Large Deviations for Random Walks on Galton-Watson Trees: Averaging and Uncertainty

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November 10, 2000; Revised: February 27, 2001

Abstract

In the study of large deviations for random walks in random environment, a key distinction has emerged between *quenched* asymptotics, conditional on the environment, and *annealed* asymptotics, obtained from averaging over environments. In this paper we consider a simple random walk $\{X_n\}$ on a Galton-Watson tree $\mathbf{T}$, i.e., on the family tree arising from a supercritical branching process. Denote by $|X_n|$ the distance between the node X_n and the root of $\mathbf{T}$. Our main result is the almost sure equality of the large deviation rate function for $|X_n|/n$ under the "quenched measure" (conditional upon $\mathbf{T}$), and the rate function for the same ratio under the "annealed measure" (averaging on $\mathbf{T}$ according to the Galton-Watson distribution). This equality hinges on a concentration of measure phenomenon for the *momentum* of the walk. (The momentum at level n, for a specific tree $\mathbf{T}$, is the average, over random walk paths, of the forward drift at the hitting point of that level). This concentration, or *certainty*, is a consequence of the *uncertainty* in the location of the hitting point. We also obtain similar results when $\{X_n\}$ is a λ-biased walk on a Galton-Watson tree, even though in that case there is no known formula for the asymptotic speed. Our arguments rely at several points on a "ubiquity" lemma for Galton-Watson trees, due to Grimmett and Kesten (1984).

KEY WORDS: Random walk in random environment, large deviations, Galton-Watson tree.
AMS (2000) subject classifications: 60K37,60F10,60J80,82C44.

1 Introduction

In the last decade, asymptotics for large-deviation probabilities involving one-dimensional, nearest-neighbor random walk in random environment (RWRE) have been determined quite precisely; see [6, 2, 4, 14, 13, 1], and [5] for an overview. Striking partial results were obtained by Sznitman and Zerner [19, 21, 16, 17] in the more difficult setting of multidimensional RWRE.

From these studies, a key distinction has emerged between **quenched** asymptotics (conditional on a typical environment), first considered by Greven and den Hollander [6], and **annealed** asymptotics (averaged over environments), first considered in [2]. In one dimension, Comets, Gantert and Zeitouni [1] showed that the quenched and annealed rate functions typically differ when the annealed rate function is nonzero. In $\mathbb{Z}^d$ for $d > 1$, the situation is less clear, and much is unknown: while there exist events for which quenched and annealed exponential asymptotics differ, it is tempting to conjecture that for d large enough there are

*Research partially supported by NSF grant #DMS-0072331 and by a US-Israel BSF grant.
**Research partially supported by the DFG
§Research partially supported by NSF grant #DMS-9803597 and by a US-Israel BSF grant
¶Research partially supported by a US-Israel BSF grant and by the fund for promotion of research at the Technion.

other events for which the quenched and annealed rate functions are non zero and coincide. This conjecture is related to the corresponding conjectures for the case of random potential, see the discussion in [18, Page 326].

In this paper we consider this question when the random environment consists of a Galton-Watson tree, i.e., the family tree arising from a supercritical Galton-Watson branching process. Since the growth of these trees is exponential, this can be viewed as an infinite dimensional setting, but the lack of cycles makes it more tractable than a random environment in $\mathbb{Z}^d$ for $d > 1$. Our main result is the equality of quenched and annealed rate functions for random walk on Galton-Watson trees; this can be viewed as a strong *averaging* present in almost every Galton-Watson tree. This equality hinges on a concentration of measure phenomenon for the *momentum* of the walk. (The momentum at level n, for a specific tree $\mathbf{T}$, is the average, over random walk paths, of the forward drift $[\deg(v) - 1]/[\deg(v) + 1]$ at the hitting point v of that level). This concentration, or *certainty*, is a consequence of the *uncertainty* in the location of the hitting point; more details are given below.

A crucial technical tool in our proofs is Lemma 2.2, a variant of an important, but little-known "ubiquity" lemma due to Grimmett and Kesten [7]. This lemma asserts, roughly speaking, that any tree property which is sufficiently likely under the Galton-Watson measure, is almost surely satisfied by the subtrees determined by a positive fraction of the vertices, along any ray emanating from the root.

