Quenched, Annealed and Functional Large Deviations for One-Dimensional Random Walk in Random Environment

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Abstract: Suppose that the integers are assigned random variables $\{\omega_i\}$ (taking values in the unit interval), which serve as an environment. This environment defines a random walk $\{X_n\}$ (called a RWRE) which, when at i, moves one step to the right with probability ω_i, and one step to the left with probability $1-\omega_i$. When the $\{\omega_i\}$ sequence is i.i.d., Greven and den Hollander (1994) proved a large deviation principle for X_n/n, conditional upon the environment, with deterministic rate function. We consider in this paper large deviations, both conditioned on the environment (*quenched*) and averaged on the environment (*annealed*), for the RWRE, which forces us to consider also the ergodic environment case. The annealed rate function is the solution of a variational problem involving the quenched rate function and specific relative entropy. We also give a detailed qualitative description of the resulting rate functions. Our techniques differ from those of Greven and den Hollander, and allow us to present also a trajectorial (quenched) large deviation principle.

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1 Introduction and Statement of Results

1.1 Introduction and background

Let $\Sigma = [0,1]^{\mathbb{Z}}$, and let $\omega = (\omega_i)_{i\in\mathbb{Z}} \in \Sigma$ be a collection of random variables which serve as an environment. For each $\omega \in \Sigma$, we denote by P_ω the distribution of the nearest neighbor random walk $(X_n)_{n=0,1,2,\ldots}$ in the environment ω, which, when at location i, moves to $i+1$ with probability ω_i and to $i-1$ with probability $1-\omega_i$. In the case when ω is a realization of a stationary, ergodic sequence, X is called the random walk in random environment (RWRE). This RWRE can serve as a model for diffusion and transport phenomena, in a medium which is locally inhomogeneous, but homogeneous on large scales. In parallel to the case of classical random walks, natural questions for the RWRE arise: transience vs. recurrence, law of large numbers, limit theorems for the distribution, large deviations,.... Periodic environments strongly relate to homogeneous ones, via homogeneization techniques, though environments with more randomness produce a richer behavior for the walk. Fluctuations of the environment have a strong influence on the long time asymptotics of the walk.

Define $\rho_i = \rho_i(\omega) = (1-\omega_i)/\omega_i, i \in \mathbb{Z}$. Depending on the (ergodic) distribution η of the environment, the random walk (X_n) is either recurrent for η-a.e. ω (if $\int \log\rho_0(\omega)\eta(d\omega) = 0$), or transient for η-a.e. ω with $X_n \to +\infty$ [resp. $-\infty$] (if $\int \log\rho_0(\omega)\eta(d\omega) < 0$ [resp. > 0]), see [16, Chap. IV, Theorem 2.3 and Corollary 2.4] or [1]. Let $Z_i^- := \rho_i + \rho_i\rho_{i-1} + \rho_i\rho_{i-1}\rho_{i-2} + \ldots$, and note that $Z_i^- < \infty$, η-a.s. when the walk is transient to the right. Further, if

$$v_\eta^{-1} := \int (1+2Z_0^-)\eta(d\omega) < \infty\,, \tag{1}$$

then the random walk has the positive speed v_η, i.e. for η-a.e. ω, we have $X_n/n \to v_\eta$, P_ω-a.s., c.f. [1]. If η is a product measure, this was observed by Solomon [20], who proved that in this case, $\int(1+2Z_0^-)\eta(d\omega) < \infty$ if $\langle\rho\rangle := \int \rho_0(\omega)\eta(d\omega) < 1$ and then

$$v_\eta = \frac{1-\langle\rho\rangle}{1+\langle\rho\rangle}\,. \tag{2}$$

A transparent derivation of (1) appears below Lemma 1. We refer to the introduction sections of [11] and [3], as well as to [12] and [19], for more about the history of the model and a description of limit laws not mentioned above.

Recall that a sequence of probability measures μ_n on a topological space satisfies the Large Deviation Principle (LDP) with rate function $I(\cdot)$ if $I(\cdot)$ is non-negative, lower semicontinuous, and for any measurable set G,

$$-\inf_{x\in G^o} I(x) \le \liminf_{n\to\infty} \frac{1}{n}\log\mu_n(G) \le \limsup_{n\to\infty} \frac{1}{n}\log\mu_n(G) \le -\inf_{x\in\bar{G}} I(x)\,. \tag{3}$$

(Here, G^o denotes the interior of G and $\bar{G}$ its closure. We refer to [4] for general background and definitions concerning large deviations). For a product measure η, the study of large deviations for the law of X_n/n was initiated by Greven and den Hollander in [11], where a Large Deviation Principle for the distributions of X_n/n under P_ω was derived, for η -a.e. ω, with a deterministic rate function I_η^q. (Of course, $I_\eta^q(v_\eta) = 0$). We refer in the sequel to such statements as *quenched* statements, while statements concerning probabilities with respect to the law $P = \eta(d\omega) \otimes P_\omega$ are referred to as *annealed* results. Random environments may create some long "traps" which slow down the walk, resulting in large deviations probabilities whose rate of decay is slower than exponential. Subexponential asymptotics, both quenched and annealed, are presented in [3], [9], [18], [17]. In this paper we will focus on exponential rates of decay.

The approach of [11] to large deviation statements involves looking at the RWRE as a Markov chain in the space of environments, and the quenched LDP is obtained by an appropriate contraction. More precisely, the rate function is the solution of a variational problem and is shown to be the Legendre transform of certain Lyapunov exponents. Our goal in this paper is to suggest a different point of view for obtaining large deviation theorems, both annealed and quenched, for the general ergodic η. We do so by building on recursion ideas which can be traced back to [15], [14], and formed the key to [3], leading here to rather simple proofs of the LDP's. As an application of our methods, we show how functional LDP's can be obtained by essentially the same methods. As a by product of our method, we are able to deduce qualitative results concerning the shape of the resulting rate functions.

After the bulk of this work was completed, we received a preprint of Zerner [24], where he uses similar recursion ideas to analyze certain multi-dimensional RWRE's. Among other results, Zerner shows how to re-derive some of Greven and den Hollander's results using a hitting time decomposition similar to ours. In contrast with our results, the annealed case is not treated in [24].

1.2 Statement of main results

Turning to the description of our results, a crucial role in our approach is played by certain hitting times. Let $T_k = \inf\{n : X_n = k\}$, $k = 0, \pm 1, \pm 2, \cdots$ and

$$\begin{aligned} \tau_k &= T_k - T_{k-1} \qquad k > 0 \\ \tau_k &= T_k - T_{k+1} \qquad k < 0\,, \end{aligned}$$

with the convention that $\infty - \infty = \infty$ in this definition. It turns out that Large Deviation Principles for T_n/n are key to the LDP's for X_n/n. We introduce the functions

$$\varphi(\lambda,\omega) := E_\omega[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}]\,,\ \ f(\lambda,\omega) := \log\varphi(\lambda,\omega)\,,\ \ G(\lambda,\eta,u) := \lambda u - \int f(\lambda,\omega)\eta(d\omega)\,. \tag{4}$$

A characterization of $\varphi(\lambda,\omega)$ in terms of continued fraction expansions is provided in Section 2, Lemma 1. Define

$$I_\eta^{\tau,q}(u) = \sup_{\lambda\in\mathbb{R}} G(\lambda,\eta,u)\,. \tag{5}$$

As will be seen in Section 2, $I_\eta^{\tau,q}$ is the rate function for the quenched LDP associated with T_n/n.

Let $M_1(\Sigma)$, $M_1^s(\Sigma)$ and $M_1^e(\Sigma)$ be the spaces of probability measures, stationary probability measures, and ergodic probability measures, on Σ. Further, denote by $M_1^e(\Sigma)^+ := \{\eta \in M_1^e(\Sigma) : \int \log\rho_0(\omega)\eta(d\omega) \le 0\}$ the set of distributions for the environment making the walk recurrent or transient to the right. Let $K \subset (0,1)$ be some fixed compact subset of $(0,1)$. For any set $M \subset M_1(\Sigma)$, we denote $M^K = M \cap \{\eta : \mathrm{supp}(\eta_0) \subset K \subset (0,1)\}$. For $\eta \in M_1^e(\Sigma)^{+,K}$, define

$$I_\eta^q(v) = \begin{cases} vI_\eta^{\tau,q}\left(\frac{1}{v}\right), & 0 \le v \le 1 \\ |v|\left(I_\eta^{\tau,q}\left(\frac{1}{|v|}\right) - \int\log\rho_0(\omega)\eta(d\omega)\right), & -1 \le v \le 0\,, \end{cases} \tag{6}$$

where the value at $v = 0$ is taken as $I_\eta^q(0) = \lim_{v\to 0} vI_\eta^{\tau,q}(1/v)$. Let $\mathrm{Inv} : \Sigma \to \Sigma$ denote the map satisfying $(\mathrm{Inv}\,\omega)_i = 1 - \omega_{-i}$, and let $\eta^{\mathrm{Inv}} = \eta\circ\mathrm{Inv}^{-1}$. For $\eta \in M_1^e(\Sigma)^K \setminus M_1^e(\Sigma)^{+,K}$, note that $\eta^{\mathrm{Inv}} \in M_1^e(\Sigma)^{+,K}$ and define $I_\eta^q(v) = I_{\eta^{\mathrm{Inv}}}^q(-v)$.

Our first main result is a quenched LDP for the distribution of X_n/n. It turns out that, even if one is interested in the annealed LDP for the i.i.d. case only, one is forced to consider the quenched LDP for certain ergodic, non product measures. This motivates the following extension of the quenched LDP of [11] which is derived there in the case where η is a product measure.

Theorem 1 *Assume $\eta \in M_1^e(\Sigma)^K$. For η-a.e. ω, the distributions of X_n/n under P_ω satisfy a large deviation principle with convex, good rate function I_η^q.*

Our approach allows us to prove also an annealed LDP. Untypical environments will come into play, so this requires some extra assumptions on the distribution α of the sequence ω, allowing to compute large deviations of the environment itself. We say that $\alpha \in M_1^e(\Sigma)$ is *locally equivalent to the product of its marginals* if its restriction $\alpha^{(n)}$ to $M_1([0,1]^n)$ is equivalent to $\prod_{i=1}^n \alpha_i$ for arbitrary n, i.e. if for any measurable $A \subset [0,1]^n$, $\alpha^{(n)}(A) = 0$ if and only if $\prod \alpha_i(A) = 0$. Now, let $\theta : \Sigma \to \Sigma$ denote the shift on Σ, given by $(\theta\omega)(i) = \omega(i+1)$, and let $h(\cdot|\alpha)$ denote the specific relative entropy with respect to any $\alpha \in M_1(\Sigma)$. We say that α satisfies the process level LDP if the distributions of $R_n := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\theta^j\omega}$ under α satisfy the LDP in $M_1(\Sigma)$, equipped with the topology of weak convergence, with rate function $h(\cdot|\alpha)$. Let $\mathcal{F}_n := \sigma(\{\omega_0, \ldots, \omega_n\})$. We will use the following Assumption (A) on α:

Assumption (A):
A1: *α satisfies the process level LDP.*
A2: *α is locally equivalent to the product of its marginals and, for each $\eta \in M_1^s(\Sigma)^K$, there is a sequence $\{\eta^n\}$ of ergodic measures with $\eta^n \underset{n\to\infty}{\longrightarrow} \eta$ weakly and $h(\eta^n|\alpha) \to h(\eta|\alpha)$.*

Product measures and Markov processes with bounded transition kernels satisfy **A1**, c.f. [7], as well as **A2**, c.f. [8], Lemma 4.8. For $u \geq 1$, let

$$I_\alpha^{\tau,a}(u) = \inf_{\eta \in M_1^e(\Sigma)} \left[I_\eta^{\tau,q}(u) + h(\eta|\alpha) \right]. \tag{7}$$

Let now

$$I_\alpha^a(v) = \begin{cases} v I_\alpha^{\tau,a}\left(\frac{1}{v}\right), & 0 \leq v \leq 1 \\ |v| I_{\alpha^{\mathrm{Inv}}}^{\tau,a}\left(\frac{1}{|v|}\right), & -1 \leq v \leq 0\,. \end{cases} \tag{8}$$

The following annealed LDP can be considered as the main result of this paper.

Theorem 2 *Assume $\alpha \in M_1^e(\Sigma)^K$ satisfies Assumption (A). Then, the distributions of X_n/n under P satisfy a LDP with convex, good rate function I_α^a.*

We note that the quenched rate function I_η^q and the annealed rate function I_α^a are related by the following variational formula:

$$I_\alpha^a(v) = \inf_{\eta \in M_1^e(\Sigma)} \left[I_\eta^q(v) + |v| h(\eta|\alpha) \right]\,. \tag{9}$$

where $vh(\eta|\alpha) = \infty$ if $h(\eta|\alpha) = \infty$. In particular, we always have $I_\alpha^a \leq I_\alpha^q$. Properties of the rate function I_η^q, and I_α^a for product measures α, are studied in Section 5 and summarized in a series of figures. An interesting feature, first discovered in [11] in the quenched case with i.i.d. environment, is the occurence of *linear pieces* of the rate function, which we explain in the next subsection. We show that a similar property holds for the annealed rate function. Already at this point, the reader may have a glance at Figures 6 to 9, Section 5. We also present in Section 5 qualitative properties of the rate functions encountered in this paper, quenched and annealed. Note also that the minimizers η in (9) describe the environments favorable to large deviations of the walk. In Section 5 we show that in general, these measures are one-dimensional Gibbs measures, with a summable interaction related to the approximants of the continuous fractions φ.

We conclude this section by a functional LDP. Let $S_n(t) := n^{-1} X_{\lfloor nt \rfloor}$, $t = 0, 1/n, 2/n, \ldots, 1$, linearly interpolated elsewhere. Throughout, we use the symbol $\mathcal{L}$ to denote the class of Lipschitz functions of

Lipschitz constant bounded by 1, equipped with the supremum topology. Define the functional $I_\eta^{\mathrm{traj},q} : \mathcal{L} \to [0,\infty]$ by

$$I_\eta^{\mathrm{traj},q}(\phi) \triangleq \int_0^1 I_\eta^q(\dot{\phi}(t))dt\,.$$

Theorem 3 *Let $\eta \in M_1^e(\Sigma)^K$.*
1. $I_\eta^{\mathrm{traj},q}$ is a good rate function on $\mathcal{L}$.
2. For η -a.e. ω, the distributions of $S_n(\cdot)$ under P_ω satisfy in $\mathcal{L}$ a LDP with rate function $I_\eta^{\mathrm{traj},q}$.

The organization of the article is as follows: In the rest of this introduction, we describe our strategy for proving Theorems 1 – 3, state auxiliary LDP's for the hitting times T_n/n, and introduce some notations and conventions. In Section 2 [resp. Section 3] we provide the proofs of the quenched [resp. annealed] LDP for hitting times. Section 4 is devoted to the proof of Theorems 1 – 3. In Section 5 we describe the various rate functions in the paper classifying their shapes, and we study also the environments which lead to an (annealed) large deviation. Finally, Section 6 describes some questions and open problems.

1.3 General strategy and statements of associated hitting times LDP's

We begin with a heuristic description of our approach, followed by the statement of some crucial auxiliary LDP's for certain hitting times. Recall the hitting times $\{\tau_k\}$, $\{T_k\}$, and note that $T_n = \sum_{k=1}^n \tau_k$. Under P_ω, i.e. in the quenched setting, the hitting times $\{\tau_k\}$ are independent, although not identically distributed. Therefore,

$$E_\omega[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}] = \prod_{k=1}^n E_\omega[e^{\lambda \tau_k}\mathbf{1}_{\tau_k<\infty}\,|\tau_{k-1}<\infty] = \prod_{k=1}^n \varphi(\lambda, \theta^{k-1}\omega)\,,$$

with θ denoting the shift as before, and therefore, disregarding technical conditions, one expects by the ergodic theorem that

$$\Lambda(\lambda) := \lim_{n\to\infty} n^{-1}\log E_\omega[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}] = \int \log\varphi(\lambda,\omega)\eta(d\omega)\,,\ \eta - a.s.$$

Therefore, if $\Lambda(\lambda)$ were essentially smooth, one could expect to deduce a LDP for T_n/n by the Gärtner-Ellis theorem, c.f. [4], with (convex) rate function $I_\eta^{\tau,q}(\cdot)$. Unfortunately, the required smoothness can fail at the boundary of the domain of $\Lambda(\lambda)$, and some extra care is needed in deriving the LDP lower bound.

We note that the random variables $\{\tau_k\}$ can be heavy tailed, i.e. they may not have exponential moments, or they may possess only certain finite exponential moments. If they were i.i.d., this would imply that the corresponding rate function in the LDP is not strictly convex and possesses linear pieces. Although the $\{\tau_k\}$ sequence is not identically distributed, this heuristics suggests that the same is true of $I_\eta^{\tau,q}(\cdot)$. Indeed, we show in Section 5 that $I_\eta^{\tau,q}(\cdot)$ possesses linear pieces due to the blowup of certain exponential moments of the hitting times.

Having derived the quenched LDP for T_n/n, a simple duality argument allows one to derive the quenched LDP for X_n/n. Indeed, the event

$$\{X_n/n < x\} \text{ is comparable to } \{T_{nx} > n\} = \{T_{nx}/nx > 1/x\}\,,$$

and this will lead to (6) and (8). Of course, c.f. (6) and (8), linear pieces in the rate function for the hitting times yield linear pieces in the rate function for the position.

The derivation of the annealed LDP in Theorem 2 starts also with the evaluation of the logarithmic moment generating function of the hitting times T_n. Then,

$$E_\omega[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}] = \exp\left(\sum_{k=1}^{n}\log\varphi(\lambda,\theta^{k-1}\omega)\right) = \exp\left(n\int\log\varphi(\lambda,\omega)R_n(d\omega)\right).$$

Thus, if R_n satisfies the LDP under the ergodic measure α, invoking the abstract Laplace principle (Varadhan's lemma) one expects that

$$\lim_{n\to\infty}\frac{1}{n}\log E[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}] = \sup_{\eta\in M_1^s(\Sigma)}\left(\int\log\varphi(\lambda,\omega)\eta(d\omega) - h(\eta|\alpha)\right).$$

From this point, the derivation of the annealed LDP for T_n/n, with rate function $I_\alpha^{\tau,a}(\cdot)$ given by the variational formula (9), is based on convexity considerations and in particular on a min-max argument. This forces us to study certain properties of the (quenched) rate function, en route to obtaining the variational representation (7) of the annealed rate function for the hitting times. Note that even if α is a product measure, one cannot a priori (and, as it turns out, also a-posteriori, c.f. Section 5.2) restrict the infimum in (9) to product measures η, and hence one has to consider the quenched LDP for non i.i.d. environments. As in the quenched case, the LDP for the position X_n/n (Theorem 2) follows from the LDP for T_n/n by simple duality arguments.

We now turn to state explicitly the LDPs for the hitting times T_n/n which are needed in the program described above. Define

$$\tau_\omega := E_\omega[\tau_1|\tau_1<\infty] \tag{10}$$

(with the value $+\infty$ allowed). Recall that a sequence of probability measures μ_n satisfies the *weak* LDP with rate function $I(\cdot)$ if the upper bound in (3) holds merely for compact sets.

Theorem 4 *Assume $\eta\in M_1^e(\Sigma)^K$. Then, for η-a.e. ω, the distributions of T_n/n under P_ω satisfy a weak LDP with deterministic, convex rate function $I_\eta^{\tau,q}$. Further, $I_\eta^{\tau,q}(\cdot)$ is decreasing on $[1,\int\tau_\omega\eta(d\omega)]$ and increasing on $[\int\tau_\omega\eta(d\omega),\infty)$.*

Theorem 4 obviously implies also a LDP for T_{-n}/n, simply by symmetry (i.e., space reversal of the measure η). An intermediate step in our proof of Theorem 4, relating the rate function for the LDP of T_{-n}/n to the one of T_n/n, is provided by the following:

Proposition 1 *Assume $\eta\in M_1^e(\Sigma)^K$. Then,*

$$\int\log E_\omega[e^{\lambda\tau_{-1}}\mathbf{1}_{\tau_{-1}<\infty}]\eta(d\omega) = \int\log E_\omega[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}]\eta(d\omega) + \int\log\rho_0(\omega)\eta(d\omega). \tag{11}$$

Further, if $\eta\in M_1^e(\Sigma)^{+,K}$, then the distributions of T_{-n}/n under P_ω satisfy, for η -a.e. ω, a weak LDP with deterministic rate function

$$I_\eta^{-\tau,q}(u) := I_\eta^{\tau,q}(u) - \int\log\rho_0(\omega)\eta(d\omega),\quad 1\le u<\infty. \tag{12}$$

We note that both in Theorem 4 and Proposition 1, the LDP's are weak due to possible positive probability mass at $+\infty$. The LDP of Theorem 4 can be strengthened to a full LDP if $\eta\in M_1^e(\Sigma)^{+,K}$.

