}{\delta t}\right)^\kappa\sum_{n=0}^\infty(n+1)^2\cdot\mathbb{E}_+\left(\left|B_{\varsigma_A+n}\right|^\kappa\,\big|\mathcal{F}_{\varsigma_A}\right).\end{aligned}$$

Since [14, p. 164] $\mathbb{E}_+\left(\left|B_{\varsigma_A+n}\right|^\kappa\,\big|\mathcal{F}_{\varsigma_A}\right)\le K_{13}Z_{\varsigma_A}^\kappa E_P\left(\prod_{i=\varsigma_A}^{j-2}\rho_{-i}^{\kappa/2}\right)$, it follows from Condition C_κ that for some $K_{14}>0$,

$$\begin{aligned}&\mathbb{P}_+\left(\Big|\sum_{j=\varsigma_A}^\infty S_{\varsigma_A,j}-Z_{\varsigma_A}\widetilde S_{\varsigma_A}\Big|\ge\delta t;\varsigma_A<\tilde\nu\right)\le\left(\frac{K_{14}}{t\delta}\right)^\kappa\mathbb{E}_+\left(Z_{\varsigma_A}^{\kappa/2};\varsigma_A<\tilde\nu\right)\\ &\le\left(\frac{K_{14}}{t\delta\sqrt{A}}\right)^\kappa\mathbb{E}_+\left(Z_{\varsigma_A}^\kappa;\varsigma_A<\tilde\nu\right)\le\delta t^{-\kappa}\mathbb{E}_+\left(Z_{\varsigma_A}^\kappa;\varsigma_A<\tilde\nu\right),\end{aligned}$$

for $A\ge A_2(\delta)$. $\square$

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