In the analysis of RWRE in $\mathbb{Z}^d$, neutral pockets often play a key role, see, e.g., [17]. For simple random walk on Galton-Watson trees, this role is played by *pipes*, relatively long paths in the tree where each vertex has precisely one child.

We start with a branching process which defines an infinite Galton-Watson tree. Let Z be an integer-valued random variable, with $p_k = P(Z = k), k = 0, 1, 2, \ldots$. We always assume that $p_0 = 0$ and

$$m := \sum_{k=1}^{\infty} k p_k > 1 \, .$$

Define $d_{\min} = \min\{k : p_k > 0\} \geq 1$. Starting from the first ancestor (called the root, and denoted $\mathbf{o}$), we consider a supercritical branching process where particles independently produce children, such that the number of children has the law of Z. We draw edges between parents and their children. This defines the measure $\mathrm{GW}(d\omega)$ on rooted Galton-Watson trees.

Let $\mathcal{T}$ denote the ensemble of all rooted trees with no leaves, i.e., with at least one child for each vertex. For any $\mathbf{T} \in \mathcal{T}$ and any vertex $j \in \mathbf{T}$, we let $|j|$ denote the distance from the root (i.e., the number of edges on the unique path $r(j)$ connecting the root to j). Let k_j denote the number of children of a vertex $j \in \mathbf{T}$. We let $D_n(\mathbf{T})$ denote the vertices at the n-th generation, i.e. at distance n from the root. For any vertex $j \in D_n(\mathbf{T})$, $n > 0$, we let j^* denote the parent of j, i.e. the vertex $r(j) \cap D_{n-1}(\mathbf{T})$. The children of a vertex $j \in D_n(\mathbf{T})$ are those vertices in $D_{n+1}(\mathbf{T})$ connected to j. We often write D_n instead of $D_n(\mathbf{T})$ if it is clear from the context which tree $\mathbf{T}$ we mean. More definitions and notations related to $\mathcal{T}$ are introduced in Section 2.

Our main object of interest in this paper are λ-biased random walks on the Galton-Watson tree, defined as follows. Given ω, the λ-biased random walk $\{X_n\}$ taking values in the vertices of ω, with distribution $P_{\lambda,\omega}$, is the Markov chain with $X_0 = \mathbf{o}$ and, with $j_1, \ldots, j_k$, denoting the children of j,

$$\begin{aligned} P_{\lambda,\omega}\Big(X_{n+1} = j^* | X_n = j\Big) &= \frac{\lambda}{\lambda + k} \, , \qquad j \neq \mathbf{o} \, , \\ P_{\lambda,\omega}\Big(X_{n+1} = j_i | X_n = j\Big) &= \frac{1}{\lambda + k} \, , \qquad j \neq \mathbf{o}, \; i = 1, 2, \ldots, k. \end{aligned}$$

$$P_{\lambda,\omega}\left(X_{n+1}=j_i|X_n=j\right)=\frac{1}{k}\,,\qquad j=\mathbf{o},\ i=1,2,\ldots,k\,.$$

(we refer to [11] for the ergodic theory of such walks). We call the law $P_{\lambda,\omega}$ the *quenched* law. We also let

$$P_\lambda(\cdot):=\int P_{\lambda,\omega}(\cdot)GW(d\omega) \tag{1.1}$$

and call the resulting measure on the process $\{X_n\}$ the *annealed* law.

It was shown in [11, Theorem 3.1] that if $0<\lambda<m$, then

$$\lim_{n\to\infty}\frac{|X_n|}{n}=v_\lambda>0\,,\qquad P_\lambda-\ \text{a.s.}\,, \tag{1.2}$$

where v_λ depends only on λ and on the distribution of Z. An explicit evaluation of v_λ is not available in general, except in the case $\lambda=1<m$, where $v_1=\sum_k p_k(k-1)/(k+1)$, see [10, Theorem 3.2]. If $\lambda\geq m$, then $\{X_n\}$ is recurrent P_λ- a.s. (see [9, Theorem 4.2]), and of course $v_\lambda=0$.