With $I_\alpha^{\tau,a}$ defined in (7), the annealed statement corresponding to Theorem 4 is the following:

Theorem 5 *Let $\alpha\in M_1^e(\Sigma)^K$ satisfy Assumption (A). Then the distributions of T_n/n under P satisfy a (weak) LDP with convex rate function $I_\alpha^{\tau,a}$.*

1.4 Notations and conventions

We collect here various notations and conventions used throughout the paper. We will consider the following sets of probability measures:

$$M_1^e(\Sigma)^+ = \{\eta \in M_1^e(\Sigma) : \int \log \rho_0(\omega)\eta(d\omega) \le 0\}\,,\; M_1^e(\Sigma)^- := M_1^e(\Sigma) \setminus M_1^e(\Sigma)^+\,.$$

Recall that $K \subset (0,1)$ is some fixed compact subset of $(0,1)$, and that for any set $M \subset M_1(\Sigma)$, $M^K = M \cap \{\eta : \mathrm{supp}(\eta_0) \subset K \subset (0,1)\}$. Set $\omega_{\min} = \omega_{\min}(\eta) := \min\{z : z \in \mathrm{supp}\,\eta_0\}$ where η_0 denotes the marginal of η, $\omega_{\max} = \omega_{\max}(\eta) := \max\{z : z \in \mathrm{supp}\,\eta_0\}$, and let $\rho_{\max} = \rho_{\max}(\eta) := (1-\omega_{\min})/\omega_{\min}$. Then, we define

$$M^{1/2} := \{\eta \in M_1^e(\Sigma) : \omega_{\min}(\eta_0) \le 1/2, \omega_{\max}(\eta_0) \ge 1/2\}\,. \tag{13}$$

Throughout, all spaces of probability measures are given the topology of weak convergence.

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2 Properties of $\varphi(\lambda,\omega)$ and proofs of the quenched LDP for hitting times.

In proving Theorem 4, it is useful to consider first the case $\eta \in M_1^e(\Sigma)^{+,K}$, and prove for it a LDP for the hitting times. Note that in this case, $\tau_1 < \infty$, η-a.s. Our strategy for handling the case $P_\omega(\tau_1 = \infty) > 0$ will then be to first consider for $\eta \in M_1^e(\Sigma)^{+,K}$ the LDP for the hitting times T_{-n}/n, and then use a space reversal.

As is often the case, certain properties of the moment generating function $\varphi(\lambda,\omega)$ play an important role in the proof of the LDP. Recall that when $\eta \in M_1^e(\Sigma)^{+,K}$, it holds that $\varphi(\lambda,\omega) = E_\omega[e^{\lambda\tau_1}]$.

Lemma 1 *For any $\lambda \in \mathbb{R}$, we have that whenever $\varphi(\lambda,\omega) < \infty$ a.s. then*

$$\varphi(\lambda,\omega) = \frac{1|}{e^{-\lambda}(1+\rho_0(\omega))} - \frac{\rho_0(\omega)|}{e^{-\lambda}(1+\rho_{-1}(\omega))} - \frac{\rho_{-1}(\omega)}{\cdots}\,. \tag{14}$$

Further, for $\eta \in M_1^e(\Sigma)^{+,K}$ and $1 < u < E_\eta[\tau_1] := \int E_\omega[\tau_1]\eta(d\omega) \le \infty$, there exists a unique $\lambda_0 = \lambda_0(u,\eta)$ such that $\lambda_0 < 0$ and

$$u = \int \frac{d}{d\lambda} \log \varphi(\lambda,\omega)\Big|_{\lambda=\lambda_0} \eta(d\omega)\,. \tag{15}$$

Finally, for u as above

$$\inf_{\eta \in M_1^e(\Sigma)^{+,K}} \lambda_0(u,\eta) > -\infty\,. \tag{16}$$

Proof of Lemma 1. Pathwise decomposition yields the following formula for τ_1:

$$\tau_1 = \mathbf{1}_{X_1=1} + (\tau_1' + \tau_1'' + 1)\mathbf{1}_{X_1=-1} \tag{17}$$

where $\tau_1'+1$ is the first hitting time of 0 after time 1 (possibly infinite) and $\tau_1'+\tau_1''+1$ is the first hitting time of $+1$ after time $\tau_1'+1$. Note that, under P_ω, the law of τ_1' conditioned on the event $X_1=-1$ is $P_{\theta^{-1}\omega}(\tau_1\in\cdot)$ and, conditioned on the event $\tau_1'<\infty$, τ_1'' is independent of τ_1' and has law $P_\omega(\tau_1\in\cdot)$. Therefore, we have

$$\begin{aligned}
\varphi(\lambda,\omega) &= E_\omega[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}]\\
&= P_\omega[X_1=1]\,E_\omega\left[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}|X_1=1\right]+P_\omega[X_1=-1]\,E_\omega\left[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}|X_1=-1\right]\\
&= \omega_0e^\lambda+(1-\omega_0)E_\omega[e^{\lambda(\tau_1\circ\theta^{-1})}\mathbf{1}_{\tau_1\circ\theta^{-1}<\infty}]\,E_\omega[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}]e^\lambda\\
&= \omega_0e^\lambda+(1-\omega_0)e^\lambda\varphi(\lambda,\theta^{-1}\omega)\varphi(\lambda,\omega)\;.
\end{aligned}$$

Hence, if $\varphi(\lambda,\omega)<\infty$ then $\varphi(\lambda,\theta^{-1}\omega)<\infty$, and

$$\varphi(\lambda,\omega)=\frac{\omega_0e^\lambda}{1-(1-\omega_0)e^\lambda\varphi(\lambda,\theta^{-1}\omega)}=\frac{1}{(1+\rho_0(\omega))e^{-\lambda}-\rho_0(\omega)\varphi(\lambda,\theta^{-1}\omega)}\;. \tag{18}$$

In the same way,

$$\varphi(\lambda,\theta^{-1}\omega)=\frac{1}{(1+\rho_{-1})e^{-\lambda}-\rho_{-1}\varphi(\lambda,\theta^{-2}\omega)}\,.$$

By iteration, we get the representation of φ as a continued fraction, i.e., (14). (For a reference on continued fractions, see [13], [23]).

Let now $\eta\in M_1^e(\Sigma)^{+,K}$. Then, the indicator can be dropped in the definition of $\varphi(\lambda,\omega)$, and, with $\lambda<0$ and

$$g(\lambda):=\int\frac{E_\omega[\tau_1e^{\lambda\tau_1}]}{E_\omega[e^{\lambda\tau_1}]}\eta(d\omega)=\int\frac{d}{d\lambda}\log\varphi(\lambda,\omega)\eta(d\omega)\,. \tag{19}$$

we have

$$g(0)=\int E_\omega[\tau_1]\eta(d\omega)=E_\eta[\tau_1]\,,$$

and the strictly increasing, continuous function $g(\cdot)$ satisfies $g(\lambda)\geq1$ and $g(\lambda)\underset{\lambda\to-\infty}{\longrightarrow}1$. This implies (15).

To complete the proof of (16), note that

$$\begin{aligned}
1\leq\frac{E_\omega[\tau_1e^{\lambda\tau_1}]}{E_\omega[e^{\lambda\tau_1}]} &= \frac{P_\omega[\tau_1=1]e^\lambda+E_\omega[\tau_1e^{\lambda\tau_1}\mathbf{1}_{\tau_1\geq2}]}{P_\omega[\tau_1=1]e^\lambda+E_\omega[e^{\lambda\tau_1}\mathbf{1}_{\tau_1\geq2}]}\\
&\leq \frac{\omega_0e^\lambda+e^{3\lambda/2}E_\omega[\tau_1e^{\lambda\tau_1-3\lambda/2}\mathbf{1}_{\tau_1\geq2}]}{\omega_0e^\lambda}\leq1+\frac{ce^{\lambda/3}}{\omega_0}\,,
\end{aligned} \tag{20}$$

for some constant c independent of ω or λ. Taking $\lambda\to-\infty$ yields the uniform convergence of the right hand side of (20) to 1, and hence (16). □

Remark : In the same way, taking expectations in (17) and iterating yields $E_\omega[\tau_1]=1+2Z_0^-$, cf (1).

We may now deal in more details with the behavior of $\varphi(\lambda,\omega)$ for positive λ:

Lemma 2 *Let $\eta\in M_1^e(\Sigma)^{+,K}$. Then*

(i) There is a deterministic $\infty>\lambda_{\mathrm{crit}}\geq0$, depending only on η, such that for $\lambda<\lambda_{\mathrm{crit}}$, $\varphi(\lambda,\omega)<\infty$ for η -a.e. ω, and for $\lambda>\lambda_{\mathrm{crit}}$, $\varphi(\lambda,\omega)=\infty$ for η -a.e. ω.

(ii) Let $u_{\text{crit}} = \infty$ if $\int E_\omega[\tau_1 e^{\lambda_{\text{crit}}\tau_1}]\ \eta(d\omega) = \infty$ and $u_{\text{crit}} := \int \frac{d}{d\lambda}\log\varphi(\lambda,\omega)\Big|_{\lambda=\lambda_{\text{crit}}}\eta(d\omega)$ else. For $E[\tau_1] \le u < u_{\text{crit}}$, there exists a unique $\lambda_0 = \lambda_0(u,\eta)$ such that $\lambda_0 \ge 0$ and (15) holds.

Remark: u_{crit} can be infinite in the general ergodic case, for instance in the periodic case, e.g. for $\eta = \frac{1}{2}\delta_{(\ldots,\omega_1,\omega_2,\omega_1,\omega_2,\ldots)} + \frac{1}{2}\delta_{(\ldots,\omega_2,\omega_1,\omega_2,\omega_1,\ldots)}$ with $\omega_1 \ge 1/2, \omega_2 > 1/2$.

Proof of Lemma 2.
(i) Let $\lambda_c(\omega) := \sup\{\lambda : E_\omega[e^{\lambda\tau_1}] < \infty\}$. Since, using (17), $E_{\theta\omega}[e^{\lambda\tau_1}] \ge (1-\omega_1)E_\omega[e^{\lambda\tau_1}]$, we have $\lambda_c(\theta\omega) \le \lambda_c(\omega)$. But $\lambda_c(\theta\omega)$ and $\lambda_c(\omega)$ have the same distribution, hence $\lambda_c(\theta\omega) = \lambda_c(\omega)$ for η-a.e. ω, i.e. λ_c is shift-invariant. Since η is ergodic, this implies that $\lambda_c(\omega) = \int \lambda_c(\omega)\eta(d\omega) := \lambda_{\text{crit}}$ for η -a.e. ω.
(ii) With $g(\lambda)$ as in (19), we have that g is strictly increasing and continuous in λ for $\lambda < \lambda_{\text{crit}}$, $g(0) = E_\eta[\tau_1] < \infty$, and $g(\lambda) \underset{\lambda\to\lambda_{\text{crit}}}{\longrightarrow} u_{\text{crit}}$. □

Turning to the main business of this section, we have the:

Proof of Theorem 4 for $\eta \in M_1^e(\Sigma)^{+,K}$.
The claims on the convexity and monotonicity of $I_\eta^{\tau,q}(\cdot)$ are a direct consequence of the definition and Lemmas 1 and 2. Considering the bounds themselves, we start by showing that for $1 < u \le E_\eta[\tau_1]$, and $\eta \in M_1^e(\Sigma)^{+,K}$,

$$\limsup_{n\to\infty} \frac{1}{n}\log P_\omega\left[\frac{1}{n}\sum_{j=1}^n \tau_j \le u\right] \le -\sup_{\lambda\le 0} G(\lambda,\eta,u) = -I_\eta^{\tau,q}(u)\,, \tag{21}$$

with G and $I_\eta^{\tau,q}$ defined in (4) and (5). Indeed, Chebyshev's inequality implies that, for $\lambda \le 0$,

$$P_\omega\left[\frac{1}{n}\sum_{j=1}^n \tau_j \le u\right] \le E_\omega\left[e^{\lambda\sum_{j=1}^n \tau_j}\right]e^{-\lambda n u}\,.$$

Note that, because for η-a.e ω, $\tau_1,\ldots,\tau_n$ are finite and therefore independent under P_ω,

$$\begin{aligned}\frac{1}{n}\log E_\omega\left[e^{\lambda\sum_{j=1}^n \tau_j}\right] &= \frac{1}{n}\sum_{j=1}^n \log E_\omega[e^{\lambda\tau_j}] = \frac{1}{n}\sum_{j=0}^{n-1}\log E_{\omega\circ\theta^j}[e^{\lambda\tau_1}]\\ &= \frac{1}{n}\sum_{j=0}^{n-1}\log\varphi(\lambda,\theta^j\omega) \underset{n\to\infty}{\longrightarrow} \int \log\varphi(\lambda,\omega)\eta(d\omega) \quad \eta\text{ -a.e. } \omega\,,\end{aligned} \tag{22}$$

due to the ergodic theorem for any fixed λ. Let $\Omega_{u.b.}$ be the set of ω's such that (22) holds for all rational λ and for $\lambda = \lambda_{\text{crit}}$. Then $\eta(\Omega_{u.b.}) = 1$. Since the map $\lambda \mapsto \log\varphi(\lambda,\omega)$ is increasing, the limit in (22) holds on $\Omega_{u.b.}$ simultaneously for all real λ. This proves (21) for $\omega \in \Omega_{u.b.}$. Now, Lemma 1 implies that the supremum in (21) is attained for $\lambda = \lambda_0(u)$, and further is equal to the supremum over $\lambda \in \mathbb{R}$.

Still with $\eta \in M_1^e(\Sigma)^{+,K}$, let $u \ge E_\eta[\tau_1]$, and note that for $\lambda \ge 0$, we have

$$P_\omega\left[\frac{1}{n}\sum_{j=1}^n \tau_j \ge u\right] \le E_\omega\left[e^{\lambda\sum_{j=1}^n \tau_j}\right]e^{-\lambda n u}$$

and

$$\frac{1}{n}\log P_\omega\left[\frac{1}{n}\sum_{j=1}^n \tau_j \ge u\right] \le \frac{1}{n}\sum_{j=0}^{n-1}\log\varphi(\lambda,\theta^j\omega) - \lambda u \underset{n\to\infty}{\longrightarrow} \int \log\varphi(\lambda,\omega)\eta(d\omega) - \lambda u \text{ for } \eta\text{-a.e. } \omega,$$

due to the ergodic theorem. Since $\lambda \geq 0$ was arbitrary, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega \left[\frac{1}{n} \sum_{j=1}^{n} \tau_j \geq u \right] \leq -\sup_{\lambda \geq 0} G(\lambda, \eta, u)\,. \tag{23}$$

Now Lemma 2 implies that the supremum in (23) is attained (with $u < u_{\mathrm{crit}}$) for $\lambda = \lambda_0(u)$, and at $\lambda = \lambda_{\mathrm{crit}}$ otherwise, and further is equal to the supremum over all $\lambda \in \mathbb{R}$. Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega \left[\frac{1}{n} \sum_{j=1}^{n} \tau_j \geq u \right] \leq -I_\eta^{\tau,q}(u)\,. \tag{24}$$

Taking ω in $\Omega_{u.b.}$, the upper bound in the LDP for $\eta \in M_1^e(\Sigma)^{+,K}$ follows from (21), (24) and the convexity of $I_\eta^{\tau,q}(\cdot)$.

To prove the LDP lower bound in Theorem 4 for $\eta \in M_1^e(\Sigma)^{+,K}$, we follow a standard change of measure, using independence of the τ_i's under P_ω. See [4, Pg. 31–33] for a similar argument. Fix $u \in (1, \infty)$, $M \in (u+1, \infty)$ (eventually, we will take $M \to \infty$, and in fact for $u < E[\tau_1]$ we could take $M = \infty$ throughout). Let

$$A_{n,M} = \{\tau_j \leq M, j = 1, \ldots, n\}\,,$$

and let $\tilde{P}_{\omega,n}(\cdot)$ denote the law of $\{\tau_i\}_{i=1}^n$, conditioned on $A_{n,M}$. Note that $\{\tau_i\}_{i=1}^n$ are still independent, although not identically distributed, under the law $\tilde{P}_{\omega,n}$. We let

$$\varphi_M(\lambda, \omega) := E_\omega[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 \leq M}]\ ,\ \tilde{\varphi}_M(\lambda, \omega) := \tilde{E}_\omega[e^{\lambda \tau_1}] = \frac{\varphi_M(\lambda, \omega)}{P_\omega[\tau_1 \leq M]}\,,$$

$$\log \varphi(\lambda) := \int \log(\varphi(\lambda, \omega))\eta(d\omega)\,, \log \varphi_M(\lambda) := \int \log(\varphi_M(\lambda, \omega))\eta(d\omega)\,, C_M := \int \log P_\omega[\tau_1 \leq M]\eta(d\omega)\,,$$

and

$$\tilde{\Lambda}_M(\lambda) := \log \tilde{\varphi}_M(\lambda) = \log \varphi_M(\lambda) - C_M\,.$$

Note that $0 \geq C_M \geq \int \log \omega_0 \eta(d\omega) > -\infty$ and $C_M \to_{M\to\infty} 0$ because $\tau_1 < \infty$, η-a.s. We have

$$\frac{1}{n} \log P_\omega \left[\frac{1}{n} \sum_{j=1}^{n} \tau_j \in (u-\delta, u+\delta) \right] \geq \frac{1}{n} \log \tilde{P}_\omega \left[\frac{1}{n} \sum_{j=1}^{n} \tau_j \in (u-\delta, u+\delta) \right] + \frac{1}{n} \log P_\omega[A_{n,M}]$$

where $\tilde{P}_\omega$ is the standard extension of $(\tilde{P}_{\omega,n})$. Using now the fact that $\tilde{\varphi}_M(\lambda, \omega)$ is smooth and convex, we define, for M large enough, $\lambda_M(u)$ such that

$$u = \int \frac{d \log \tilde{\varphi}_M(\lambda, \omega)}{d\lambda}\Big|_{\lambda = \lambda_M(u)} \eta(d\omega)\,.$$

Define $\tilde{Q}_\omega = \tilde{Q}_{\omega, M, \lambda_M(u)}$ such that, for each n,

$$\frac{d\tilde{Q}_\omega}{d\tilde{P}_{\omega,n}} = \frac{1}{Z_{n,\omega}} \exp(\lambda_M(u) \sum_{j=1}^{n} \tau_j)\,,$$

with $Z_{n,\omega} = \prod_{j=1}^n \tilde\varphi_M(\lambda_M(u), \theta^j\omega)$. Note that $\tilde Q_\omega$ is a product measure, and we have

$$\begin{aligned} &\tilde P_\omega \left[\frac{1}{n}\sum_{j=1}^n \tau_j \in (u-\delta, u+\delta)\right] \\ &\geq \exp\left(-n\lambda_M(u)(u+\delta) + \sum_{j=1}^n \log\tilde\varphi_M(\lambda_M(u),\theta^j\omega)\right) \tilde Q_\omega \left[\left| \frac{1}{n}\sum_{j=1}^n \tau_j - u \right| \leq \delta \right] . \end{aligned} \tag{25}$$

By the ergodic theorem, with the last equality due to our choice of $\lambda_M(u)$,

$$\int \frac{1}{n}\sum_{j=1}^n \tau_j d\tilde Q_\omega \underset{n\to\infty}{\longrightarrow} \int d\eta \int \tau_1 d\tilde Q_\omega = u \tag{26}$$

on a set of η-measure 1 which depends on u and M. Similar to the upper bound, we consider the set $\Omega_{l.b.}$ of all ω's such that, for all rational u and all integers $M \in (u+1, \infty)$, (26) holds true, and also (22) with $\tilde\varphi_M(\lambda_M(u), .)$ instead of $\varphi(\lambda, .)$. Then $\eta(\Omega_{l.b.}) = 1$, (26) holds on this set for all $u \in (1,\infty)$ by monotonicity of $\lambda \mapsto \int \tau_j d\tilde Q_{\omega,M,\lambda}$, as well as (22) with $\tilde\varphi_M(\lambda_M(u), \cdot)$ instead of $\varphi(\lambda,\cdot)$ again by monotonicity.
The independence of the τ_j under $\tilde Q_\omega$ implies that, for n large enough,

$$\int \left(\frac{1}{n}\sum_{j=1}^n (\tau_j - E_{\tilde Q_\omega}[\tau_j])\right)^4 d\tilde Q_\omega \leq \frac{3M^4}{n^2} .$$

Hence, using (26) and the Borel-Cantelli lemma, we obtain for $\omega \in \Omega_{l.b.}$

$$\tilde Q_\omega \left[\left| \frac{1}{n}\sum_{j=1}^n \tau_j - u \right| \geq \delta \right] \underset{n\to\infty}{\longrightarrow} 0 .$$

Substituting in (25), taking logarithms, dividing by n, and letting $n\to\infty$ and then $\delta \to 0$, we conclude that

$$\begin{aligned} \lim_{\delta\to 0}\liminf_{n\to\infty} \frac{1}{n}\log \tilde P_\omega \left[\frac{1}{n}\sum_{j=1}^n \tau_j \in (u-\delta,u+\delta)\right] &\geq -\left(\lambda_M(u)u - \liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log\tilde\varphi_M(\lambda_M(u),\theta^j\omega)\right) \\ &= -\left(\lambda_M(u)u - \tilde\Lambda_M(\lambda_M(u))\right) \\ &= -\left(\lambda_M(u)u - \log\varphi_M(\lambda_M(u))\right) - C_M \\ &\geq -\sup_{\lambda\in\mathbb{R}} \left(\lambda u - \log\varphi_M(\lambda)\right) - C_M := -I_M(u) - C_M . \end{aligned} \tag{27}$$

Let $I^*(u) = \limsup_{M\to\infty} I_M(u)$. Because $\varphi_M(\cdot)$ is non-decreasing in M, so is $-I_M(\cdot)$, implying that $I^*(u) \geq 0$ and, because $I_M(u) < \infty$ for large M, also $I^*(u) < \infty$. Hence, the level sets $\{\lambda : \lambda u - \log\varphi_M(\lambda) \geq I^*(u)\}$ are non-empty, compact, nested sets and hence contain some $\lambda^* < \infty$ in their intersection. By Lebesgue's monotone convergence, we get

$$\log\varphi(\lambda^*) = \lim_{M\to\infty} \log\varphi_M(\lambda^*) \leq \lambda^* u - I^*(u) ,$$

implying that $I_\eta^{\tau,q}(u) := \sup_{\lambda\in\mathbb{R}}(\lambda u - \log\varphi(\lambda)) \geq I^*(u)$ and hence, in conjunction with (27), the lower bound with rate function $I_\eta^{\tau,q}(u)$ for all $\omega \in \Omega_{l.b.}$. ◻

As mentioned in the outline at the beginning of this section, we turn next to handle the hitting times T_{-n}/n. Define

$$\tilde{\varphi}(\lambda,\omega) = E_\omega\left[e^{\lambda\tau_{-1}}\mathbf{1}_{\tau_{-1}<\infty}\right]. \tag{28}$$

Let $\bar{\tau}_{-1},\bar{\tau}_{-2},\bar{\tau}_{-3},\ldots,\bar{\tau}_{-N}$ have the distribution of $\tau_{-1},\ \tau_{-2},\ \tau_{-3},\ldots,\tau_{-N}$, conditioned on $T_{-N}<\infty$. In fact the law of $\bar{\tau}_{-1}$ does not depend on N: the distributions of $X_0^{T_{-N}} := (X_0,\ldots,X_{T_{-N}})$ under P_ω, conditioned on $T_{-N}<\infty$, $N=1,2,\ldots$ form a consistent family whose extension is again a Markov chain. To see this, let $P_{\bar{\omega},N} := P_\omega[\cdot|T_{-N}<\infty]$, restricted to $X_0^{T_{-N}}$. Denoting $x_1^n := (x_1,\ldots,x_n)$, compute (with $x_i > -N$),

$$\begin{aligned}
P_{\bar{\omega},N}[X_{n+1}=x_n+1|X_1^n=x_1^n] &= \frac{P_{\bar{\omega},N}[X_{n+1}=x_n+1,X_1^n=x_1^n]}{P_{\bar{\omega},N}[X_1^n=x_1^n]}\\
&= \frac{P_\omega[X_{n+1}=x_n+1,X_1^n=x_1^n,T_{-N}<\infty]}{P_\omega[X_1^n=x_1^n,T_{-N}<\infty]}\\
&= \frac{P_\omega[X_{n+1}=x_n+1,X_1^n=x_1^n]P_{\theta^{x_n+1}\omega}[T_{-N-x_n-1}<\infty]}{P_\omega[X_1^n=x_1^n]P_{\theta^{x_n}\omega}[T_{-N-x_n}<\infty]}\\
&= P_\omega[X_{n+1}=x_n+1|X_1^n=x_1^n]P_{\theta^{x_n+1}\omega}[T_{-1}<\infty] = \omega_{x_n}P_{\theta^{x_n+1}\omega}[T_{-1}<\infty],
\end{aligned}$$

where we used the Markov property in the third and in the fourth equality. The last term depends neither on N nor on x_1^{n-1}. Therefore, the extension of $(P_{\bar{\omega},N})_{N\geq 1}$ is the distribution of the Markov chain with transition probabilities $\bar{\omega}_i = \omega_i P_{\theta^{i+1}\omega}[T_{-1}<\infty], i\in\mathbb{Z}$. In particular, $\bar{\tau}_{-1},\bar{\tau}_{-2},\bar{\tau}_{-3},\ldots$ are independent under P_ω and form a stationary sequence under P. Let

$$\bar{\varphi}(\lambda,\omega) := E_\omega[e^{\lambda\bar{\tau}_{-1}}] = \frac{\tilde{\varphi}(\lambda,\omega)}{P_\omega[T_{-1}<\infty]} \tag{29}$$

We will show below the following LDP.

Theorem 6 *With $\eta\in M_1^e(\Sigma)^{+,K}$, the distributions of $\frac{1}{n}\sum_{j=1}^n \bar{\tau}_{-j}$ under P_ω satisfy, for η -a.e. ω, a LDP with deterministic rate function $I_\eta^{\tau,q}$.*

An important step in the proof of Theorem 6 will be:

Lemma 3 *For any $\eta\in M_1^e(\Sigma)^+$, we have, with $\varphi(\lambda,\omega)=E_\omega[e^{\lambda\tau_1}]$ and $\bar{\varphi}(\lambda,\omega)=E_\omega[e^{\lambda\bar{\tau}_{-1}}]$,*

$$\int \log\bar{\varphi}(\lambda,\omega)\eta(d\omega) = \int\log\varphi(\lambda,\omega)\eta(d\omega)\,. \tag{30}$$

We next recall that if $(\omega_x)_{x\in\mathbb{Z}}$ is such that $X_n\to+\infty$ P_ω-a.s. then

$$P_\omega[\min_k X_k\leq -n] = \frac{\sum_{j=0}^\infty\prod_{i=-n+1}^j\rho_i}{1+\sum_{j=-n+1}^\infty\prod_{i=-n+1}^j\rho_i}\,,$$

for a proof, see e.g. [2, Pg. 65–71]. Hence, we have for $\eta\in M_1^e(\Sigma)^{+,K}$ that

$$\int\log P_\omega[\tau_{-1}<\infty]\eta(d\omega) = \lim_{n\to\infty}\frac{1}{n}\log P_\omega[T_{-n}<\infty] = \int\log\rho_0(\omega)\eta(d\omega)\,,\ \eta-a.s. \tag{31}$$

Equipped with Theorem 6 and Lemma 3, we can give now the:
Proof of Proposition 1 Note that for $A \subseteq [1,\infty)$, we have

$$P_\omega \left[\frac{1}{n} \sum_{j=1}^{n} \tau_{-j} \in A, T_{-n} < \infty \right] = P_\omega \left[\frac{1}{n} \sum_{j=1}^{n} \bar{\tau}_{-j} \in A \right] P_\omega[T_{-n} < \infty]$$

Proposition 1 now follows from Theorem 6, Lemma 3, (29) and (31). In particular, Lemma 3 and (31) imply (11) for $\eta \in M_1^e(\Sigma)^{+,K}$. If $\eta \in M_1^e(\Sigma)^{-,K}$, applying (11) for $\eta^{\text{Inv}} \in M_1^e(\Sigma)^{+,K}$ implies (11) for η, since $\rho_0(\text{Inv }\omega) = \rho_0(\omega)^{-1}$. $\square$

Still assuming Theorem 6 and Lemma 3, we can now complete the:
Proof of Theorem 4 for $\eta \in M_1^e(\Sigma)^{-,K}$

Clearly, $\int \log \rho_0(\omega)\eta(d\omega) = -\int \log \rho_0(\omega)\eta^{\text{Inv}}(d\omega)$, and further the law of $\sum_{j=1}^n \tau_j$ under η is the same as the law of $\sum_{j=1}^n \tau_{-j}$ under $\eta^{\text{Inv}} \in M_1^e(\Sigma)^{+,K}$. Hence, the distributions of $\frac{1}{n}\sum_{j=1}^n \tau_j$ under P_ω satisfy, η-a.s., the LDP with rate function

$$I^{\tau,q}_{\eta^{\text{Inv}}}(u) + \int \log \rho_0(\omega)\eta(d\omega)\,.$$

The conclusion follows by (11). $\square$

Proof of Lemma 3. Considering (29) and (31), (30) is equivalent to

$$\int \log \tilde{\varphi}(\lambda,\omega)\eta(d\omega) = \int \log \varphi(\lambda,\omega)\eta(d\omega) + \int \log \rho_0(\omega)\eta(d\omega) \tag{32}$$

To prove (32), define $\Lambda = \{\lambda : \int \log \varphi(\lambda,\omega)\eta(d\omega) < \infty\}$. To circumvent integrability problems, we fix $M < \infty$, which could be taken as ∞ in the case $\lambda \le 0$, yielding a more transparent proof in this case. Consider the event $D_M := \{\tau_{-1} < T_M\}$, and define

$$\tilde{\varphi}^M(\lambda,\omega) = E_\omega[e^{\lambda\tau_{-1}}; D_M]$$

for $\lambda \in \Lambda$. Note that on D_M, $T_M = \tau_{-1} + \hat{\tau}_1' + T_M'$ where $\tau_{-1} + \hat{\tau}_1'$ is the first hitting time of 0 after τ_{-1} and T_M' is independent of τ_{-1} and $\hat{\tau}_1'$, with the same distribution as T_M. This path decomposition now yields, similarly to (17), that

$$E_\omega[e^{\lambda T_M}; D_M] = E_\omega[e^{\lambda T_M}]\tilde{\varphi}^M(\lambda,\omega)\varphi(\lambda,\theta^{-1}\omega)\,, \tag{33}$$

implying, for $\lambda \in \Lambda$ and all ω with $P_\omega[T_M < \infty] = 1$,

$$1 \ge \frac{E_\omega[e^{\lambda T_M}; D_M]}{E_\omega[e^{\lambda T_M}]} = \tilde{\varphi}^M(\lambda,\omega)\varphi(\lambda,\theta^{-1}\omega)\,, \tag{34}$$

and hence, for $\lambda \in \Lambda$,

$$-\log \varphi(\lambda,\theta^{-1}\omega) \ge \log \tilde{\varphi}^M(\lambda,\omega)\,,$$

implying by monotonicity also that $-\log\varphi(\lambda,\theta^{-1}\omega) \ge \log\tilde{\varphi}(\lambda,\omega)$. Since $\log\tilde{\varphi}(\lambda,\omega) \ge \log\tilde{\varphi}^M(\lambda,\omega) \ge \lambda + \log(1-\omega_0)$, it follows that both $\log\tilde{\varphi}(\lambda,\omega)$ and $\log\tilde{\varphi}^M(\lambda,\omega)$ are integrable for $\lambda \in \Lambda$.

Next, using again path decomposition one finds that, η-a.s.,

$$E_\omega\left[e^{\lambda\tau_{-1}}; D_M\right] = (1-\omega_0)e^\lambda + \omega_0 e^\lambda E_{\theta\omega}\left[e^{\lambda\tau_{-1}}; D_{M-1}\right] E_\omega\left[e^{\lambda\tau_{-1}}; D_M\right]\,.$$

Hence, η-a.s,

$$\tilde{\varphi}^M(\lambda,\omega)\tilde{\varphi}^{M-1}(\lambda,\theta\omega) = \frac{e^{-\lambda}\tilde{\varphi}^M(\lambda,\omega)}{\omega_0} - \rho_0(\omega)\,,$$

and similarly, by (18),

$$\rho_0(\omega)\varphi(\lambda,\omega)\varphi(\lambda,\theta^{-1}\omega) = \frac{e^{-\lambda}\varphi(\lambda,\omega)}{\omega_0} - 1\,.$$

Then, η-a.s,

$$\begin{aligned}\rho_0(\omega)\Big(1-\tilde{\varphi}^M(\lambda,\omega)\varphi(\lambda,\theta^{-1}\omega)\Big)\varphi(\lambda,\omega) &= \rho_0(\omega)\varphi(\lambda,\omega) - \tilde{\varphi}^M(\lambda,\omega)\frac{e^{-\lambda}\varphi(\lambda,\omega)}{\omega_0} + \tilde{\varphi}^M(\lambda,\omega)\\ &= \Big(1-\varphi(\lambda,\omega)\tilde{\varphi}^{M-1}(\lambda,\theta\omega)\Big)\tilde{\varphi}^M(\lambda,\omega)\,.\end{aligned}$$

Therefore, η-a.s,

$$\log\rho_0(\omega) + \log\varphi(\lambda,\omega) - \log\tilde{\varphi}^M(\lambda,\omega) = \log(1-\tilde{\varphi}^{M-1}(\lambda,\theta\omega)\varphi(\lambda,\omega)) - \log(1-\tilde{\varphi}^M(\lambda,\omega)\varphi(\lambda,\theta^{-1}\omega))\,,$$

and averaging over $M=2$ to K and taking expectations (using stationarity and (33)!) yields, for $\lambda\in\Lambda$,

$$\begin{aligned}&E[\log\rho_0] + E[\log\varphi(\lambda,\omega)] - (K-1)^{-1}\sum_{M=2}^{K} E\left[\log\tilde{\varphi}^M(\lambda,\omega)\right]\\ =\;& -(K-1)^{-1}\left(E\left[\log(1-\tilde{\varphi}^K(\lambda,\omega)\varphi(\lambda,\theta^{-1}\omega)) - \log(1-\tilde{\varphi}^1(\lambda,\theta\omega)\varphi(\lambda,\omega))\right]\right)\\ =\;& -(K-1)^{-1}\left(E\left[\log\frac{E_\omega[e^{\lambda T_K};D_K^c]}{E_\omega[e^{\lambda T_K}]}\right] + \text{const}\right)\,.\end{aligned} \tag{35}$$

But, using again the Markov property and stationarity of η,

$$(K-1)^{-1}E\left[\log E_\omega[e^{\lambda T_K};D_K^c]\right] = (K-1)^{-1}\sum_{M=1}^{K} E\left[\log E_\omega[e^{\lambda\tau_1}\mathbf{1}_{\tau_{-M}>\tau_1}]\right] \underset{K\to\infty}{\longrightarrow} E[\log\varphi(\lambda,\omega)]\,,$$

due to monotone convergence, implying that the right hand side of (35) vanishes for $K\to\infty$. Substituting in (35) and using monotone convergence again, we get (32) for $\lambda\in\Lambda$. To get that the left hand side of (32) is $+\infty$ for $\lambda\in\Lambda^c$, assume otherwise, and reverse the role of $\tilde{\varphi}$ and φ in the above proof, while replacing D_M by $\bar{D}_M=\{\tau_1<T_{-M}<\infty\}$. $\square$

Proof of Theorem 6. Note that all that is needed in order to mimic the argument given in the proof of Theorem 4 for $\eta\in M_1^e(\Sigma)^{+,K}$ is the almost sure convergence of $n^{-1}\sum_{i=1}^n\log\bar{\varphi}(\lambda,\theta^{-i}\omega)$ to $\int\log\bar{\varphi}(\lambda,\omega)\eta(d\omega)$, which is ensured by the ergodicity of η. $\square$

Remarks:

1. In the recurrent case, $\{\bar{\tau}_{-i}\}$ has the same law as $\{\tau_{-i}\}$.
2. Lemma 3 implies, by differentiating (30) at zero, that the cumulants of τ_1 and $\bar{\tau}_{-1}$ have the same expectation under η. In particular, $E_\eta[\bar{\tau}_{-1}]=E_\eta[\tau_1]$. We note that Lemma 3 resembles results of [5], although we do not see a direct relation between the two.

For future reference, we note some easy properties of the rate function $I_\eta^{\tau,q}(\cdot)$. The reader is advised to skip this part in first reading. Recall the notations introduced in (4). With $\tau_\omega=E_\omega[\tau_1|\tau_1<\infty]$, let $M_u:=\{\eta\in M_1^e(\Sigma)^K : E_\eta[\tau_\omega]\geq u\}$. Then, for $\eta\in M_u$, one has by Lemma 1 (for $\eta\in M_1^e(\Sigma)^{+,K}\cap M_u$) and Proposition 1 (for $\eta\in M_u\setminus M_1^e(\Sigma)^{+,K}$) that

$$I_\eta^{\tau,q}(u) = \sup_{\lambda\in\mathbb{R}}[G(\lambda,\eta,u)] = \sup_{\lambda\leq 0}[G(\lambda,\eta,u)]\,. \tag{36}$$

Similarly, let $M_u^- := \{\eta \in M_1^e(\Sigma)^K : E_\eta[\tau_\omega] \le u\}$. Then, for $\eta \in M_u^-$, one has by Lemma 2 (for $\eta \in M_1^e(\Sigma)^{+,K} \cap M_u^-$) and Proposition 1 (for $\eta \in M_u^- \setminus M_1^e(\Sigma)^{+,K}$) that

$$I_\eta^{\tau,q}(u) = \sup_{\lambda \in \mathbb{R}}[G(\lambda,\eta,u)] = \sup_{\lambda \ge 0}[G(\lambda,\eta,u)]\,. \tag{37}$$

Next, if $\eta \in M_u^-$ then, by Jensen's inequality,

$$\begin{aligned}\sup_{\lambda\le 0} G(\lambda,\eta,u) &\le \sup_{\lambda \le 0}\Big[\lambda u - \lambda \int E_\omega[\tau_1|\tau_1<\infty]\eta(d\omega)\Big] - \int \log P_\omega[\tau_1<\infty]\eta(d\omega)\\ &= -\int \log P_\omega[\tau_1<\infty]\eta(d\omega)\end{aligned}$$

where the last equality is due to the fact that the last supremum is achieved at $\lambda=0$. On the other hand, the substitution $\lambda = 0$ in the above reveals that

$$\sup_{\lambda\le 0} G(\lambda,\eta,u) \ge -\int \log P_\omega[\tau_1<\infty]\eta(d\omega)\,.$$

Hence, due to (31),

$$\sup_{\lambda\le 0} G(\lambda,\eta,u) = \int \log \rho_0(\omega) d\eta_0(\omega) \vee 0\,, \tag{38}$$

Similarly, if $\eta \in M_u$ then

$$\sup_{\lambda\ge 0} G(\lambda,\eta,u) = \int \log \rho_0(\omega) d\eta_0(\omega) \vee 0\,. \tag{39}$$

We also note that the rate function $I_\eta^{\tau,q}(\cdot)$ is convex, with minimum value $\int \log\rho_0(\omega)\eta(d\omega)\vee 0$ achieved at $E_\eta[\tau_\omega]$. Hence, for all $\eta \in M_1^e(\Sigma)^K$,

$$\sup_{\lambda\le 0} G(\lambda,\eta,u) = \inf_{w\le u} I_\eta^{\tau,q}(w)\,, \tag{40}$$

and

$$\sup_{\lambda\ge 0} G(\lambda,\eta,u) = \inf_{w\ge u} I_\eta^{\tau,q}(w)\,. \tag{41}$$

We conclude this section with some properties of $\varphi(\lambda,\omega)$ in the particular case that η is locally equivalent to the product of its marginals, as defined before Assumption (A). These properties are needed in the study of the annealed case.