Our main results, concerning decay rates for the probability of atypical behavior of the random walk, follow. Note that the exponential decay rates under the annealed law and the quenched law, coincide.

Theorem 1.1 Speedup probabilities - exponential decay *Assume $m<\infty$. Let $\lambda\geq 0$. Then, there exists a continuous, convex, strictly increasing function $I_\lambda:[v_\lambda,1]\mapsto\mathbb{R}_+$, with $I_\lambda(v_\lambda)=0$ and*

$$I_\lambda(1)=-\log\sum_{k=1}^{\infty}\frac{k}{k+\lambda}p_k\,, \tag{1.3}$$

satisfying, for $b>a$, $a\in(v_\lambda,1]$,

$$\lim_{n\to\infty}\frac{1}{n}\log P_{\lambda,\omega}\left(\frac{|X_n|}{n}\in[a,b)\right)=\lim_{n\to\infty}\frac{1}{n}\log P_\lambda\left(\frac{|X_n|}{n}\in[a,b)\right)=-I_\lambda(a)\,,\quad GW-a.s.$$

The situation is similar with respect to slowdown probabilities, except that the rate of decay need not be exponential in all cases.

Theorem 1.2 Slowdown probabilities - exponential decay *Assume $\lambda<m<\infty$ and either $d_{\min}\geq 2$ or $\lambda\geq 1$. Then, there exists a convex, decreasing function $I_\lambda:[0,v_\lambda]\mapsto\mathbb{R}_+$, with $I_\lambda(v_\lambda)=0$, satisfying, for $0\leq b<a<v_\lambda$,*

$$\lim_{n\to\infty}\frac{1}{n}\log P_{\lambda,\omega}\left(\frac{|X_n|}{n}\in[b,a)\right)=\lim_{n\to\infty}\frac{1}{n}\log P_\lambda\left(\frac{|X_n|}{n}\in[b,a)\right)=-I_\lambda(a)\,,\quad GW-a.s. \tag{1.4}$$

Further, if $\lambda\geq d_{\min}$ then $I_\lambda:[0,v_\lambda]\mapsto 0$, whereas if $\lambda<d_{\min}$ then I_λ is strictly decreasing on $[0,v_\lambda]$. Finally, if $d_{\min}\geq 2$ and $\lambda<d_{\min}$ then

$$I_\lambda(0):=\lim_{a\downarrow 0}I_\lambda(a)=H\left(\frac{1}{2}\middle|\frac{d_{\min}}{d_{\min}+\lambda}\right)\,, \tag{1.5}$$

where $H(s|t):=s\log\frac{s}{t}+(1-s)\log\frac{1-s}{1-t}$.

Recalling that $v_\lambda=0$ for $\lambda\geq m$, one has by combining Theorems 1.1 and 1.2 that

Corollary 1.1 Large Deviation Principle *Assume $m < \infty$ and either $d_{\min} \geq 2$ or $\lambda \geq 1$. Then, the random variables $|X_n|/n$ satisfy, under both P_λ and $P_{\lambda,\omega}$, the large deviation principle on $[0,1]$ with speed n and the same (convex) continuous, rate function $I_\lambda(\cdot)$.*

We pause now to explain in more detail the "uncertainty" mentioned in the title. Consider the first time T_n that the walk hits D_n and its location X_{T_n} at that time. Conditioning on X_{T_n}, and even on all vertices of the tree up to level n, $k_{X_{T_n}}$ is distributed according to $\{p_k\}$. Thus, certainty in the position of the walk yields uncertainty on its expected drift, that is on the "momentum" of the walk. Our work can be seen as a converse to this statement: the growth of the tree provides a large uncertainty in the value of X_{T_n}, even when conditioning on the tree up to level n and on T_n being atypical. (A precise quantification of this uncertainty is contained in Propositions 4.1 and 5.1 below). This uncertainty then leads to concentration, and hence certainty, for the expected momentum of the walk at time T_n. This certainty is at the heart of the equality between the quenched and annealed asymptotics.