Lemma 4 *Let $\eta \in M_1^e(\Sigma)^{+,K}$ be locally equivalent to the product of its marginals. Then,*

(i) If $\rho_{\max} < 1$, then $\lambda_{\text{crit}} = \bar\lambda := -\frac{1}{2}\log(4\omega_{\min}(1-\omega_{\min})) > 0$ and $\varphi(\lambda,\omega) = E_\omega[e^{\lambda\tau_1}] < \infty$ iff $\lambda \le \lambda_{\text{crit}}$. Further, $u_{\text{crit}} := \int \frac{d}{d\lambda}\log\varphi(\lambda,\omega)\Big|_{\lambda=\lambda_{\text{crit}}}\eta(d\omega) < \infty$ unless η is degenerate, i.e. unless $\omega = const$ η-a.s.

(ii) If $\rho_{\max} \ge 1$, we have $\lambda_{\text{crit}} = 0$.

Note that without the condition of local equivalence to the product of marginals, one can have $\lambda_{\text{crit}} > \bar\lambda$, c.f. the example in the remark following Lemma 2.

The next lemma is needed in the proof of Lemma 4. It can also be used to show that, if $\rho_{\max} \le 1$ and $\eta(\omega_0 \ne 1/2) > 0$, the random walk has a positive speed $v_\eta > 0$.

Lemma 5 *Let $(b_1(\omega), b_2(\omega), b_3(\omega), \ldots)$ be a stationary, ergodic sequence with $0 < b_1(\omega) \le 1$, η - a.s., and $E_\eta[b_1(\omega)] < 1$. Then we have $E_\eta\left[\sum_{n=1}^\infty b_1 \cdots b_n\right] < \infty$.*

Proof: Fix γ such that $0 < \gamma < 1$ and $\eta(b_1 \le \gamma) > 0$. Let $t_0 := 0$, $t_1 := \inf\{n \ge 1 : b_n \le \gamma\}$ and $t_{k+1} = \inf\{n > t_k : b_n \le \gamma\}$. Due to our assumption on γ, the ergodic theorem implies that $E_\eta[t_1] < \infty$. Clearly, $b_1 \cdots b_{t_k} \le \gamma^k$ and therefore

$$\sum_{n=1}^\infty b_1 \cdots b_n \le \sum_{k=1}^\infty \gamma^k (t_k - t_{k-1}). \tag{42}$$

But $E_\eta[t_k - t_{k-1}] = E_\eta[t_1]$ due to stationarity, and taking expectations in (42) yields

$$E_\eta\left[\sum_{n=1}^\infty b_1 \cdots b_n\right] \le E_\eta[t_1] \sum_{k=1}^\infty \gamma^k < \infty\,.$$

$\square$

Proof of Lemma 4. Throughout, we take $\lambda \ge 0$.
(i) Note that for $\overline{\omega}_{\min} = (\ldots, \omega_{\min}, \omega_{\min}, \omega_{\min}, \ldots)$, we have (by standard coupling) that

$$E_\omega[e^{\lambda\tau_1}] \le E_{\overline{\omega}_{\min}}[e^{\lambda\tau_1}]\,. \tag{43}$$

Let $\bar\varphi(\lambda) := E_{\overline{\omega}_{\min}}[e^{\lambda\tau_1}]$. Note that, by the same recursion used to derive (18), based on (17), one knows that if $\bar\varphi(\lambda) < \infty$ then

$$\bar\varphi(\lambda) = \omega_{\min} e^\lambda + e^\lambda (1 - \omega_{\min})(\bar\varphi(\lambda))^2\,, \tag{44}$$

leading to

$$\bar\varphi(\lambda) = \frac{1 - \sqrt{1 - e^{2(\lambda - \bar\lambda)}}}{2e^\lambda(1 - \omega_{\min})}\,,$$

as long as $\lambda \le \bar\lambda = -\frac{1}{2}\log(4\omega_{\min}(1 - \omega_{\min}))$. We have to show that for $\lambda > \bar\lambda$, $E_\omega[e^{\lambda\tau_1}] = \infty$ for η -a.a. ω. Assume $\eta_0(\omega_{\min}) > 0$. In a first step, we show that for each $K > 0$, there is $A_K \subset \Sigma$ with $\eta(A_K) > 0$ and $E_\omega[e^{\lambda\tau_1}] \ge K$ for $\omega \in A_K$. Let $B_M := \{\omega : \omega_0 = \omega_{-1} = \omega_{-2} = \ldots = \omega_{-M} = \omega_{\min}\}$. If η is a product measure, $\eta(B_M) = (\eta_0(\omega_{\min}))^M > 0$. If η is not a product measure, our assumption on η implies that $\eta(B_M) > 0$ also, since B_M depends only on $\omega_0, \omega_{-1}, \ldots, \omega_{-M}$. For $\omega \in B_M$, we have, using a coupling argument, that

$$E_\omega[e^{\lambda\tau_1}] \ge E_{\overline{\omega}_{\min}}[e^{\lambda\tau_1} \mathbf{1}_{\min_k X_k \ge -M}] \ge K \tag{45}$$

for $M = M(K)$ big enough, since

$$\lim_{M\to\infty} E_{\overline{\omega}_{\min}}[e^{\lambda\tau_1} \mathbf{1}_{\min_k X_k \ge -M}] = E_{\overline{\omega}_{\min}}[e^{\lambda\tau_1}] = \infty\,.$$

This proves the first step by taking $A_K = B_{M(K)}$. Let now $\omega \in B_{M(K)+1}$. If $E_\omega[e^{\lambda\tau_1}] < \infty$ then

$$E_\omega[e^{\lambda\tau_1}] \ge \omega_{\min} e^\lambda + (1 - \omega_{\min}) e^\lambda K E_\omega[e^{\lambda\tau_1}]$$

and this is a contradiction if $(1 - \omega_{\min})e^\lambda K > 1$. If $\eta_0(\omega_{\min}) = 0$, one has to approximate.
In order to show $u_{\text{crit}} < \infty$, it is enough to prove that $\int E_\omega[\tau_1 e^{\lambda_{\text{crit}}\tau_1}]\eta(d\omega) < \infty$. Let $\psi(\lambda, \omega, C) := E_\omega[(\tau_1 \wedge C)e^{\lambda\tau_1}]$. The same recursion as in (17) yields, for $\lambda \le \lambda_{\text{crit}}$,

$$\begin{aligned}\psi(\lambda, \omega, C) \quad &\le \quad \omega_0 e^\lambda + (1 - \omega_0)e^\lambda \varphi(\lambda, \theta^{-1}\omega)\varphi(\lambda, \omega) \\ &\qquad + (1 - \omega_0)e^\lambda \psi(\lambda, \theta^{-1}\omega, C)\varphi(\lambda, \omega) + (1 - \omega_0)e^\lambda \psi(\lambda, \omega, C)\varphi(\lambda, \theta^{-1}\omega)\end{aligned}$$

Note that for $\lambda \le \lambda_{\rm crit}$, $\varphi(\lambda,\omega) \le \overline{\varphi}(\lambda)$ for all ω. This implies

$$\psi(\lambda,\omega,C) \le a(\lambda,\omega_0) + b(\lambda,\omega_0)\psi(\lambda,\theta^{-1}\omega,C) \tag{46}$$

where

$$a(\lambda,\omega_0) = \frac{\omega_0 e^{\lambda} + (1-\omega_0)e^{\lambda}(\overline{\varphi}(\lambda))^2}{1-(1-\omega_0)e^{\lambda}\overline{\varphi}(\lambda)}\,,\ \ b(\lambda,\omega_0) = \frac{(1-\omega_0)e^{\lambda}\overline{\varphi}(\lambda)}{1-(1-\omega_0)e^{\lambda}\overline{\varphi}(\lambda)}\,. \tag{47}$$

Iteration of (46) yields, taking $\lambda = \lambda_{\rm crit}$,

$$\psi(\lambda_{\rm crit},\omega_0,C) \le a(\lambda_{\rm crit},\omega_0) + \sum_{j=0}^{\infty} b(\lambda_{\rm crit},\omega_0)\cdots b(\lambda_{\rm crit},\omega_{-j})a(\lambda_{\rm crit},\omega_{-j-1}) \tag{48}$$

But we know that $a(\lambda_{\rm crit},\cdot)$ is bounded, because, using the value $\overline{\varphi}(\lambda_{\rm crit}) = (\frac{\omega_{\rm min}}{1-\omega_{\rm min}})^{1/2}$, we have from (47) that

$$a(\lambda_{\rm crit},\omega_0) = \frac{\omega_0 e^{\lambda_{\rm crit}} + (1-\omega_0)e^{\lambda_{\rm crit}}(\bar{\varphi}(\lambda_{\rm crit}))^2}{1-(1-\omega_0)\frac{1}{2(1-\omega_{\rm min})}} \le \frac{e^{\lambda_{\rm crit}} + e^{\lambda_{\rm crit}}(\bar{\varphi}(\lambda_{\rm crit}))^2}{1/2} < \infty\,.$$

Further, using the same substitution,

$$b(\lambda_{\rm crit},\omega_0) = \frac{1-\omega_0}{1-\omega_0+2(\omega_0-\omega_{\rm min})}. \tag{49}$$

In particular, $0 < b(\lambda_{\rm crit},\omega_0) \le 1$ and, if η is not degenerate, $E_\eta[b(\lambda_{\rm crit},\omega_0)] < 1$. Lemma 5 now enables us to integrate (48) with η and we see that $\int \psi(\lambda_{\rm crit},\omega,C)\eta(d\omega)$ is bounded uniformly in C.
Note that $u_{\rm crit} = \infty$ in the degenerate case since

$$\lim_{\lambda\to\lambda_{\rm crit}} \frac{d}{d\lambda}\log\varphi(\lambda,\overline{\omega}_{\rm min}) = \infty\,.$$

(ii) We use the same argument as in the proof of (i). Assume $\eta_0(\omega_0 \le 1/2) > 0$. Let $\lambda > 0$. In a first step, we show that for each $K > 0$, there is $A_K \subset \Sigma$ with $\eta(A_K) > 0$ and $E_\omega[e^{\lambda\tau_1}] \ge K$ for $\omega \in A_K$. Let $B'_M := \{\omega : \omega_0 \le 1/2,\ \omega_{-1} \le 1/2,\ \omega_{-2} \le 1/2,\ \ldots, \omega_{-M} \le 1/2\}$. If η is a product measure, $\eta(B'_M) = (\eta_0(\omega_{\rm min}))^M > 0$. If η is not a product measure, our assumption on η implies that $\eta(B'_M) > 0$ also, since B'_M depends only on $\omega_0, \omega_{-1}, \ldots, \omega_{-M}$. For $\omega \in B'_M$, we have

$$E_\omega[e^{\lambda\tau_1}] \ge E_{(\ldots,1/2,1/2,1/2,\omega_1,\omega_2,\ldots)}[e^{\lambda\tau_1}\mathbf{1}_{\min_k X_k \ge -M}] \ge K$$

for $M = M(K)$ big enough, since

$$\lim_{M\to\infty} E_{(\ldots,1/2,1/2,1/2,\omega_1,\omega_2,\ldots)}[e^{\lambda\tau_1}\mathbf{1}_{\min_k X_k \ge -M}] = E_{(\ldots,1/2,1/2,1/2,\omega_1,\omega_2,\ldots)}[e^{\lambda\tau_1}] = \infty\ .$$

This proves the first step by taking $A_K = B'_{M(K)}$. Let now $\omega \in B'_{M(K)+1}$. If $E_\omega[e^{\lambda\tau_1}] < \infty$ then

$$E_\omega[e^{\lambda\tau_1}] \ge \frac{1}{2}e^{\lambda} + (1-\frac{1}{2})e^{\lambda}KE_\omega[e^{\lambda\tau_1}]$$

and this is a contradiction if $\frac{1}{2}e^{\lambda}K > 1$. If $\eta_0(\omega_0 \le 1/2) = 0$, one has to approximate. □

Remark: An inspection of the proof reveals that part i) of Lemma 4 still holds for any $\eta \in M_1^e(\Sigma)^{+,K}$ satisfying $\eta(\{\omega_i \in [\omega_{\rm min},\omega_{\rm min}+\varepsilon]\}_{i=0}^{M_0}) > 0$ for all $\varepsilon > 0$, and $M_0 = M(K_0)$ such that (45) holds with $K_0 = K_0(\omega_{\rm min}) := [2(\omega_{\rm min}/(1-\omega_{\rm min}))^{1/2}]$.

3 Proofs - annealed LDP's for hitting times.

Recall the notation $f(\lambda,\omega) := \log E_\omega[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}] = \log\varphi(\lambda,\omega)$. In what follows, $\omega_{\min}$, $\rho_{\max}$, etc. are always defined in terms of α, whereas if $\alpha \in M_1^e(\Sigma)^{+,K}$ then λ_{crit} is defined as in Lemma 2, while if $\alpha \in M_1^e(\Sigma)^K \setminus M_1^e(\Sigma)^{+,K}$ then $\lambda_{\text{crit}} := \lambda_{\text{crit}}(\alpha^{\text{Inv}})$. Also, unless denoted otherwise, expectations are taken with respect to α or P_α. We recall that $M_1(\Sigma)$ is equipped with the topology of weak convergence, and define the compact set

$$\mathcal{D}_\alpha := \{\mu \in M_1^s(\Sigma)^K : \operatorname{supp}\mu_0 \subseteq \operatorname{supp}\alpha_0\}.$$

Lemma 6 *Assume $\alpha \in M_1^e(\Sigma)^K$ satisfies Assumption (A) and is non-degenerate. Then, the function $(\mu,\lambda) \to \int f(\lambda,\omega)\mu(d\omega)$ is continuous on $\mathcal{D}_\alpha \times (-\infty,\lambda_{\text{crit}}]$.*

Proof of Lemma 6. For $\kappa > 1$, decompose $\varphi(\lambda,\omega)$ as follows:

$$E_\omega[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}] = E_\omega[e^{\lambda\tau_1};\tau_1<\kappa] + E_\omega[e^{\lambda\tau_1};\infty>\tau_1\geq\kappa] := \varphi_1^\kappa(\lambda,\omega) + \varphi_2^\kappa(\lambda,\omega)\,, \tag{50}$$

where $(\lambda,\omega) \to \log\varphi_1^\kappa(\lambda,\omega)$ is bounded and continuous. We also have

$$0 \leq \log\left(1+\frac{\varphi_2^\kappa(\lambda,\omega)}{\varphi_1^\kappa(\lambda,\omega)}\right) \leq \log\left(1+\frac{\varphi_2^\kappa(\lambda_{\text{crit}},\omega)}{\omega_{\min}e^\lambda}\right).$$

Hence, the required continuity of the function $(\mu,\lambda) \to \int f(\lambda,\omega)\mu(d\omega)$ will follow from (50) as soon as we show that for any fixed constant $C_1 < 1$,

$$\lim_{\kappa\to\infty}\sup_{\mu\in\mathcal{D}_\alpha}\int\log\left(1+\frac{\varphi_2^\kappa(\lambda_{\text{crit}},\omega)}{C_1}\right)\mu(d\omega) = 0\,. \tag{51}$$

If $\alpha \in M^{1/2}$ (recall (13)), then $\lambda_{\text{crit}} = 0$ and then one finds for each $\epsilon > 0$ a $\kappa_\mu = \kappa(\epsilon,\mu)$ large enough such that,

$$E_\mu\left[\log\left(1+\frac{P_\omega[\infty>\tau_1>\kappa_\mu]}{P_\omega[\tau_1<\infty]}\right)\right] < \epsilon\,.$$

Further, in this situation, for ergodic μ, c.f. (31),

$$\int f(0,\omega)\mu(d\omega) = \Big(-\int\log\rho_0(\omega)\mu(d\omega)\Big)\wedge 0\,. \tag{52}$$

In particular, $\mu \mapsto \int f(0,\omega)\mu(d\omega)$, being linear, is uniformly continuous on the compact set $\mathcal{D}_\alpha$. Therefore, using (50), one sees that for each $\mu \in \mathcal{D}_\alpha$ one can construct a neighborhood B_μ of μ such that, for each $\nu \in B_\mu \cap \mathcal{D}_\alpha$,

$$E_\nu\left[\log\left(1+\frac{P_\omega[\infty>\tau_1>\kappa_\mu+1]}{P_\omega[\tau_1<\infty]}\right)\right] < \epsilon\,.$$

By compactness, it follows that there exists an $\kappa = \kappa(\epsilon)$ large enough such that, for all $\mu \in \mathcal{D}_\alpha$,

$$E_\mu\left[\log\left(1+\frac{P_\omega[\infty>\tau_1>\kappa]}{P_\omega[\tau_1<\infty]}\right)\right] < \epsilon\,.$$

Using the inequality $\log(1+cx) \leq c\log(1+x)$, valid for $x \geq 0$, $c \geq 1$, one finds that for κ large enough,

$$\sup_{\mu\in\mathcal{D}_\alpha}\int\log\left(1+\frac{\varphi_2^\kappa(0,\omega)}{C_1}\right)\mu(d\omega) \leq \epsilon/C_1\,,$$

proving (51) for $\alpha \in M^{1/2}$.

The case $\alpha \notin M^{1/2}$ is simpler: suppose $\omega_{\min} > 1/2$. Then, with $\lambda \geq 0$, because $\operatorname{supp} \mu_0 \subseteq \operatorname{supp} \alpha_0$,

$$E_\omega[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty}] \leq E_{\overline{\omega}_{\min}}[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty}] < \infty$$

for μ-a.e. ω, where $\overline{\omega}_{\min} = (\ldots, \omega_{\min}, \omega_{\min}, \omega_{\min}, \ldots)$, and the last inequality is due to Lemma 4. On the other hand, we have $f(\lambda, \omega) \geq \lambda + \log \omega_0$. We show that $(\lambda, \omega) \mapsto \varphi(\lambda, \omega)$ is continuous, which is enough to complete the proof of the lemma. Write as before

$$E_\omega[e^{\lambda \tau_1} \mathbf{1}_{\tau_1 < \infty}] = E_\omega[e^{\lambda \tau_1}; \tau_1 < \kappa] + E_\omega[e^{\lambda \tau_1}; \infty > \tau_1 \geq \kappa] \tag{53}$$

and observe that the first term in the right hand side of (53) is continuous as a function of ω and the second term goes to 0 for $\kappa \to \infty$, uniformly in ω. More precisely,

$$E_\omega[e^{\lambda \tau_1}; \infty > \tau_1 \geq \kappa] \leq E_{\overline{\omega}_{\min}}[e^{\lambda_{\text{crit}} \tau_1}; \tau_1 \geq \kappa]$$

where $E_{\overline{\omega}_{\min}}[e^{\lambda_{\text{crit}} \tau_1}] < \infty$ and therefore $P_{\overline{\omega}_{\min}}[\tau_1 \geq \kappa] \to_{\kappa \to \infty} 0$ due to the transience of the random walk under the measure $\eta = \delta_{\overline{\omega}_{\min}}$. If $\omega_{\max} < 1/2$, apply the same arguments for α^{Inv}. $\square$

Proof of Theorem 5.

Upper bounds: We begin by proving an upper bound for $\frac{1}{n} \log P\left[\frac{1}{n} \sum_{j=1}^n \tau_j \leq u\right]$, where $1 < u < \infty$. We have, for $\lambda \leq 0$,

$$P\left[\frac{1}{n} \sum_{j=1}^n \tau_j \leq u\right] \leq E\left[\exp\left(\lambda \sum_{j=1}^n \tau_j\right) \mathbf{1}_{\tau_j < \infty, j=1,\ldots,n}\right] e^{-\lambda n u} \tag{54}$$

But,

$$\begin{aligned} E\left[\exp\left(\lambda \sum_{j=1}^n \tau_j\right) \mathbf{1}_{\tau_j < \infty, j=1,\ldots,n}\right] &= E\left[\prod_{j=1}^n E_\omega\left[e^{\lambda \tau_j} \mathbf{1}_{\tau_j < \infty}\right]\right] \\ &= E\left[\exp\left(\sum_{j=0}^{n-1} f(\lambda, \theta^j \omega)\right)\right] = E\left[\exp\left(n \int f(\lambda, \omega) R_n(d\omega)\right)\right] \end{aligned}$$

where $R_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\theta^j \omega} \in M_1(\Sigma)$ denotes the empirical field.

By assumption, the distributions of R_n satisfy a LDP with rate function $h(\cdot|\alpha)$. Lemma 6 ensures that we can apply Varadhan's lemma (see [4, Lemma 4.3.6]) to get

$$\limsup_{n \to \infty} \frac{1}{n} \log E\left[\exp\left(n \int f(\lambda, \omega) R_n(d\omega)\right)\right] \leq \sup_{\eta \in M_1^s(\Sigma)} \left[\int f(\lambda, \omega) \eta(d\omega) - h(\eta|\alpha)\right]. \tag{55}$$

Going back to (54), this yields the upper bound

$$\begin{aligned} \limsup_{n \to \infty} \frac{1}{n} \log P\left[\frac{1}{n} \sum_{j=1}^n \tau_j \leq u\right] &\leq \inf_{\lambda \leq 0} \sup_{\eta \in M_1^s(\Sigma)} \left[\int f(\lambda, \omega) \eta(d\omega) - h(\eta|\alpha) - \lambda u\right] \\ &= -\sup_{\lambda \leq 0} \inf_{\eta \in M_1^s(\Sigma)} \left[G(\lambda, \eta, u) + h(\eta|\alpha)\right]. \end{aligned} \tag{56}$$

Since $\mu \to -\int f(\lambda,\omega)\mu(d\omega) + h(\mu|\alpha)$ is lower semi-continuous and $M_1(\Sigma)$ is compact, the infimum in (56) is achieved for each λ, on measures with support of their marginal included in K, for otherwise $h(\eta|\alpha) = \infty$. Further, by (16), the supremum over λ can be taken over a compact set (recall that $\infty > u > 1$!). Hence, by the Minimax Theorem (see [4, Pg. 151] for Sion's version), the min-max is equal to the max-min in (56). Further, since taking first the supremum in λ in the right hand side of (56) yields a lower semicontinuous function, an achieving $\bar{\eta}$ exists, and then, due to compactness, there exists actually an achieving pair $\bar{\lambda}, \bar{\eta}$. We will show below that the infimum may be taken over ergodic measures only, that is

$$\inf_{\eta \in M_1^s(\Sigma)^K} \sup_{\lambda \le 0} (G(\lambda,\eta,u) + h(\eta|\alpha)) = \inf_{\eta \in M_1^e(\Sigma)^K} \sup_{\lambda \le 0} (G(\lambda,\eta,u) + h(\eta|\alpha)) \tag{57}$$

Then,

$$(56) = -\inf_{\eta \in M_1^e(\Sigma)^K} \sup_{\lambda \le 0} (G(\lambda,\eta,u) + h(\eta|\alpha)) = -\inf_{\eta \in M_1^e(\Sigma)^K} \inf_{w \le u} \left[I_\eta^{\tau,q}(w) + h(\eta|\alpha) \right] , \tag{58}$$

where the second equality is due to (40). Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left[\frac{1}{n} \sum_{j=1}^n \tau_j \le u \right] \le -\inf_{w \le u} \inf_{\eta \in M_1^e(\Sigma)^K} \left[I_\eta^{\tau,q}(w) + h(\eta|\alpha) \right] = -\inf_{w \le u} I_\alpha^{\tau,a}(w). \tag{59}$$

Turning to the proof of (57), we have, due to Assumption (A2), a sequence of ergodic measures with $\eta^n \to \bar{\eta}$ and $h(\eta^n|\alpha) \to h(\bar{\eta}|\alpha)$. Let λ_n be the maximizers in (57) corresponding to η^n. We have

$$\inf_{\eta \in M_1^e(\Sigma)^K} \sup_{\lambda \le 0} \left(\left[\lambda u - \int f(\lambda,\omega)\eta(d\omega) \right] + h(\eta|\alpha) \right) \le \left[\lambda_n u - \int f(\lambda_n,\omega)\eta^n(d\omega) \right] + h(\eta^n|\alpha) \tag{60}$$

W.l.o.g. we can assume, by taking a subsequence, that $\lambda_n \to \lambda^* \le 0$. Using the joint continuity in Lemma 6, we have, for $\epsilon > 0$ and $n \ge N_0(\epsilon)$,

$$\begin{aligned}
\lambda_n u - \int f(\lambda_n,\omega)\eta^n(d\omega) + h(\eta^n|\alpha) &\le \left[\lambda^* u - \int f(\lambda^*,\omega)\bar{\eta}(d\omega) \right] + h(\bar{\eta}|\alpha) + \epsilon \\
&\le \inf_{\eta \in M_1^s(\Sigma)^K} \sup_{\lambda \le 0} \left(\left[\lambda u - \int f(\lambda,\omega)\eta(d\omega) \right] + h(\eta|\alpha) \right) + \epsilon
\end{aligned}$$

But this shows the equality in (57), since the reverse inequality there is trivial. This completes the proof of the upper bound for the lower tail (the case $u = 1$ being handled directly by noting that $n^{-1}\sum_{j=1}^n \tau_j \le 1$ implies that $\tau_j = 1, j = 1, \ldots, n$).

We next turn our attention to the upper bound for the upper tail, that is to $\frac{1}{n} \log P\left[\infty > \frac{1}{n} \sum_{j=1}^n \tau_j \ge u \right]$, where $1 < u < \infty$. We have, for $\lambda \ge 0$,

$$P\left[\infty > \frac{1}{n} \sum_{j=1}^n \tau_j \ge u \right] \le E\left[\exp\left(\lambda \sum_{j=1}^n \tau_j \right) \mathbf{1}_{\tau_j < \infty, j=1,\ldots,n} \right] e^{-\lambda n u} \tag{61}$$

But

$$\begin{aligned}
E\left[\exp\left(\lambda \sum_{j=1}^n \tau_j \right) \mathbf{1}_{\tau_j < \infty, j=1,\ldots,n} \right] &= E\left[\prod_{j=1}^n E_\omega \left[e^{\lambda \tau_j} \mathbf{1}_{\tau_j < \infty} \right] \right] \\
&= E\left[\exp\left(\sum_{j=0}^{n-1} f(\lambda, \theta^j \omega) \right) \right] = E\left[\exp\left(n \int f(\lambda,\omega) R_n(d\omega) \right) \right] .
\end{aligned}$$

Lemma 6 now ensures that we can apply Varadhan's lemma (see [4, Lemma 4.3.6]) to get

$$\limsup_{n\to\infty} \frac{1}{n} \log E\left[\exp\left(n\int f(\lambda,\omega)R_n(d\omega)\right)\right] \le \sup_{\eta\in M_1^s(\Sigma)} \left[\int f(\lambda,\omega)\eta(d\omega) - h(\eta|\alpha)\right]. \tag{62}$$

(The r.h.s. in (62) is $+\infty$ if $\lambda > \lambda_{\rm crit}(\alpha)$, simply by choosing $\eta=\alpha$). Going back to (61), this yields the upper bound

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log P\left[\infty > \frac{1}{n}\sum_{j=1}^{n}\tau_j \ge u\right] &\le \inf_{\lambda\ge 0}\sup_{\eta\in M_1^s(\Sigma)} \left[\int f(\lambda,\omega)\eta(d\omega) - h(\eta|\alpha) - \lambda u\right] \\ &= -\sup_{\lambda\ge 0}\inf_{\eta\in M_1^s(\Sigma)} \left[G(\lambda,\eta,u) + h(\eta|\alpha)\right]. \end{aligned} \tag{63}$$

Since $\mu \to -\int f(\lambda,\omega)\mu(d\omega) + h(\mu|\alpha)$ is lower semi-continuous and $M_1(\Sigma)$ is compact, the infimum in (63) is achieved for each λ, on measures with support of their marginal included in K, for otherwise $h(\eta|\alpha)=\infty$. Since (as can be checked using $\eta=\alpha$), the supremum over λ can be taken over the compact set $[0,\lambda_{\rm crit}]$ which depends only on α, there exists a pair $\bar\lambda,\bar\eta$ which achieves the infimum and the supremum in (63). The Minimax Theorem (see [4, Pg. 151]) implies that the infimum and the supremum in (63) can be exchanged. Exactly as we showed (57), we prove that

$$\inf_{\eta\in M_1^s(\Sigma)^K}\sup_{\lambda\ge 0}\left(G(\lambda,\eta,u)+h(\eta|\alpha)\right) = \inf_{\eta\in M_1^e(\Sigma)^K}\sup_{\lambda\ge 0}\left(G(\lambda,\eta,u)+h(\eta|\alpha)\right) \tag{64}$$

Then,

$$(63) = -\inf_{\eta\in M_1^e(\Sigma)^K}\sup_{\lambda\ge 0}\left(G(\lambda,\eta,u)+h(\eta|\alpha)\right) = -\inf_{\eta\in M_1^e(\Sigma)^K}\inf_{w\ge u}\left[I_\eta^{\tau,q}(w)+h(\eta|\alpha)\right], \tag{65}$$

where the last equality is due to (41). Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left[\infty > \frac{1}{n}\sum_{j=1}^{n}\tau_j \ge u\right] \le -\inf_{w\ge u}\inf_{\eta\in M_1^e(\Sigma)^K}\left[I_\eta^{\tau,q}(w)+h(\eta|\alpha)\right] = -\inf_{w\ge u} I_\alpha^{\tau,a}(w). \tag{66}$$

This completes the proof of the upper bound for the upper tail. Since we show below that $I_\alpha^{\tau,a}(\cdot)$ is convex, the upper bound in Theorem 5 is established.

Proof of the lower bounds. We will use the following standard argument.

Lemma 7 *Let P be a probability distribution, $(\mathcal{F}_n)$ be an increasing sequence of σ-fields and A_n be $\mathcal{F}_n$-measurable sets, $n=1,2,3,\ldots$. Let (Q_n) be a sequence of probability distributions such that $Q_n[A_n]\to 1$ and*

$$\limsup_{n\to\infty} \frac{1}{n} H(Q_n|P)\Big|_{\mathcal{F}_n} \le h$$

where $H(\cdot|P)\Big|_{\mathcal{F}_n}$ denotes the relative entropy w.r.t. P on the σ-field $\mathcal{F}_n$ and h is a positive number. Then we have

$$\liminf_{n\to\infty} \frac{1}{n} \log P[A_n] \ge -h.$$

Proof of Lemma 7. From the basic entropy inequality ([6], p. 423),

$$Q_n[A_n] \le \frac{\log 2 + H(Q_n|P)\Big|_{\mathcal{F}_n}}{\log(1+1/P[A_n])}, \quad A_n\in\mathcal{F}_n,$$

we have $-Q_n[A_n]\log P[A_n] \le \log 2 + H(Q_n|P)\Big|_{\mathcal{F}_n}$. Dividing by n and taking limits we obtain the desired result. $\square$

For $\eta \in M_1^e(\Sigma)^+$, fix $u+1 < M < \infty$, define $\tilde{Q}_\omega$ as in the proof of the lower bound of Theorem 4, and let $\tilde{Q}_\eta = \tilde{Q}_\omega \otimes \eta(d\omega)$. Let $A_n = \{|n^{-1}\sum_{j=1}^n \tau_j - u| < \delta\}$. We know already that

$$\tilde{Q}_\omega[A_n^c] \underset{n\to\infty}{\longrightarrow} 0\,,\ \eta - \text{a.s.}\,,$$

and this implies

$$\tilde{Q}_\eta[A_n^c] \underset{n\to\infty}{\longrightarrow} 0\,.$$

Let $\mathcal{F}_n := \sigma(\{\tau_i\}_{i=1}^n, \{\omega_j\}_{j=-M}^n)$, $\mathcal{F}_n^\omega = \sigma(\{\omega_j\}_{j=-M}^n)$. Note that

$$\tilde{Q}_\eta|_{\mathcal{F}_n} = \tilde{Q}_\omega|_{\mathcal{F}_n} \otimes \eta|_{\mathcal{F}_n^\omega}\,.$$

Hence,

$$H(\tilde{Q}_\eta|P)\Big|_{\mathcal{F}_n} = H(\eta|\alpha)\Big|_{\mathcal{F}_n^\omega} + \int H(\tilde{Q}_\omega|P_\omega)\Big|_{\mathcal{F}_n}\eta(d\omega)\,. \tag{67}$$

Considering the second term in (67), we have

$$\begin{aligned}\frac{1}{n}\int H(\tilde{Q}_\omega|P_\omega)\Big|_{\mathcal{F}_n}\eta(d\omega) \quad &= -\frac{1}{n}\int \log Z_{n,\omega}\eta(d\omega) + \lambda_M(u)\int \frac{1}{n}\sum_{j=1}^n \tau_j d\tilde{Q}_\omega\eta(d\omega)\\ &= -\frac{1}{n}\int\sum_{j=1}^n \log\tilde{\varphi}_M(\lambda_M(u),\theta^{j-1}\omega)\eta(d\omega) + \lambda_M(u)\int\frac{1}{n}\sum_{j=1}^n \tau_j d\tilde{Q}_\omega\eta(d\omega)\end{aligned}$$

and we see, as in the proof of the lower bound of Theorem 4, that

$$\frac{1}{n}\int H(\tilde{Q}_\omega|P_\omega)\Big|_{\mathcal{F}_n}\eta(d\omega) \underset{n\to\infty}{\longrightarrow} \lambda_M(u)u - \tilde{\Lambda}_M(\lambda_M(u)) \le I_M(u) - C_M$$

We already know that

$$\limsup_{M\to\infty}(I_M(u) - C_M) \le I_\eta^{\tau,q}(u)\,,$$

while, considering the first term in (67), we know that

$$\limsup_{n\to\infty}\frac{1}{n}H(\eta|\alpha)\Big|_{\mathcal{F}_n^\omega} = h(\eta|\alpha)\,.$$

Hence,

$$\limsup_{M\to\infty}\limsup_{n\to\infty}\frac{1}{n}H(\tilde{Q}_\eta|P)\Big|_{\mathcal{F}_n} \le I_\eta^{\tau,q}(u) + h(\eta|\alpha)\,.$$

and we can now apply the standard argument. As in the quenched case, one derives the LDP lower bound for $\eta \in M_1^e(\Sigma)^K \setminus M_1^e(\Sigma)^{+,K}$ by repeating the above argument with the required (obvious) modifications.

Finally, we prove the convexity of $I_\alpha^{\tau,a}(\cdot)$. Note that the function

$$\sup_{\lambda\in\mathbb{R}}\inf_{\eta\in M_1^s(\Sigma)^K}[G(\lambda,\eta,u) + h(\eta|\alpha)] = \sup_{\lambda\in\mathbb{R}}\left[\lambda u + \inf_{\eta\in M_1^s(\Sigma)^K}\left(-\int f(\lambda,\omega)\eta(d\omega) + h(\eta|\alpha)\right)\right], \tag{68}$$

being a supremum over affine functions in u, is clearly convex in u, while one shows, exactly as in (57), that

$$\inf_{\eta\in M_1^s(\Sigma)^K}\sup_{\lambda\in\mathbb{R}}\left[G(\lambda,\eta,u)+h(\eta|\alpha)\right]=\inf_{\eta\in M_1^e(\Sigma)^K}\sup_{\lambda\in\mathbb{R}}\left[G(\lambda,\eta,u)+h(\eta|\alpha)\right] \tag{69}$$

and therefore

$$\inf_{\eta\in M_1^s(\Sigma)^K}\sup_{\lambda\in\mathbb{R}}\left[G(\lambda,\eta,u)+h(\eta|\alpha)\right]=\inf_{\eta\in M_1^e(\Sigma)^K}\left[I_\eta^{\tau,q}(u)+h(\eta|\alpha)\right]=I_\alpha^{\tau,a}(u)\,.$$

Recalling that supremum and infimum in (68) can be exchanged, this completes the proof of Theorem 5. □

4 Proofs - LDP's for X_n and functional LDP's.

The results in this section are relatively straightforward applications of the work done previously. We thus emphasize in the proofs only the new elements which need to be introduced.

Proof of Theorem 1. By symmetry, it is enough to consider $\eta\in M_1^e(\Sigma)^{+,K}$.

1. Note that $I_\eta^q(0^+)=\lambda_{\text{crit}}$ by Lemma 2, (5) and (6). Further,

$$\lim_{v\to 0^-} I_\eta^q(v)=\lim_{v\to 0}|v|I_\eta^{-\tau,q}(\frac{1}{|v|})=\lim_{v\to 0^+}\left[vI_\eta^{\tau,q}(\frac{1}{v})+v\int\log\rho_0(\omega)\eta_0(d\omega)\right],$$

   and hence $I_\eta^q(\cdot)$ is continuous at 0. Using the convexity of $I_\eta^{\tau,q}$ and the fact that $x\mapsto xf(1/x)$ is convex if f is convex, one sees that I_η^q is convex on $(0,1]$ and on $[-1,0)$, separately. Finally, $(I_\eta^q)'(0^+)=(I_\eta^q)'(0^-)-\int\log\rho_0(\omega)\eta_0(d\omega)\ge(I_\eta^q)'(0^-)$, establishing the convexity of I_η^q on $[-1,1]$.

2. Let $v>v_\eta$. We have

$$P_\omega\left[\frac{X_n}{n}\ge v\right]\le P_\omega\left[T_{\lfloor nv\rfloor}\le n\right]=P_\omega\left[\frac{1}{\lfloor nv\rfloor}\sum_{j=1}^{\lfloor nv\rfloor}\tau_j\le\frac{n}{\lfloor nv\rfloor}\right].$$

   Theorem 4 now implies

$$\limsup_{n\to\infty}\frac{1}{n}\log P_\omega\left[\frac{X_n}{n}\ge v\right]\le -vI_\eta^{\tau,q}\left(\frac{1}{v}\right).$$

   In the same way, we have for any $|v-v_\eta|/2>\delta>0$ and $0<\epsilon<\delta/2$,

$$P_\omega\left[(v+\delta)\ge\frac{X_n}{n}\ge(v-\delta)\right]\ge P_\omega\left[(1-\epsilon)n\le T_{\lceil nv\rceil}\le n\right],$$

   hence, Theorem 4 implies

$$\liminf_{n\to\infty}\frac{1}{n}\log P_\omega\left[\frac{X_n}{n}\in(v-\delta,v+\delta)\right]\ge -vI_\eta^{\tau,q}\left(\frac{1-\epsilon}{v}\right),\ \eta-a.e.\,\omega\,,$$

   and the lower bound follows by letting $\epsilon\to 0$.

3. Assume $v_\eta > 0$. Let $0 < v < v_\eta$. We have for any $\delta/2 > \epsilon > 0$, and $\delta < |v - v_\eta|/2$,
$$P_\omega\left[(v-\delta) \le \frac{X_n}{n} \le (v+\delta)\right] \ge P_\omega\left[n(1+\epsilon) \ge T_{\lfloor nv \rfloor} \ge n(1-\epsilon)\right].$$
The lower bound follows from Theorem 4.