Returning to our results, it is evident from the statement of Theorem 1.2 that slowdown probabilities may decay slower than exponential. We present below only a partial analysis of that case, pertaining to $d_{\min} = \lambda$. While we show that in this case, the quenched and annealed subexponential decay rates are of the same order, we cannot show they are actually equal. See Section 7 for comments on the case $d_{\min} \neq \lambda$.

Theorem 1.3 Slowdown probabilities - subexponential regime *Assume $d_{\min} = \lambda < m < \infty$ and $0 \leq b < a < v_\lambda$.*
(i). If $d_{\min} = 1$ then there exist finite constants $C_1, C_2 > 0$ such that

$$-C_1 \leq \liminf_{n\to\infty} \frac{\log P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in (b,a)\right)}{n^{1/3}} \leq \limsup_{n\to\infty} \frac{\log P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in (b,a)\right)}{n^{1/3}} \leq -C_2\,, \quad GW - a.s.$$

(ii). If $d_{\min} > 1$ then there exist finite constants $C_1, C_2 > 0$ such that

$$-C_1 \leq \liminf_{n\to\infty} \frac{\log P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in (b,a)\right)}{n/(\log n)^2} \leq \limsup_{n\to\infty} \frac{\log P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in (b,a)\right)}{n/(\log n)^2} \leq -C_2\,, \quad GW - a.s.$$

Both parts of the theorem apply with $P_{\lambda,\omega}$ replaced by P_λ.

A comparison with the subexponential slowdown regime for RWRE in $\mathbb{Z}^d$ is again in order: for $d = 1$, quenched and annealed estimates differ sharply, see the review in [5]. In higher dimensions, Sznitman [17] shows that for a class of transient walks ("neutral or biased to the right"), the quenched slowdown probabilities are of order $\exp(-Cn/(\log n)^{2/d})$ whereas the annealed ones are of order $\exp(-Cn^{d/(d+2)})$.

The structure of the article is as follows: we present some auxiliary lemmas in Section 2. The computation of the extreme large deviations (corresponding to speeds 0 and 1), and the proof that quenched and annealed asymptotics coincide in this extreme case, are provided in Section 3. Reading this section is an opportunity to appreciate our use of uncertainty in a situation where it is not hidden by technicalities. The speedup estimate Theorem 1.1 is proved in Section 4, while the slowdown Theorem 1.2 is proved in Section 5. The subexponential rates for slowdown contained in Theorem 1.3 are discussed in Section 6. Finally, Section 7 presents additional comments and open problems.

2 Generalities and auxiliary results

We begin with a useful, well known general lemma relating quenched and annealed rates of decay.

Lemma 2.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(E, \mathcal{E})$ be a Polish space equipped with the Borel σ-field. For each $\omega \in \Omega$ let P_ω be a measure on E, such that the map $\omega \mapsto P_\omega(A) \in [0,1]$ is measurable for each $A \in \mathcal{E}$. For any $A_n \in \mathcal{E}$, both*

$$\limsup_{n\to\infty} \frac{1}{n} \log \int P_\omega(A_n) P(d\omega) \geq \limsup_{n\to\infty} \frac{1}{n} \log P_\omega(A_n), \quad P-a.s.$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \int P_\omega(A_n) P(d\omega) \geq \liminf_{n\to\infty} \frac{1}{n} \log P_\omega(A_n), \quad P-a.s.$$

Next, we introduce several notations in $\mathcal{T}$. For $\mathbf{T} \in \mathcal{T}$ and any vertex $j \in D_n(\mathbf{T})$, we let $\mathbf{T}^j$ denote the subtree rooted at j consisting of all descendants of j in $\mathbf{T}$. For $j \in D_1(\mathbf{T})$, we let $\mathbf{T}_j^R$ denote the subtree $\mathbf{T} \setminus \mathbf{T}^j$ ($\mathbf{T}_j^R$ consists of the root if the root has only one child). Finally, for $j \in D_n(\mathbf{T})$ and l an ancestor of j, we let l^j denote the unique child of l which is an ancestor of j.

Recall $\mathbf{o}$ denotes the root. We consider functions assigning to a tree and a vertex on the first level of the tree the value 0 or 1. A function $A : \mathcal{T} \times D_1(\mathbf{T}) \mapsto \{0,1\}$ is called R-defined if, for $j \in D_1(\mathbf{T})$, it holds that $A(\mathbf{T}, j)$ is measurable with respect to the σ-field generated by $\mathbf{T}_j^R$, and $A(\mathbf{T}, j) = 0$ if $k_{\mathbf{o}} = 1$.

A key technical tool which enters in different places in our arguments is the following variant of a lemma contained in [7].

Lemma 2.2 *For any R-defined $A(\cdot,\cdot)$ consider the random variables*

$$N_n^A(j) = \Big| \{l : l \text{ is an ancestor of } j \text{ such that } A(\omega^l, l^j) = 1\} \Big| ,$$

where $j \in D_n(\omega)$. Assume that $m < \infty$ and

$$E_{GW} \left(\sum_{j \in D_1(\omega)} \mathbf{1}_{\{A(\omega, j) = 0\}} \right) < 1 \, .$$

Then, there exists a $\beta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log GW \left(\min_{j \in D_n} \frac{N_n^A(j)}{n} < \beta \right) < 0 \, , \tag{2.1}$$

which implies also that

$$\liminf_{n\to\infty} \min_{j \in D_n} \frac{N_n^A(j)}{n} \geq \beta \, , \quad GW - a.s. \tag{2.2}$$

Proof of Lemma 2.2: Define $\zeta_n = \sum_{j \in D_n} e^{-\theta N_n^A(j)}$. Then,

$$\begin{aligned} E_{GW}(\zeta_n) &= E_{GW} \left(\sum_{v \in D_{n-1}} e^{-\theta N_{n-1}^A(v)} \cdot E_{GW} \left(\sum_{j \in D_1(\omega^v)} \mathbf{1}_{\{A(\omega^v, j) = 0\}} + \mathbf{1}_{\{A(\omega^v, j) = 1\}} e^{-\theta} \right) \right) \\ &:= c(\theta) E_{GW} \left(\zeta_{n-1} \right) \end{aligned} \tag{2.3}$$

where

$$c(\theta) = E_{GW}\left(\sum_{j\in D_1(\omega)} \mathbf{1}_{\{A(\omega,j)=0\}} + \mathbf{1}_{\{A(\omega,j)=1\}}e^{-\theta}\right) = E_{GW}(\zeta_1)\,. \tag{2.4}$$

We have $c(\theta) < \infty$ due to our assumption $m < \infty$. For any $\theta > 0$, Markov's inequality implies

$$\begin{aligned} GW\left(\min_{v\in D_n} N_n^A(v) \le \beta n\right) &\le e^{\beta\theta n} E_{GW}\left(\sum_{v\in D_n} e^{-\theta N_n^A(v)}\right) \\ &= e^{n[\beta\theta + \log c(\theta)]} := e^{n\alpha(\theta)}\,. \end{aligned}$$

Since for $\beta = 0$, $\lim_{\theta\to\infty}\alpha(\theta) < 0$, it follows that for some $\beta_0 > 0$, there still exists a θ such that $\alpha(\theta) < 0$, proving (2.1). Then, the Borel-Cantelli lemma completes the proof of (2.2). ☐

3 Extreme exponential rates - proof of (1.3) and (1.5)

As a warm-up for the proofs of Theorems 1.1 and 1.2, and to exhibit some of the ideas which occur there, we begin by proving (1.3) and (1.5).

Proof of (1.3): We let $\mathcal{F}_n = \sigma(\cup_{i=0}^n D_i)$. Then,

$$\begin{aligned} P_\lambda(|X_n| = n) &= P_\lambda(|X_{n-1}| = n-1) P_\lambda(|X_n| = n \,\big|\, |X_{n-1}| = n-1) \\ &= P_\lambda(|X_{n-1}| = n-1)\sum_{k=1}^\infty \frac{k}{k+\lambda} p_k = \left(\sum_{k=1}^\infty \frac{k}{k+\lambda}p_k\right)^n \end{aligned}$$

where the last equality is obtained by iterations, and this shows that

$$\lim_{n\to\infty}\frac{1}{n}\log P_\lambda(|X_n| = n) = \log\sum_{k=1}^\infty \frac{k}{k+\lambda}p_k\,.$$