We now prove the upper bound for $P_\omega[X_n/n \le v], v < v_\eta$. The proof is technically more involved (except if η is locally equivalent to the product of its marginals, see (79)). We start with the case $v = 0$. Let $\epsilon, \delta > 0$, with $\delta < v_\eta$. Then,
$$\begin{aligned} P_\omega[X_n \le 0] &\le P_\omega\left[T_{[n\delta]} \ge n\right] + P_\omega\left[T_{[n\delta]} < n, \frac{X_n}{n} \le 0\right] \\ &\le P_\omega\left[T_{[n\delta]} \ge n\right] + \sum_{1/\epsilon \le k,l;(k+l)\epsilon \le 1/\delta} P_\omega\left[\frac{T_{[n\delta]}}{n\delta} \in [k\epsilon,(k+1)\epsilon[\right] \times \\ &\quad P_{\theta^{[n\delta]}\omega}\left[\frac{T_{-[n\delta]}}{n\delta} \in [l\epsilon,(l+1)\epsilon[\right] \sup_{-2n\delta\epsilon \le m - n(1-(k+l)\delta\epsilon) \le 0} P_\omega[X_m \le 0] \end{aligned} \tag{70}$$
by the strong Markov property. Define the random variable
$$a = \limsup_{n\to\infty} \frac{1}{n} \sup_{m:-2n\delta\epsilon \le m-n \le 0} \log P_\omega[X_m \le 0] \quad ,$$
and note, using the inequality $P_\omega[X_n \le 0] \ge P_\omega[X_m \le 0] \inf_{i\le 0} P_{\theta^i\omega}[X_{n-m} = -(n-m)]$ with a worst-environment estimate, that
$$a - C\delta\epsilon \le \limsup_{n\to\infty} \frac{1}{n} \log P_\omega[X_n \le 0] \le a \tag{71}$$
with $C = -2\log(1-\omega_{\max}) > 0$. The first two probabilities in the right-hand side of (70) will be estimated using Theorem 4. By convexity, the rate functions $I_\eta^{\pm\tau,q}$ are continuous, so that the oscillation
$$w(\delta;\epsilon) = \max\{|I_\eta^{\tau,q}(u) - I_\eta^{\tau,q}(u')| + |I_\eta^{-\tau,q}(u) - I_\eta^{-\tau,q}(u')|; u, u' \in [1, 1/\delta], |u-u'| \le \epsilon\}$$
tends to 0 with ϵ, for all fixed δ. From the proof of Theorem 4, it is not difficult to see that the third term in the right-hand side of (70) can be estimated similarly (it does not cause problems to consider $P_{\theta^{[n\delta]}\omega}$ instead of P_ω):
$$\limsup_{n\to\infty} \frac{1}{n} \log P_{\theta^{[n\delta]}\omega}\left[\frac{T_{-[n\delta]}}{n\delta} \in [l\epsilon,(l+1)\epsilon[\right] \le -\delta\left(I_\eta^{-\tau,q}(l\epsilon) - w(\delta;\epsilon)\right) \quad \eta - a.e.\omega.$$
Finally, we get from (71) and (70)
$$a \le C\delta\epsilon + \max\{-I_\eta^q(\delta), \max_{1/\epsilon \le k,l;(k+l)\epsilon \le 1/\delta} [-\delta\epsilon(kI_\eta^q(1/k\epsilon) + lI_\eta^q(-1/l\epsilon)) + 2\delta w(\delta;\epsilon) + (1-(k+l+2)\delta\epsilon)a]\} \; .$$
By convexity and since $\delta \le v_\eta$, it holds $kI_\eta^q(1/k\epsilon) + lI_\eta^q(-1/l\epsilon) \ge (k+l)I_\eta^q(0) \ge (k+l)I_\eta^q(\delta)$, and therefore $a' := a + I_\eta^q(\delta)$ is such that
$$a' \le C\delta\epsilon + \left(\max_{1/\epsilon \le k,l;(k+l)\epsilon \le 1/\delta} [2\delta w(\delta;\epsilon) + 2\delta\epsilon I_\eta^q(\delta) + (1-(k+l+2)\delta\epsilon)a']\right)^+ \; .$$
Computing the maximum for positive a', we derive that $2a' \le C\epsilon + 2(w(\delta;\epsilon) + \epsilon I_\eta^q(\delta))$. Letting now $\epsilon \to 0$ and $\delta \to 0$, we conclude that
$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega\left[X_n \le 0\right] \le -I_\eta^q(0) \quad , \quad \eta - a.s. \tag{72}$$

For an arbitrary $v \in [0, v_\eta[$, we write

$$P_\omega\Big[\frac{X_n}{n} \le v\Big] \le P_\omega\Big[T_{[nv]} \ge n\Big] + \sum_{k: v/\epsilon \le k \le 1/\epsilon} P_\omega\Big[\frac{T_{[nv]}}{n} \in [k\epsilon, (k+1)\epsilon[\Big] \sup_{m: n(1-(k+1)\epsilon) \le m \le n(1-k\epsilon)} P_{\theta^{[nv]}\omega}\Big[X_m \le 0\Big] \tag{73}$$

where the two first probabilities in the right-hand side can be estimated using Theorem 4, and the last one as in (72), following the lines above. This yields

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log P_\omega\Big[X_n \le nv\Big] &\le \limsup_{\epsilon\to 0} \Big(I_\eta^q(v) \vee \max_{v/\epsilon \le k \le 1/\epsilon} [-k\epsilon I_\eta^q(v/k\epsilon) - (1-k\epsilon) I_\eta^q(0)] \Big) \\ &= -I_\eta^q(v) \ , \end{aligned} \tag{74}$$

by convexity.

4. The upper bound for general subsets of $[0,1]$ follows again by noting that the rate function $I_\eta^q(\cdot)$ is convex.

5. The proof concerning deviations to the left follows the same path, replacing T_n by T_{-n}. □

Proof of Theorem 2. All the statements follow from Theorem 5 by a rerun of the derivation of Theorem 1 from Theorem 4, except for the convexity of I_α^a and also the upper bound similar to (74). From the convexity of $I_\alpha^{\pm\tau,a}$ it is clear that I_α^a is convex separately on $[-1,0]$ and on $[0,1]$. If $\lambda_{\rm crit} = 0$ we have $0 \le I_\alpha^a(0) \le I_\alpha^q(0) = 0$, and then I_α^a is convex on $[-1,1]$ in this case. It remains to consider the case $\lambda_{\rm crit} > 0$. We will assume that $\rho_{\max} < 1$, the case $\rho_{\min} > 1$ being proved with the same arguments for $\alpha^{\rm Inv}$ instead of α. Then for any η with $h(\eta|\alpha) < \infty$ (and in particular, $\rho_{\max}(\eta) < 1$),

$$I_\eta^{\tau,q}(u) \ge \lambda_{\rm crit} u - \int \log \varphi(\lambda_{\rm crit}, \omega)\eta(d\omega) \ge \lambda_{\rm crit} u - \log \bar\varphi(\lambda_{\rm crit}) \, ,$$

where $\bar\varphi(\lambda_{\rm crit}) = E_{\bar\omega_{\min}}[e^{\lambda_{\rm crit}\tau_1}] < \infty$, as in the proof of Lemma 4. Hence,

$$I_\alpha^a(0) = \lim_{u\to\infty} u^{-1} I_\alpha^{\tau,a}(u) \ge \lambda_{\rm crit} \, .$$

Since we already know that $I_\alpha^a(0) \le I_\alpha^q(0) = \lambda_{\rm crit}$, we conclude that $I_\alpha^a(0) = \lambda_{\rm crit}$. Due to separate convexity it is enough, in order to prove convexity of I_α^a on $[-1,1]$, to show that for $v > 0$

$$I_\alpha^a(v) + I_\alpha^a(-v) \ge 2 I_\alpha^a(0) \tag{75}$$

since this will imply that $I_\alpha^{a\prime}(0^-) \le I_\alpha^{a\prime}(0^+)$. But

$$\begin{aligned} I_\alpha^a(v) &= v I_\alpha^{\tau,a}(\frac{1}{v}) = v \inf_{\eta \in M_1^e(\Sigma)^K} \Big[I_\eta^{\tau,q}(\frac{1}{v}) + h(\eta|\alpha) \Big] \\ &\ge v \inf_{\eta \in M_1^e(\Sigma)^K} \Big[\lambda_{\rm crit} \frac{1}{v} - \int \log \varphi(\lambda_{\rm crit}, \omega)\eta(d\omega) + h(\eta|\alpha) \Big] \end{aligned}$$

by the substitution $\lambda = \lambda_{\rm crit}$ in (5). With a similar computation for $I_\alpha^a(-v)$ we then get

$$I_\alpha^a(v) + I_\alpha^a(-v) \ge 2\lambda_{\rm crit} + v \inf_{\eta,\eta' \in M_1^e(\Sigma)^K} \Big[-\int \log \varphi(\lambda_{\rm crit}, \omega)\eta(d\omega) - \int \log \tilde\varphi(\lambda_{\rm crit}, \omega)\eta'(d\omega) + h(\eta|\alpha) + h(\eta'|\alpha) \Big] \tag{76}$$

with $\tilde{\varphi}$ defined in (28), and we finally derive (75) by showing that

$$\int \log \varphi(\lambda_{\rm crit}, \omega)\eta(d\omega) + \int \log \tilde{\varphi}(\lambda_{\rm crit}, \omega)\eta'(d\omega) \leq 0 \tag{77}$$

for all $\eta, \eta' \in M_1^e(\Sigma)^K$ such that $h(\eta|\alpha) + h(\eta'|\alpha) < \infty$. Recall (34) and note that

$$\tilde{\varphi}(\lambda_{\rm crit}, \omega)\varphi(\lambda_{\rm crit}, \theta^{-1}\omega) \leq 1$$

holds for all ω with $\omega_i \geq \omega_{\min}, i \in \mathbb{Z}$. The point here is, that $\varphi(\lambda_{\rm crit}, \theta^{-1}\omega)$ [resp., $\tilde{\varphi}(\lambda_{\rm crit}, \omega)$] is measurable with respect to the σ-algebra $\mathcal{F}^-$ generated by $\omega_i, i < 0$ [resp., $\mathcal{F}^+$ generated by $\omega_i, i \geq 0$]. Taking logarithms in the last inequality and integrating for the measure $\eta_{|_{\mathcal{F}^-}} \otimes \eta'_{|_{\mathcal{F}^+}}$, we get

$$\int \log \varphi(\lambda_{\rm crit}, \theta^{-1}\omega)\eta(d\omega) + \int \log \tilde{\varphi}(\lambda_{\rm crit}, \omega)\eta'(d\omega) \leq 0$$

proving (77) since η is translation invariant. Granted with the convexity, we now complete the proof of the Theorem by showing that for $v \in [0, v_\alpha[$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P\Big[X_n \leq nv\Big] \leq -I_\alpha^a(v) \ , \tag{78}$$

the statement analogous to (74). But the strategy is quite different, and much simpler, since α is locally equivalent to the product of its marginals. Indeed we just proved $I_\alpha^a(0) = \lambda_{\rm crit}$, and from Lemma 4, if $\omega_{\min} \leq 1/2$ then $\lambda_{\rm crit} = 0$ and the claim is trivial, but in the opposite case $\omega_{\min} > 1/2$ that we consider now, we have $\lambda_{\rm crit} = \bar{\lambda} := -\frac{1}{2}\log(4\omega_{\min}(1-\omega_{\min})) > 0$. We will use the exponential martingale M_n for the walk in a fixed environment ω,

$$M_n := \exp\left(sX_n - \sum_{k=0}^{n-1} \Gamma(s, \omega_{X_k})\right)$$

with $\Gamma(s,w) = \log(we^s + (1-w)e^{-s})$. Taking $s := (1/2)\log((1-\omega_{\min})/\omega_{\min}) < 0$ we see that $\Gamma(s,.)$ is decreasing, so that a.s., $\Gamma(s, \omega_{X_k}) \leq \Gamma(s, \omega_{\min}) = -\bar{\lambda}$. Therefore,

$$P_\omega[X_m \leq 0] \leq E_\omega[\exp(sX_m)] \leq E_\omega\left[\exp(sX_m - \sum_{k=0}^{m-1}\Gamma(s, \omega_{X_k}))\right]\exp(-m\bar{\lambda}) = \exp(-mI_\alpha^a(0)) \tag{79}$$

for α-a.e. ω. Inserting this in (73) and taking the average over the medium, we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\log P\Big[X_n \leq nv\Big] \leq \limsup_{\epsilon\to 0}\left([-I_\alpha^a(v)] \vee \max_{v/\epsilon \leq k \leq 1/\epsilon}[-k\epsilon I_\alpha^a(v/k\epsilon) - (1-k\epsilon)I_\alpha^a(0)]\right) = -I_\alpha^a(v) \ ,$$

using again convexity. This is (78), and the proof is complete.

□

Proof of Theorem 3. Fix $\Delta > 0$ (eventually, $\Delta \to 0$). For $\phi \in \mathcal{L}$, let

$$\theta_0 = 0,\ \theta_j = \min\{t \geq \theta_{j-1} : |\phi(t) - \phi(\theta_{j-1})| = \Delta\} \wedge 1\ ,\ j = 1, \ldots, J\ ,$$

define $Y_j = \phi(\theta_j)$, and say that $1 \leq j \in I^+$ if $Y_j > Y_{j-1}$ and $j \in I^-$ otherwise. Define next the random times

$$\xi_0 = 0\,,\, \xi_j = \min\left\{k > \xi_{j-1} : X_k = n\Delta\lfloor\frac{Y_j}{\Delta}\rfloor\right\} \wedge n\,, j = 1, \ldots, J\,.$$

Consider the event

$$A^{\phi}_{\Delta,\delta} := \bigcap_{j=1}^{J} \left\{ \left| \frac{1}{n}(\xi_j - \xi_{j-1}) - (\theta_j - \theta_{j-1}) \right| \le \delta \right\} .$$

We begin by proving the

Lemma 8

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_\omega \left[A^{\phi}_{\Delta,\delta} \right] \le - \sum_{j\in I^+} \Delta I^{\tau,q}_{\eta}(\theta_j - \theta_{j-1}) - \sum_{j\in I^-} \Delta I^{-\tau,q}_{\eta}(\theta_j - \theta_{j-1}) \,, \eta \text{ a.e.}, \tag{80}$$

and

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log P_\omega \left[A^{\phi}_{\Delta,\delta} \right] \ge - \sum_{j\in I^+} \Delta I^{\tau,q}_{\eta}(\theta_j - \theta_{j-1}) - \sum_{j\in I^-} \Delta I^{-\tau,q}_{\eta}(\theta_j - \theta_{j-1}) \,, \eta \text{ a.e.}. \tag{81}$$

Proof of Lemma 8. The proof is no more than an exercise in book-keeping. Indeed, let $M^+ = \max Y_j$, $M^- = -\min Y_j$, and note that $[-M^-, M^+] = \cup_{k=1}^{(M^+ + M^-)/\Delta} L_k$, where $L_k = [-M^- + (k-1)\Delta, -M^- + k\Delta]$. With $R_j = [Y_j \wedge Y_{j+1}, Y_j \vee Y_{j+1}]$, one obtains a partition of $j \in I^+$ ($j \in I^-$) into sets K^+_k, (K^-_k), such that $R_j = L_k$ for $j \in K^+_k$ ($j \in K^-_k$). Note that $|K^+_k| - |K^-_k| = 0$ or 1, and $|K^+_k| \le \Delta^{-1}$.

Next, let $\{\tau^\ell_i\}_{\ell=1}^\infty$ be independent (given ω) copies of the random variable τ_i, and, with $\tilde\tau_{-i} = \inf\{t \ge T_i : X_t = i-1\} - T_i$, let $\{\tilde\tau^\ell_{-i}\}_{\ell=1}^\infty$ denote independent (given ω) copies of $\tilde\tau_{-i}$. Then, with respect to P_ω,

$$\begin{aligned} A^{\phi}_{\Delta,\delta} &= \bigcap_{k=1}^{(M^+ + M^-)/\Delta} \left(\left\{ \bigcap_{\ell \in K^+_k} \{ \frac{1}{n} \sum_{i=Y_\ell}^{Y_{\ell+1}} \tau^\ell_i \in (\theta_\ell - \theta_{\ell-1} - \delta, \theta_\ell - \theta_{\ell-1} + \delta) \} \right\} \right. \\ &\qquad \left. \bigcap \left\{ \bigcap_{\ell \in K^-_k} \{ \frac{1}{n} \sum_{i=Y_\ell}^{Y_{\ell+1}} \tilde\tau^\ell_{-i} \in (\theta_\ell - \theta_{\ell-1} - \delta, \theta_\ell - \theta_{\ell-1} + \delta) \} \right\} \right) . \end{aligned}$$

An application of Theorem 4 now yields the lemma. $\square$

Lemma 8 possesses an analogue stated in terms of the process X_t itself. Its proof repeats the same argument and is therefore omitted. For simplicity in notations, we assume that Δ^{-1} is integer valued. Define (note that Δ now denotes discretization in time, not space!)

$$B^{\phi}_{\Delta,\delta} := \bigcap_{j=1}^{1/\Delta} \left\{ \left| \frac{1}{n} X_{nj\Delta} - \phi(j\Delta) \right| \le \delta \right\} .$$

Lemma 9

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_\omega \left[B^{\phi}_{\Delta,\delta} \right] \le - \sum_{j=1}^{1/\Delta} \Delta I^q_\eta\left(\frac{\phi(j\Delta) - \phi((j-1)\Delta)}{\Delta}\right) \,,\ \eta \ - \text{a.e.}, \tag{82}$$

and

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log P_\omega \left[B^{\phi}_{\Delta,\delta} \right] \ge - \sum_{j=1}^{1/\Delta} \Delta I^q_\eta\left(\frac{\phi(j\Delta) - \phi((j-1)\Delta)}{\Delta}\right) \,,\ \eta \ - \text{a.e.} \tag{83}$$

We may now return to the proof of Theorem 3.

1. In view of the compactness of $\mathcal{L}$, the only issue is the lower-semicontinuity of $I_\eta^{q,\mathrm{traj}}$. This however is obvious due to the convexity of $I_\eta^q(\cdot)$ on $\mathbb{R}$.

2. In view of the compactness of $\mathcal{L}$ and the projective limits method, c.f. [4, Ch. 5.1], having established Lemma 9, all that is needed is to prove that for any $\phi \in \mathcal{L}$,

$$\lim_{\Delta\to 0}\sum_{j=1}^{1/\Delta} \Delta I_\eta^q\Big(\frac{\phi(j\Delta)-\phi((j-1)\Delta)}{\Delta}\Big) = \int_0^1 I_\eta^q(\dot\phi(t))dt\,. \tag{84}$$

But this is obvious from dominated convergence since ϕ is differentiable a.e. (Lebesgue) with derivative bounded in absolute value by 1. $\square$

5 Properties of the rate functions and the environment

We gather in this section some detailed properties of the various annealed and quenched rate functions $I_\eta^{\tau,q}$, $I_\alpha^{\tau,a}$, I_η^q, I_α^a encountered in this paper, and properties of the environment in the annealed setup which leads to a large deviation. Throughout, we assume that η is ergodic and locally equivalent to the product of its marginals, whereas $\alpha \in M_1^e(\Sigma)^K$ will be taken to be a **product measure**. Note that under these assumptions, all the above rate functions are convex and (c.f. (7) and (9)),

$$I_\alpha^{\tau,a}(u) \le I_\alpha^{\tau,q}(u)\,, u \ge 1\,, \quad \text{and} \quad I_\alpha^a(v) \le I_\alpha^q(v)\,, v \in [-1,1]\,.$$

5.1 Properties and shape of the rate functions

With some abuse, we say that a measure $\eta \in M_1^e(\Sigma)^K$ is transient to the right (transient to the left, recurrent) if X_n is transient to $+\infty$ (transient to $-\infty$, recurrent), η-a.s. We also introduce the notation $\langle\tau\rangle_\eta = E_\eta[\tau_\omega]$, where we recall that $\tau_\omega = E_\omega[\tau_1|\tau_1 < \infty]$. The following summarizes our main results concerning the quenched rate functions. Additional details, e.g. the precise slopes of certain linear pieces of the rate functions, are mentioned inside the proofs. We remind the reader that the term "increasing" includes the case of not strictly increasing, etc. The reader may wish at this point to look at Figures 1 - 9 that summarize graphically our results.

Proposition 2 *Assume that η is ergodic, locally equivalent to the product of its marginals and non-degenerate, i.e. not concentrated on one point. Then,*

Case A. *$\int \log\rho_0(\omega)\eta(d\omega) = 0$, i.e. η is recurrent. Then, $I_\eta^{\tau,q}$ and I_η^q are strictly convex, $I_\eta^{\tau,q}$ is decreasing on $[1,\infty)$ with $\lim_{u\to\infty} I_\eta^{\tau,q}(u) = 0$, while $I_\eta^q(0) = 0$ and I_η^q increasing on $[0,1]$, decreasing on $[-1,0]$ and I_η^q is symmetric (see Figures 1 and 6).*

Case B. *$\int \log\rho_0(\omega)\eta(d\omega) < 0$, $\langle\tau\rangle_\eta = \infty$, i.e. η is transient to the right with zero speed. Then, $I_\eta^{\tau,q}$ and I_η^q have the same properties as in case A except that I_η^q is not symmetric (see Figure 7).*

Case C. *$\eta \in M^{1/2}$, $\int \log\rho_0(\omega)\eta(d\omega) < 0$, and $\langle\tau\rangle_\eta < \infty$, i.e. η is transient to the right with mixed drifts and positive speed. Then, $I_\eta^{\tau,q}$ is strictly convex and decreasing on $[1,\langle\tau\rangle_\eta]$, while $I_\eta^{\tau,q} = 0$ on $[\langle\tau\rangle_\eta,\infty)$. I_η^q*

is monotone increasing on $[0,1]$*, monotone decreasing on* $[-1,0]$*, strictly convex on* $[-1,-v_\eta]\cup[v_\eta,1]$*,* $I^q_\eta(v)=|v|\left|\int\log\rho_0(\omega)\eta(d\omega)\right|$ *for* $v\in[-v_\eta,0]$*, and* $I^q_\eta=0$ *on* $[0,v_\eta]$ *(see Figures 3 and 8).*

Case D. $\rho_{\max}<1$*, i.e. all drifts point to the right and the walk is transient to* $+\infty$*. Define* $\lambda_{\rm crit}$ *and* $u_{\rm crit}$ *as in Lemma 4. Then,* $I^{\tau,q}_\eta$ *is strictly convex and decreasing on* $[1,\langle\tau\rangle_\eta]$*, is strictly convex and increasing on* $\langle\tau\rangle_\eta,u_{\rm crit}]$*, and is linear on* $[u_{\rm crit},\infty)$*. Further,* $I^{\tau,q}_\eta(v_\eta^{-1})=0$*. The rate function* I^q_η *is decreasing and strictly convex on* $[-1,-u_{\rm crit}^{-1}]$*, decreasing linearly on* $[-u_{\rm crit}^{-1},0]$*, decreasing linearly (with a smaller slope) on* $[0,u_{\rm crit}^{-1}]$*, and strictly convex on* $[u_{\rm crit}^{-1},1]$*, with* $I^q_\eta(v_\eta)=0$ *(see Figures 4 and 9).*

Case E. $\eta\in M^{1/2}$*,* η *is transient to* $-\infty$*, with* $\langle\tau\rangle_\eta$ *either finite or infinite. Then* $I^{\tau,q}_\eta$ *is strictly convex and decreasing on* $[1,\langle\tau\rangle_\eta)$*, and* $I^{\tau,q}_\eta(u)=E_\eta[\log\rho_0]>0$ *for* $u\geq\langle\tau\rangle_\eta$ *(see Figure 2). For* I^q_η*, simply consider Cases B and C under the transformation* $v\mapsto -v$*.*

Case F. $\omega_{\max}<1/2$*, i.e. all drifts point to the left. With* $\langle\tau\rangle_\eta<\infty$*,* $I^{\tau,q}_\eta$ *is strictly convex and decreasing on* $[1,\langle\tau\rangle_\eta]$*, strictly convex and increasing on* $[\langle\tau\rangle_\eta,u_{\rm crit}(\eta^{\rm Inv})]$*, and linearly increasing on* $[u_{\rm crit}(\eta^{\rm Inv}),\infty)$ *(see Figure 5). The rate function* I^q_η *is obtained from Case D by the transformation* $v\mapsto -v$*.*

We note that we do not discuss the regularity properties of I^q_η at 0. In the case of η a product measure, some information on analyticity, obtained by considering the continued fraction defining $\varphi(\lambda,\omega)$, may be found in [11].

Proof of Proposition 2. In Theorem 4 we have already shown that $I^{\tau,q}_\eta$ is decreasing and convex on $[1,E_\eta[\tau_\omega]]$ and increasing and convex on $[E_\eta[\tau_\omega],\infty)$. It follows from Lemma 1, Lemma 2 and Lemma 4 that if $\eta\in M^e_1(\Sigma)^+$ then $I^{\tau,q}_\eta$ is strictly convex on $[1,u_{\rm crit}]$, and that for $u>u_{\rm crit}$ one has that

$$I^{\tau,q}_\eta(u)=\lambda_{\rm crit}u-\int\log\varphi(\lambda_{\rm crit},\omega)\eta(d\omega)\,. \tag{85}$$

Note also that $\lambda_{\rm crit}=0$ in cases A, B, C. Proposition 1 then allows one to make the appropriate transfer of the results to all $\eta\in M^e_1(\Sigma)^K$. Finally, the results for I^q_η follow those for $I^{\tau,q}_\eta$ by using the representation (6), which allows for the transfer of strict convexity from the time variable to the space variable. $\square$

We turn next to the annealed rate functions. Introduce the product measure $\hat\alpha\in M^e_1(\Sigma)^K$ as follows:

$$\frac{d\hat\alpha_0}{d\alpha_0}=\frac{1}{\rho_0}\left(\int\frac{1}{\rho_0(\omega_0)}\alpha_0(d\omega_0)\right)^{-1}\,. \tag{86}$$

Let $u^*:=\langle\tau\rangle_{\hat\alpha}=E_{\hat\alpha}[\tau_\omega]\in[1,\infty]$. Note that the formula in the remark following Lemma 1 implies that $u^*<\infty$ if $\omega_{\max}<1/2$, and define

$$b=\begin{cases} u^*\,, & \omega_{\max}<1/2\,,\\ \infty\,, & \alpha\in M^{1/2}\,,\\ \langle\tau\rangle_\alpha\,, & \omega_{\min}>1/2\,.\end{cases}\quad,\quad b'=\begin{cases}\langle\tau\rangle_\alpha\,, & \alpha\in M^e_1(\Sigma)^+\,,\\ u^*\,, & \hat\alpha\in M^e_1(\Sigma)^-\,,\\ \infty\,, & \text{otherwise}\,.\end{cases}\quad.$$

Always, $b\geq\langle\tau\rangle_\alpha$ and $b'\leq b$. Note that $\hat\alpha\in M^e_1(\Sigma)^-$ implies that

$$0\leq\int\log\rho_0(\omega_0)\hat\alpha_0(d\omega_0)=\frac{\int\rho_0^{-1}\log\rho_0(\omega_0)\alpha_0(d\omega_0)}{\int\rho_0^{-1}(\omega_0)\alpha_0(d\omega_0)}\leq\int\log\rho_0(\omega_0)\alpha_0(d\omega_0)\,,$$

(where we used Jensen's inequality to show the last inequality) and hence $\alpha\in M^e_1(\Sigma)^-$.

Proposition 3 *Assume $\alpha \in M_1^e(\Sigma)^K$ is a product measure, and non-degenerate. Then $I_\alpha^{\tau,a}(\cdot)$ is increasing on $[b,\infty)$, constant on $[b',b]$, and decreasing on $[1,b']$.*

Remark: In fact, one sees from the proof below that whenever $b \neq b'$ then $I_\alpha^{\tau,a}(u) = I_\alpha^{\tau,a}(b')$ for $u > b'$.

More detailed information is also available. The classification of different cases follows the one in Proposition 2.

Proposition 4 *Assume $\alpha \in M_1^e(\Sigma)^K$ is a product measure, and not concentrated on one point.*

Case A. $I_\alpha^{\tau,a}$ is strictly decreasing with limit 0 at infinity. I_α^a is strictly decreasing on $[-1,0]$ and strictly increasing on $[0,1]$, with $I_\alpha^a(0) = 0$ (and is not necessarily symmetric!).

Case B. Same as Case A.

Case C. $I_\alpha^{\tau,a}$ is strictly decreasing on $[1,\langle\tau\rangle_\alpha]$, and is zero on $[\langle\tau\rangle_\alpha,\infty)$. I_α^a is zero on $[0,v_\alpha]$ and is strictly increasing on $[v_\alpha,1]$. Further, define $d = E_\alpha[\rho_0^2]/E_\alpha[\rho_0]$, with $v^* = (1-d)/(1+d)$ if $d < 1$ and $v^* = 0$ otherwise. Then, I_α^a is strictly decreasing on $[-1,0]$ and is linear on $[-v^*,0]$.

Case D. $I_\alpha^{\tau,a}$ is strictly decreasing on $[1,\langle\tau\rangle_\alpha]$ and strictly increasing on $[\langle\tau\rangle_\alpha,\infty)$, with $I_\alpha^{\tau,a}(\langle\tau\rangle_\alpha) = 0$. I_α^a is strictly decreasing on $[-1,v_\alpha]$ and strictly increasing on $[v_\alpha,1]$, with $I_\alpha^a(v_\alpha) = 0$ and $I_\alpha^a(0) = I_\alpha^q(0) = \lambda_{\text{crit}}$. Assume in addition that there exists a non degenerate minimizer η^+ of $\eta \mapsto -E_\eta[f(\lambda_{\text{crit}},\omega)] + h(\eta|\alpha)$ for which the conclusions of Lemma 4, part i) hold true and such that $\lambda_{\text{crit}}(\eta^+) = \lambda_{\text{crit}}$ (with $\lambda_{\text{crit}} := \lambda_{\text{crit}}(\alpha)$). In this case, $I_\alpha^{\tau,a}$ is linear on $[u^+,\infty)$ with $u^+ = u_{\text{crit}}(\eta^+) = E_{\eta^+}\left[E_\omega[\tau_1 e^{\lambda_{\text{crit}}\tau_1}]/E_\omega[e^{\lambda_{\text{crit}}\tau_1}]\right] < \infty$, and I_α^a is linear on $[0,(u^+)^{-1}]$.

Case E. Set $\rho^* = E_\alpha[\rho_0^{-2}]/E_\alpha[\rho_0^{-1}]$, and $u^* = (1+\rho^*)/(1-\rho^*)$ if $\rho^* < 1$, $u^* = \infty$ otherwise. Then, $I_\alpha^{\tau,a}$ is strictly decreasing on $[1,u^*]$ and $I_\alpha^{\tau,a}(u) = -\log E_\alpha[\rho_0^{-1}] > 0$ on $[u^*,\infty)$. For I_α^a, simply consider Cases B and C under the transformation $v \mapsto -v$.

Case F. With $\rho^* < 1$ and u^* as in Case E, $I_\alpha^{\tau,a}$ is strictly decreasing on $[1,u^*]$ and strictly increasing on $[u^*,\infty)$. Further, $I_\alpha^{\tau,a}(u^*) = -\log E_\alpha[\rho_0^{-1}] > 0$. For I_α^a, simply consider Case D under the transformation $v \mapsto -v$. Assume in addition that there exists a non degenerate minimizer η^- of $\eta \mapsto -E_\eta[f(\lambda_{\text{crit}},\omega)] + h(\eta|\alpha)$ for which the conclusions of Lemma 4, part i) hold true and such that $\lambda_{\text{crit}}(\eta^-) = \lambda_{\text{crit}}$. In this case, $I_\alpha^{\tau,a}$ is linear on $[u^-,\infty)$ with $u^- = u_{\text{crit}}(\eta^-) = E_{\eta^-}\left[E_\omega[\tau_1 e^{\lambda_{\text{crit}}\tau_1}\mathbf{1}_{\tau_1<\infty}]/E_\omega[e^{\lambda_{\text{crit}}\tau_1}\mathbf{1}_{\tau_1<\infty}]\right] < \infty$.

Remarks:
1. We will see examples at the end of this section where the additional assumption in Cases D and F is satisfied. Checking instead the stronger assumption that η^+ [resp.,η^-] is locally equivalent to the product of its marginal with $\lambda_{\text{crit}}(\eta^+) = \lambda_{\text{crit}}$ and η^+ [resp., $\lambda_{\text{crit}}(\eta^-) = \lambda_{\text{crit}}$ and η^-] non-degenerate, turns out to be far more difficult.
2. It is worthwhile to note that in Case D, if in addition a minimizer η^-, satisfying the additional assumptions, exists for α^{Inv} (which belongs to case F) then I_α^a is linear on the interval $[-(u^-)^{-1},0]$ defined in Case F.

Before proving the above propositions, we state and prove some auxiliary facts.

Lemma 10 *1. For any product measure α and any bounded continuous function Ψ,*

$$\inf_{\eta\in M_1^s(\Sigma)}\left[h(\eta|\alpha) + \int \Psi(\omega_0)\eta_0(d\omega_0)\right] = H(\tilde{\alpha}_0|\alpha_0) + \int \Psi(\omega_0)\tilde{\alpha}_0(d\omega_0)\,,$$

where $\tilde{\alpha}$ is a product measure and $d\tilde{\alpha}_0/d\alpha_0 = \exp(-\Psi(\omega_0))/\int \exp(-\Psi(\omega_0))\alpha_0(d\omega_0)$.
2. Let $\Theta = \{\eta \in M_1^e(\Sigma) : \int \Psi(\omega_0)\eta_0(d\omega_0) \geq 0\}$. If $\alpha \in \Theta$ then

$$\inf_{\eta\in M_1^s(\Sigma)} \left[h(\eta|\alpha) + \left(0 \vee \int \Psi(\omega_0)\eta_0(d\omega_0)\right)\right] = \inf_{\eta\in\Theta} \left[h(\eta|\alpha) + \left(0 \vee \int \Psi(\omega_0)\eta_0(d\omega_0)\right)\right] . \tag{87}$$

In particular, if also $\tilde{\alpha} \in \Theta$ then

$$\inf_{\eta\in M_1^s(\Sigma)} \left[h(\eta|\alpha) + \left(0 \vee \int \Psi(\omega_0)\eta_0(d\omega_0)\right)\right] = H(\tilde{\alpha}_0|\alpha_0) + \int \Psi(\omega_0)\tilde{\alpha}_0(d\omega_0) . \tag{88}$$

Proof of Lemma 10: 1. We have, with $\mathcal{F}_n = \sigma(\omega_0, \omega_1, \ldots, \omega_{n-1})$,

$$h(\eta|\alpha) = \sup_n \frac{1}{n} H(\eta|\alpha)|_{\mathcal{F}_n} .$$

Therefore,

$$\begin{aligned} h(\eta|\alpha) &\geq H(\eta_0|\alpha_0) \\ &\geq \int -\Psi(\omega_0)\eta(d\omega_0) - \log \int e^{-\Psi(\omega_0)}\alpha_0(d\omega_0) \end{aligned}$$

where the second inequality is due to the variational characterization of relative entropy, c.f. for example [4, Lemma 6.2.13]. Hence

$$h(\eta|\alpha) + \int \Psi(\omega_0)\eta(d\omega_0) \geq -\log \int e^{-\Psi(\omega_0)}\alpha(d\omega_0)$$

and equality is achieved for the measure $\tilde{\alpha}$.

2. Assume (87) does not hold true. Then there exists a $\eta^* \notin \Theta$ such that

$$h(\eta^*|\alpha) + \left(0 \vee \int \Psi(\omega_0)\eta_0^*(d\omega_0)\right) = h(\eta^*|\alpha) < \inf_{\eta\in\Theta} \left[h(\eta|\alpha) + \left(0 \vee \int \Psi(\omega_0)\eta_0(d\omega_0)\right)\right] .$$

Because $\alpha \in \Theta$, $\eta^* \neq \alpha$, and further, $\int \Psi(\omega_0)\alpha_0(d\omega_0) > 0$, for otherwise α is a global minimizer, yielding a contradiction. Take a convex combination $\eta_\theta := \theta\alpha + (1-\theta)\eta^*$ such that $\int \Psi(\omega_0)\eta_\theta(d\omega_0) = 0$. Since the product measure α satisfies Assumption (A), one can find a sequence $\eta_\theta^n \in M_1^e(\Sigma)$ such that $\eta_\theta^n \to \eta_\theta$ weakly and $h(\eta_\theta^n|\alpha) \to h(\eta_\theta|\alpha)$. Therefore,

$$\limsup_{n\to\infty} h(\eta_\theta^n|\alpha) + \left(0 \vee \int \Psi(\omega_0)\eta_\theta^n(d\omega_0)\right) = h(\eta_\theta|\alpha) < h(\eta^*|\alpha) ,$$

a contradiction. □

It is worthwhile to note that actually, one may compute explicitly the optimal η in (87) even when $\tilde{\alpha} \notin \Theta$: it is a product measure with marginal $Z_\beta^{-1}\exp(\beta\Psi(\omega_0))\alpha_0(d\omega_0)$, where $-1 \leq \beta \leq 0$ is chosen such that $\int \exp(\beta\Psi(\omega_0))\Psi(\omega_0)d\alpha_0(\omega_0) = 0$. Using this observation one may relax the assumptions in the lemma to Ψ being merely bounded measurable.

Proof of Proposition 3. Since $I_\alpha^{\tau,a}$ is convex, it is enough to show that whenever b' is finite then it is a minimizer, of $I_\alpha^{\tau,a}$, that the latter is constant on $[b', b]$, and that if $b' = \infty$ then $I_\alpha^{\tau,a}$ is decreasing. We divide the proof into the following cases:

1. $\alpha \in M_1^e(\Sigma)^+$. Then, $b = b' = \langle\tau\rangle_\alpha \in (1,\infty]$, and $I_\alpha^{\tau,a}(b') = 0$ (if $b' < \infty$) while, if $b' = \infty$, $\lim_{u\to\infty} I_\alpha^{\tau,a}(u) = 0$.

2. Assume $\hat\alpha \in M_1^e(\Sigma)^-$ (and then, as noted above, also $\alpha \in M_1^e(\Sigma)^-$). Then $b' = u^*$. Assume first $u^* < \infty$. Then,

$$I_\alpha^{\tau,a}(u^*) \le I_{\hat\alpha}^{\tau,q}(u^*) + H(\hat\alpha_0|\alpha_0) = -\log\int \rho_0(\omega_0)^{-1}\alpha_0(d\omega_0)\,, \tag{89}$$

as can be checked by an explicit computation involving the definition of $\hat\alpha$. On the other hand, using in the first equality the exact value of the minimum of $I_\eta^{\tau,q}(\cdot)$, see the comment before (40), and (88) in the second equality (with $\Psi(\rho_0) = \log\rho_0$),

$$\inf_u I_\alpha^{\tau,a}(u) = \inf_{\eta\in M_1^e(\Sigma)}\left[\left(0\vee\int\log\rho_0(\omega_0)\eta_0(d\omega_0)\right) + h(\eta|\alpha)\right] = -\log\int\rho_0(\omega_0)^{-1}\alpha_0(d\omega_0)\,. \tag{90}$$

Hence, u^* is a global minimizer of $I_\alpha^{\tau,a}$ in this case.

If $u^* = \infty$, (90) still holds true while, for any $u < \infty$,

$$\begin{aligned} I_\alpha^{\tau,a}(u) \le I_{\hat\alpha}^{\tau,q}(u) + H(\hat\alpha_0|\alpha_0) &= I_{\hat\alpha}^{\tau,q}(u) - \int\log\rho_0(\omega_0)\hat\alpha_0(d\omega_0) - \log\int\rho_0(\omega_0)^{-1}\alpha_0(d\omega_0) \\ &\underset{u\to\infty}{\longrightarrow} -\log\int\rho_0(\omega_0)^{-1}\alpha_0(d\omega_0)\,, \end{aligned}$$

since $\hat\alpha \in M_1^e(\Sigma)^-$. It follows that $\inf_u I_\alpha^{\tau,a}(u) = \lim_{u\to\infty} I_\alpha^{\tau,a}(u)$, as required.

3. Assume $\hat\alpha \in M_1^e(\Sigma)^+$ but $\alpha \in M_1^e(\Sigma)^-$. In this case $b' = \infty$, and one repeats the previous argument, using this time that

$$\lim_{u\to\infty} I_\alpha^{\tau,a}(u) \le \inf_{\{\eta\in M_1^e(\Sigma):\int\log\rho_0(\omega_0)\eta(d\omega_0)\ge 0\}}\left[\int\log\rho_0(\omega_0)\eta(d\omega_0) + h(\eta|\alpha)\right]\,,$$

while, using now (87),

$$\begin{aligned} I_\alpha^{\tau,a}(u) &\ge \inf_{\eta\in M_1^e(\Sigma)}\left[h(\eta|\alpha) + \left(0\vee\int\log\rho_0(\omega_0)\eta_0(d\omega_0)\right)\right] \\ &= \inf_{\{\eta\in M_1^e(\Sigma):\int\log\rho_0(\omega_0)\eta(d\omega_0)\ge 0\}}\left[\int\log\rho_0(\omega_0)\eta(d\omega_0) + h(\eta|\alpha)\right]\,, \end{aligned}$$

implying as before that $\inf_u I_\alpha^{\tau,a}(u) = \lim_{u\to\infty} I_\alpha^{\tau,a}(u)$. □

Proof of Proposition 4

1. Properties of $I_\alpha^{\tau,a}$. The monotonicity of $I_\alpha^{\tau,a}$ on the claimed intervals is a direct consequence of Proposition 3, while the convexity is stated in Theorem 5. Further, (7) implies that if $I_\alpha^{\tau,q}(u) = 0$ then $I_\alpha^{\tau,a}(u) = 0$, yielding the claimed zero values for $I_\alpha^{\tau,a}$.