It thus remains to compute the limit of $n^{-1}\log P_{\lambda,\omega}(|X_n| = n)$. We have

$$P_{\lambda,\omega}(|X_n| = n) = P_{\lambda,\omega}(|X_{n-1}| = n-1)\sum_{j\in D_{n-1}} a_{n,j}\frac{k_j}{k_j+\lambda} =: P_{\lambda,\omega}(|X_{n-1}| = n-1) Z_n \tag{3.1}$$

where

$$a_{n,j} := P_{\lambda,\omega}\left(X_{n-1} = j \,\Big|\, |X_{n-1}| = n-1\right)\,.$$

Note that $k_j/(k_j+\lambda)$, $j \in D_{n-1}$, are i.i.d. under P_λ and independent of $\mathcal{F}_{n-1}$ whereas $\{a_{n,j}\}_{j\in D_{n-1}}$, are $\mathcal{F}_{n-1}$-measurable and $\sum_{j\in D_{n-1}} a_{n,j} = 1$. We have

$$E(Z_n) = \sum_{k=1}^\infty \frac{k}{k+\lambda}p_k \tag{3.2}$$

and we will show that Z_n concentrates at its expectation.

(i) Assume first that $d_{\min} > 1$, then it holds that $P_{\lambda,\omega}(|X_{n-1}| = n-1) \geq (2/(2+\lambda))^{n-1}$, and therefore

$$a_{n,j} = \frac{P_{\lambda,\omega}(X_{n-1} = j)}{P_{\lambda,\omega}(|X_{n-1}| = n-1)} \leq \frac{\left(\frac{1}{2+\lambda}\right)^{n-1}}{\left(\frac{2}{2+\lambda}\right)^{n-1}} = \left(\frac{1}{2}\right)^{n-1} .$$

We have $E(Z_n|\mathcal{F}_{n-1}) = E(Z_n)$, hence $\operatorname{Var}(Z_n) = E(\operatorname{Var}(Z_n|\mathcal{F}_{n-1}))$. But

$$\operatorname{Var}(Z_n|\mathcal{F}_{n-1}) \leq \left(\max_{j\in D_{n-1}} a_{n,j}\right) \sum_{j\in D_{n-1}} a_{n,j} \operatorname{Var}\left(\frac{k_j}{k_j+\lambda}\right) \leq \left(\frac{1}{2}\right)^{n-1} \operatorname{Var}\left(\frac{k_j}{k_j+\lambda}\right) .$$

By Chebychev's inequality and the Borel-Cantelli lemma it follows that for any $\delta > 0$ and GW-a.e. ω there exists $n_0 = n_0(\delta,\omega)$ finite, such that

$$|Z_n - E(Z_n)| \leq \delta \text{ for } \; n \geq n_0 .$$

Together with (3.1) and (3.2), this proves that for GW-a.e. ω,

$$\lim_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(|X_n| = n) = \log \sum_{k=1}^{\infty} \frac{k}{k+\lambda} p_k .$$

(ii) Assume that $p_1 > 0$. Let $b(j)$ be the number of vertices on the path from 0 to j which have at least two children, and

$$N_n := \min_{j\in D_n} b(j) .$$

An application of Lemma 2.2 (taking there $A(\omega,j) = \mathbf{1}_{\{k_o\geq 2\}}$) yields that there is $\beta_0 > 0$ such that for GW-a.e. ω, $\liminf_{n\to\infty} N_n/n \geq \beta_0$. Now,

$$P_{\lambda,\omega}(|X_{n-1}| = n-1) \geq \left(\frac{1}{1+\lambda}\right)^{n-1}$$

and, as before, for $n > n_1(\omega)$ for some $n_1(\omega)$ which is finite GW-a.s.,

$$a_{n,j} = \frac{P_{\lambda,\omega}(X_{n-1} = j)}{P_{\lambda,\omega}(|X_{n-1}| = n-1)} \leq \frac{\left(\frac{1}{2+\lambda}\right)^{N_{n-1}} \left(\frac{1}{1+\lambda}\right)^{n-1-N_{n-1}}}{\left(\frac{1}{1+\lambda}\right)^{n-1}} \leq \left(\frac{1+\lambda}{2+\lambda}\right)^{N_{n-1}} \leq \left(\frac{1+\lambda}{2+\lambda}\right)^{(n-1)\beta_0} ,$$

and we proceed as in case (i). □

Proof of (1.5): By Lemma 2.1, it is enough to provide a lower bound for the quenched probabilities and an upper bound for the corresponding annealed probabilities.