To see the claimed strict monotonicity of $I_\alpha^{\tau,a}$ in case A–D, note that by convexity, it is enough to show that $I_\alpha^{\tau,a}(u) > 0$ at a point u in order to show that it is strictly monotone there. But $I_\alpha^{\tau,a}(u) = 0$ only if $I_\alpha^{\tau,q}(u) = 0$ by (7) and the fact that the infimum there is attained, leading to the monotonicity claim.

Cases E–F require slightly more work. Assume first that $u^* < \infty$, we already know, c.f. Proposition 3, that u^* is a global minimum of $I_\alpha^{\tau,a}$.

To prove the strict monotonicity of $I_\alpha^{\tau,a}$ on $[1,u^*]$ when $u^*<\infty$, in both cases E and F, we check that $I_\alpha^{\tau,a}(u) > -\log E_\alpha[\rho_0^{-1}]$ for $u<u^*$, and then the convexity of $I_\alpha^{\tau,a}$ proves the required strict monotonicity. To this end, note that $\hat\alpha$ is transient to the left (because $\rho^*<1$). But, for any η,

$$I_\eta^{\tau,q}(u)+h(\eta|\alpha)\geq E_{\eta_0}[\log\rho_0]+H(\eta_0|\alpha_0)\geq -\log\int\rho_0^{-1}(\omega)\alpha_0(d\omega)\,,$$

where the first inequality is achieved only on product measures transient to the left, and the second, due to Lemma 10, only when $\eta_0=\hat\alpha_0$. But in the latter case, the first inequality is strict because $u<u^*$ and $E_{\hat\eta}[\tau_\omega]=u^*$. Since the infimum over η is always achieved in the definition of $I_\alpha^{\tau,a}$, we conclude that necessarily

$$\inf_{\eta\in M_1^e(\Sigma)}\left[I_\eta^{\tau,q}(u)+h(\eta|\alpha)\right]>I_\alpha^{\tau,a}(u^*)\,,$$

as claimed.

The strict monotonicity on $[u^*,\infty)$ in Case F is proved similarly, using that in Case F the quenched rate function is strictly monotone, and repeating the above argument.

Finally, it remains to check the strict monotonicity on $[1,\infty)$ in case E when $u^*=\infty$. The argument given above actually shows that $I_\alpha^{\tau,a}(\infty):=\lim_{u\to\infty}I_\alpha^{\tau,a}(u)=-\log E_\alpha[\rho_0^{-1}]$, and hence it suffices to check that $I_\alpha^{\tau,a}(u)>I_\alpha^{\tau,a}(\infty)$. The argument is the same as above and therefore omitted.

We turn now to the linear part of $I_\alpha^{\tau,a}$ in Case D. We checked in the proof of Proposition 2 that if the conclusions of Lemma 4, part i) are satisfied, then $I_{\eta^+}^{\tau,q}$ is linear on $[u^+,\infty)$. More precisely, like in (85) it holds for $u\geq u^+$ that

$$I_{\eta^+}^{\tau,q}(u)=\lambda_{\text{crit}}u-\int f(\lambda_{\text{crit}},\omega)\eta^+(d\omega)$$

Hence we have

$$I_\alpha^{\tau,a}(u)\leq I_{\eta^+}^{\tau,q}(u)+h(\eta^+|\alpha)=\lambda_{\text{crit}}u-\int f(\lambda_{\text{crit}},\omega)\eta^+(d\omega)+h(\eta^+|\alpha)\,. \tag{91}$$

On the other hand, it follows from the large deviation lower bound together with the substitution $\lambda=\lambda_{\text{crit}}$ in (63) that

$$\begin{aligned}
-I_\alpha^{\tau,a}(u) &\leq \limsup_{n\to\infty}\frac{1}{n}\log P\left[\frac{T_n}{n}\geq u\right]\\
&\leq -\inf_{\eta\in M_1^s(\Sigma)}\left[\lambda_{\text{crit}}u-\int f(\lambda_{\text{crit}},\omega)\eta(d\omega)+h(\eta|\alpha)\right]\\
&= -\lambda_{\text{crit}}u+\int f(\lambda_{\text{crit}},\omega)\eta^+(d\omega)-h(\eta^+|\alpha)
\end{aligned}$$

since η^+ is a minimizer. Therefore the equality holds in (91), and $I_\alpha^{\tau,a}$ is linear on $[u^+,\infty)$. The proof of existence of a linear part in Case F is similar.

2. Properties of I_α^a. All the stated properties of I_α^a follow immediately, using (8), from the properties of $I_\alpha^{\tau,a}$, except for checking that in Case D, $I_\alpha^a(0)=\lambda_{\text{crit}}$. But this was obtained in the proof of Theorem 2. $\square$

We conclude this section by providing a class of examples where the additional assumption in Proposition 4, Cases D and F, is satisfied, resulting with the existence of linear pieces for $I_\alpha^{\tau,a}$. We concentrate on Case D, as the construction for Case F is similar.

Choose $\omega_{\min} > 1/2$, $\alpha_0(\omega_{\min}) > 0$, $\alpha_0(\omega_{\max}) > 0$, and $\omega_{\max} - \omega_{\min}$ small enough. (What is meant by small enough will become clear in the course of the construction). Due to the remark below the proof of Lemma 4 it is enough to ensure that any ergodic minimizer η^+ of the function $F(\eta) = -\int f(\lambda_{\text{crit}}, \omega)\eta(d\omega) + h(\eta|\alpha)$, satisfies, for a fixed M_0 depending on $\omega_{\min}$ only, that $\eta^+(\{\omega_i = \omega_{\min}\}_{i=0}^{M_0+1}) > 0$, and that $\eta_0^+(\omega_{\max}) > 0$. We argue by contradiction. Assume that $\eta^+(\{\omega_i = \omega_{\min}\}_{i=0}^{M_0+1}) = 0$. Then,

$$h(\eta^+|\alpha) \geq \frac{1}{M_0+2} H(\eta^+|\alpha)_{|[0,M_0+1]} \geq \frac{1}{M_0+2} \log \frac{1}{1-\alpha_0(\omega_{\min})^{M_0+2}} =: \delta\,. \tag{92}$$

Recall that $\varphi(\lambda_{\text{crit}}, \omega) \leq \overline{\varphi}(\lambda_{\text{crit}}) = (\omega_{\min}/(1-\omega_{\min}))^{1/2}$, and note that estimates similar to (44) and the substitution of the value for λ_{crit} lead to the bound

$$\begin{aligned} \varphi(\lambda_{\text{crit}}, \omega) &\geq \varphi(\lambda_{\text{crit}}, (.., \omega_{\max}, \omega_{\max}, \ldots)) \\ &= \frac{\sqrt{\omega_{\min}(1-\omega_{\min})} - \sqrt{\omega_{\min}(1-\omega_{\min}) - \omega_{\max}(1-\omega_{\max})}}{1-\omega_{\max}} := \underline{\varphi}(\lambda_{\text{crit}}) \end{aligned}$$

The estimates $F(\alpha) \leq -\log \underline{\varphi}(\lambda_{\text{crit}})$ and $F(\eta^+) \geq \delta - \log \overline{\varphi}(\lambda_{\text{crit}})$ (following from (92)) imply that if $\omega_{\max}$ is close enough to $\omega_{\min}$ then $F(\eta^+) > F(\alpha)$, a contradiction. The proof that $\eta_0^+(\omega_{\max}) > 0$ being similar, only simpler, we conclude our construction.

5.2 Description of the annealed environment leading to a large deviation

The minimizing measures for the variational problem (9) are of particular interest, for they hint at the environment which creates atypical behavior. We discuss below the simpler (though equivalent) question for the hitting times rate function. Recall that the infimum in (7), $I_\alpha^{\tau,a}(u) = \inf_{\eta \in M_1^e(\Sigma)}[I_\eta^{\tau,q}(u) + h(\eta|\alpha)]$, is achieved.

Proposition 5 *Assume that α is a product measure, not concentrated on a single point. Let $u \in (1, \infty)$, such that $I_\alpha^{\tau,a}$ does neither have a minimum at u nor a linear part at u (i.e., $I_\alpha^{\tau,a}$ is not linear in a neighborhood of u). Then the minimizers in (7) are one-dimensional Gibbs measures with summable, translation invariant interaction, and they are not product measures. In particular, this implies that $I_\alpha^{\tau,a}(u) < I_\alpha^{\tau,q}(u)$ for all such u's.*

Proposition 5 says that in general, even though α is a product measure the best environments for creating large deviations are not product measures. Here are some interesting exceptions, the first two of them we already met in Propositions 2 and 4:

1) For $u = \langle\tau\rangle_\alpha$ minimizing $I_\alpha^{\tau,a}$ in Cases C and D, the minimizer in (7) is α. The same holds in Case C for $u \geq \langle\tau\rangle_\alpha$.

2) For $u = u^*$ in Cases E, F and for $u \geq u^*$ in Case E (minimizing $I_\alpha^{\tau,a}$), the minimizer is the product measure $\hat{\alpha}$ introduced in (86).

3) Since $I_{\eta'}^{\tau,q}(1) = \int \log \omega_0 \; \eta'(d\omega)$ for all $\eta' \in M_1^e(\Sigma)$, with the supremum in (5) being for $\lambda = -\infty$, the minimizer η in (7) for $u = 1$ (or in (9) for $v = 1$) is the product measure with

$$\frac{d\eta_0}{d\alpha_0} = \frac{\omega_0}{\int \omega_0 \alpha_0(d\omega_0)}\,.$$

Note that the nature of the solutions to the variational problem remains an open question when u belongs to a linear, but non constant, part of the rate function, see the remark below the proof.

Proof of Proposition 5. We start with the case when u is smaller than the minimizer of $I_\alpha^{\tau,a}$, i.e., $u < \langle\tau\rangle_\alpha$ in Cases A to D, $u < u^*$ in Cases E, F. For such a u, we have from (56), (58),

$$I_\alpha^{\tau,a}(u) = \sup_{\lambda\leq 0} \inf_{\eta\in M_1^s(\Sigma)} \left[\lambda u - \int f(\lambda,\omega)\eta(d\omega) + h(\eta|\alpha)\right] .$$

Recall that the supremum is achieved, and note that all maximizers λ are nonzero, due to the strict monotonicity of $I_\alpha^{\tau,a}$ around u stated in Proposition 4. Let $\Lambda(u)$ be the set of maximizers λ. The set of maximizers $\eta \in M_1^e(\Sigma)$ of the function

$$\int f(\lambda,\omega)\eta(d\omega) - h(\eta|\alpha) \tag{93}$$

for λ ranging over $\Lambda(u)$ coincides with the set of minimizers in (7). We prove that such η's are Gibbs measures, constructing their potential. For $M \geq 0$ we consider the RWRE with reflection at site $-M$ (i.e., with environment $(\ldots,1,\omega_{-M+1},\omega_{-M+2},\ldots)$, and we denote by $E_\omega^{\mathrm{ref},M}$ the corresponding expectation. With a recursion we find as in (14)

$$\varphi_M(\lambda,\omega) := E_\omega^{\mathrm{ref},M}[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty}] = \frac{1|}{e^{-\lambda}(1+\rho_0(\omega))} - \frac{\rho_0(\omega)|}{\cdots} - \frac{\cdots}{e^{-\lambda}(1+\rho_{-M+1}(\omega)) - e^{\lambda}\rho_{-M+1}(\omega)}$$

that is, the M-th approximant of the continued fraction φ. Since $\lambda < 0$ and since

$$\varphi_N(\lambda,\omega) = E_\omega^{\mathrm{ref},N}[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty, X_\cdot\geq -M+1 \text{ on } [0,\tau_1]}] + E_\omega^{\mathrm{ref},N}[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty, X_\cdot \text{ hits } -M \text{ on } [0,\tau_1]}]$$

for $N = M, M+1$, we have

$$0 < \varphi_M - \varphi_{M+1} \leq e^{\lambda(2M+1)} \tag{94}$$

using that $\tau_1 \geq 2M+1$ when $X_\cdot$ hits $-M$ on $[0,\tau_1]$, and that $E_\omega^{\mathrm{ref},N}[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty,X_\cdot\geq -M+1 \text{ on } [0,\tau_1]}]$ achieves the same value for $N = M$, $N = M+1$. Recalling $\varphi_0 = e^\lambda$ we introduce the decomposition

$$f = \lambda + \sum_{M\geq 0} g_M \quad , \quad g_M(\omega) = \log\frac{\varphi_{M+1}(\lambda,\omega)}{\varphi_M(\lambda,\omega)}$$

where g_M depends only on $\omega_{-M},\ldots,\omega_0$. Combining (94) with $\omega_0 e^\lambda \leq \varphi_M \leq 1$ we see that $\|g_M\|_\infty \leq Ce^{2\lambda M}$ with some finite C depending on $\omega_{\min}$, and then $\sum_M M\|g_M\|_\infty < \infty$. This implies that the maximizers η of (93) are one-dimensional Gibbs measures, with translation invariant, summable potential $(J_V; V \subset \mathbb{Z})$ given by $J_{[i-M,i]}(\omega) = g_M(\theta^i\omega)$ and $J_V = 0$ if V is not an interval. Refer to [10] for an account on Gibbs measures, and note that the potential J_V depends on λ. We show now that the "potential at the origin" $H_0 = \sum J_{[i-M,i]}$, where the sum extends over i, M such that $i - M \leq 0 \leq i$, is not ω_0-measurable, which implies that those Gibbs measures η are not product measures ([10], Sect.2-4). The series H_0 is equal to the limit as $n \to \infty$ of

$$\sum_{i=0}^{n}\sum_{M\geq i} g_M(\theta^i\omega) = \sum_{i=0}^{n}\log\frac{\varphi(\theta^i\omega)}{\varphi_i(\theta^i\omega)} = \log\frac{E_\omega[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}]}{E_\omega^{\mathrm{ref},0}[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}]}$$

where the second equality comes from the strong Markov property. Introducing the stopping time $\zeta = \inf\{n \geq \tau_{-1}; X_n = 0\} \in (0,\infty]$, we have $E_\omega[e^{\lambda T_n}\mathbf{1}_{\zeta<T_n<\infty}] = E_\omega[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}]E_\omega[e^{\lambda\zeta}\mathbf{1}_{\zeta<T_n<\infty}]$ by the strong Markov property, and therefore

$$\phi_n := \frac{E_\omega[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}]}{E_\omega^{\mathrm{ref},0}[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}]} = \frac{E_\omega[e^{\lambda T_n}\mathbf{1}_{T_n<\infty,T_n<\zeta}]}{E_\omega^{\mathrm{ref},0}[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}]} + \phi_n\, E_\omega[e^{\lambda\zeta}\mathbf{1}_{\zeta<T_n<\infty}] .$$

Solving in ϕ_n we obtain

$$\left(1 - E_\omega[e^{\lambda\zeta}\mathbf{1}_{\zeta<T_n<\infty}]\right)^{-1} \lim_{n\to\infty} \frac{E_\omega[e^{\lambda T_n}\mathbf{1}_{T_n<\infty, T_n<\zeta}]}{E_\omega^{\mathrm{ref},0}[e^{\lambda T_n}\mathbf{1}_{T_n<\infty}]} = e^{H_0} \in (0,\infty)$$

where the first factor depends on ω_{-1} when $\lambda \neq 0$, but the second factor does not. From this we conclude that H_0 is not ω_0-measurable, which ends the proof in the first case.

In the opposite case, i.e., for u larger than the minimizer of $I_\alpha^{\tau,a}$ but not on a linear part, $I_\alpha^{\tau,a}(u)$ is given this time by (63) due to (65) and (66), and all maximizers λ belong to $(0,\lambda_{\mathrm{crit}})$. The proof works the same, except that the bounds in (94) will be replaced by

$$0 < \varphi_{M+1} - \varphi_M \leq E_\omega^{\mathrm{ref},M+1}[e^{\lambda\tau_1}\mathbf{1}_{\tau_1<\infty, X_\cdot \text{ hits } -M \text{ on } [0,\tau_1]}] \leq e^{(\lambda-\lambda')(2M+1)}\varphi(\lambda', \bar{\omega}_{\min})$$

with some $\lambda' \in (\lambda, \lambda_{\mathrm{crit}})$, and combined with $e^\lambda \leq \varphi_M \leq \varphi(\lambda, \bar{\omega}_{\min})$ which is finite. $\square$

Remarks:
1. The definition of g_M above reveals a nice interplay between the Gibbs decomposition of function f appearing in (93) as an interaction, and the approximants φ_M of the continued fraction φ. The interpretation of these approximants in terms of reflection is most natural. The key property (94), which implies summability of the potential, can be alternatively derived for $\lambda < 0$ from standard approximation results in continued fraction theory; see Pringsheim Theorem, page 92 in [13] and its proof. All this shows the particular interest of formula (14).

2. When u belongs to a linear, but not constant, part of the annealed rate function, the maximizer λ is equal to $\lambda_{\mathrm{crit}} > 0$. Exponential convergence of the series $\sum g_M$ breaks down, and we believe that the minimizers η exhibit long range dependence.

6 Concluding remarks and open problems

1. Our quenched results cover the case when η is ergodic, without being locally equivalent to the product of its marginals. However, the shape of the quenched rate function in this case can be different. For instance, one can construct examples where there are no linear pieces in $I_\eta^{\tau,q}$ in Case C above.

2. In general, we do not know how to solve the annealed variational problem in (9) more explicitly than in Proposition 5, and hence we do not have explicit expressions for I_α^a. One case where this problem can be solved is when $|v| = 1$. More precisely, for $v = 1$ we have

$$P_\omega\left[\frac{X_n}{n} = 1\right] = \prod_{i=0}^{n-1} \omega_i\,,$$

and

$$P\left[\frac{X_n}{n} = 1\right] = E\left[P_\omega\left[\frac{X_n}{n} = 1\right]\right] = E\left[\prod_{i=0}^{n-1}\omega_i\right] = \prod_{i=0}^{n-1} E_{\alpha_0}[\omega_0]\,.$$

Hence, taking logarithms, dividing by n and taking limits, one concludes that $I_\eta^q(1) = -\int \log \omega_0 \alpha_0(d\omega_0)$, $I_\alpha^a(1) = -\log\int \omega_0\alpha_0(d\omega_0)$. In particular, $I_\alpha^a(1) < I_\eta^q(1)$ as soon as α is non-degenerate.

3. We speculate that the extra assumption stated in Proposition 4, Cases D and F, is always satisfied, and is not limited to the class of example constructed at the end of the last section. Recall that these extra assumptions imply the existence of linear pieces for $I_\alpha^{\tau,a}$.

4. As in the i.i.d. environment case studied in length in [3], [9], [18], [17], one may look for refined asymptotics in the flat pieces of $I_\eta^{\tau,q}$ or I_η^q. When η is equivalent to the product of its marginals, we believe it to exhibit the same qualitative behavior as in the i.i.d. case, that is polynomial decay in the case $\omega_{\min} < 1/2 < \omega_{\max}$ and sub-exponential decay when $\omega_{\min} = 1/2$. Refined asymptotics for the multi-dimensional case were obtained in [22]. Some explicit computations are possible in the Markov environment case, we do not pursue this direction here.

5. When the support of α_0 includes the points 0 or 1, our proofs break down (even if $\alpha_0(\{0\} \cup \{1\}) = 0$). We believe that under strong enough assumptions on the rate of decay of the $\alpha_0([0,1] \setminus [\epsilon, 1-\epsilon])$, the analysis can still be pushed through.

6. The multi-dimensional case presents many challenges. Important works in this domain are [24], [22], but many questions remain open, most notably what happens when $0 \notin \operatorname{conv}\operatorname{supp}\alpha_0$, what is the annealed rate function, and what is the relation of the latter to the quenched rate function.

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