(i) We have, by coupling,

$$P_\lambda(|X_n| \leq n\varepsilon) \leq P_{\lambda,\omega_{\min}}(|X_n| \leq n\varepsilon) \tag{3.3}$$

where $\omega_{\min}$ denotes the $d_{\min}$-ary rooted tree, that is, with each vertex having $d_{\min}$ children. Fix $p \in (0,1)$, and let $S_n^{R,p}$ denote the (biased) reflected random walk on $\mathbb{Z}_+$ with $P(S_{n+1}^{R,p} = x+1|S_n^{R,p} = x) = p + (1-p)\mathbf{1}_{\{x=0\}}$. We use $P_{SRW^R(p)}$ to denote the law of $S_n^{R,p}$. For $Y_1, Y_2, \ldots$, i.i.d. with $P(Y_i = 1) = p = 1 - P(Y_i = -1)$ and $S_n^p := \sum_{i=1}^n Y_i$, $S_0 = 0$, we denote the distribution of the biased simple random walk $\{S_n^p\}$ as $P_{SRW(p)}$.

Then, under $P_{\lambda,\omega_{\min}}$, the random variable $|X_n|$ has the same distribution as $S_n^{R,p}$ for $p := d_{\min}/(d_{\min}+\lambda)$. By Cramér's Theorem (see [3, Theorem 2.2.3]), when $d_{\min} > \lambda$, that is $p > 1/2$ and ε is small enough,

$$\limsup_{n\to\infty} \frac{1}{n} \log P(S_n^{R,p} \leq n\varepsilon) \leq \limsup_{n\to\infty} \frac{1}{n} \log P(S_n^p \leq n\varepsilon) \leq -H\left(\frac{\varepsilon+1}{2}\Big|p\right) \tag{3.4}$$

Together with (3.3) and the continuity of $H(\cdot|p)$, this implies

$$\limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_\lambda\left(|X_n| \leq n\varepsilon\right) \leq -H\left(\frac{1}{2}\Big|p\right) \tag{3.5}$$

(ii) Turning to a lower bound for the quenched probabilities, in case $d_{\min} \geq 2$, fix $C < 1/(\log d_{\min})$ with $b_n := \lfloor \log n \rfloor^3$ and h_n denoting the even integer nearest to $C \log\log n$. We call a tree $\mathbf{T} \in \mathcal{T}$ *n-slow* if there exists a vertex $j \in D_{b_n}(\mathbf{T})$ such that:

H1. The finite subtree consisting of the first h_n levels of $\mathbf{T}^j$, denoted $\hat{\mathbf{T}}_n^j$, is a finite rooted $d_{\min}$-ary tree.

H2. All ancestors l of j satisfy $k_l \leq e^{(\log n)^4}$.

We next check that the assumption $m < \infty$ implies that there exists an $n_0(\omega)$ such that GW-a.e. ω is n-slow for all $n > n_0(\omega)$. Indeed, for a fixed vertex l,

$$GW(k_l \geq e^{(\log n)^4}) \leq m e^{-(\log n)^4}.$$

Consider the subtree Θ_n of ω obtained by removing, for each vertex $j \in D_i(\omega)$, $i \leq b_n$, all children (and their descendants) except for the first $d_{\min}$. Then,

$$GW\left(\max_{l\in D_i(\Theta_n), i=1,\ldots,b_n} k_l \geq e^{(\log n)^4}\right) \leq m e^{-(\log n)^4} (d_{\min})^{b_n+1},$$

whereas, for n large enough,

$$GW(\nexists j \in D_{b_n}(\Theta_n) \text{ satisfying } \mathbf{H1}) \leq \left(1 - (p_{d_{\min}})^{d_{\min}^{h_n}}\right)^{d_{\min}^{b_n}} \leq e^{-n}.$$

Therefore, for n large enough,

$$GW(\omega \text{ is not } n\text{-slow}) \leq GW(\Theta_n \text{ is not } n\text{-slow}) \leq e^{-n} + e^{-(\log n)^3},$$

and the Borel-Cantelli lemma completes the claim.

Let now ω be n-slow. Then, with $j \in D_{b_n}(\omega)$ as in the definition of n-slow, and $k_n = h_n/2 + b_n$, we have for all $t > k_n$,

$$\begin{aligned} P_{\lambda,\omega}(T_{b_n+h_n} \geq t) &\geq P_{\lambda,\omega}\left(X_{k_n} \in D_{h_n/2}(\hat{\omega}_n^j),\ X_\ell \in \hat{\omega}_n^j, \ell = k_n+1,\ldots,t\right) \\ &\geq (e^{(\log n)^4} + \lambda)^{-b_n} p^{h_n/2} P(t_n > t) \end{aligned} \tag{3.6}$$

where $p = d_{\min}/(d_{\min}+\lambda) > 1/2$ and

$$t_n := \inf\left\{j > 0 : |S_j^p| = \frac{h_n}{2}\right\}$$

denotes the exit time of $SRW(p)$ from the interval $\left(-\frac{h_n}{2}, \frac{h_n}{2}\right)$. We have, with $A_t := \{t_n \geq t\}$,

$$\liminf_{t\to\infty} t^{-1} \log P_{SRW(p)}(A_t) \geq \liminf_{t\to\infty} t^{-1} \log P_{SRW(\frac{1}{2})}(A_t) - H\left(\frac{1}{2}\Big|p\right) \tag{3.7}$$

as a consequence of the following standard argument: Let P, Q be probability distributions, $\{\mathcal{F}_t\}$ an increasing sequence of σ-fields such that $Q \ll P$ on $\mathcal{F}_t$, and $\{A_t\}$ a sequence of $\mathcal{F}_t$-measurable events such that $Q(A_t) > 0$. We have

$$\log P(A_t) \geq \log Q(A_t) - \frac{1}{Q(A_t)} \int_{A_t} \log \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} dQ\,.$$

We conclude, provided that $t^{-1} \log \frac{dQ}{dP}|_{\mathcal{F}_t}$ converges to some constant $h(Q|P)$ uniformly on A_t, that

$$\liminf_{t\to\infty} t^{-1} \log P(A_t) \geq \liminf_{t\to\infty} t^{-1} \log Q(A_t) - h(Q|P)\,.$$

Recall [15, Page 243], that

$$h_n^2 t^{-1} \log P_{SRW(\frac{1}{2})}(A_t) \to -\frac{\pi^2}{2} \tag{3.8}$$

as $n, t \to \infty$, provided $h_n^2/t \to 0$. GW-a.a. ω are n-slow for all large n, hence applying (3.6), (3.7) and (3.8) for $t = n$, imply that for any $\varepsilon > 0$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(|X_n| \leq n\varepsilon) \geq \liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(T_{\varepsilon n} \geq n) \geq -H\left(\frac{1}{2}\Big|p\right)\,, \tag{3.9}$$

which together with (3.5) and Lemma 2.1 completes the proof of (1.5). □

4 Proof of Theorem 1.1

As mentioned in the introduction, our main tool is the analysis of the hitting times T_n. It turns out also that a crucial ingredient is the fact that the exponential rate of decay of atypical behavior of T_n is not changed significantly when one also asks that the walk does not return to the root. This latter fact is of course easier to demonstrate in the case of speed-up considered here.

Throughout the proof, we use j to denote a vertex of the rooted Galton-Watson tree ω. Let ω^j denote the subtree of ω consisting of j and its descendants, and let $\overline{\omega}^j$ denote the graph consisting of ω^j and an additional edge connecting j to itself. We start a λ-biased simple random walk on $\overline{\omega}^j$, and denote its law by $P_{\lambda,\omega,j}$ (here, at the root j, we have

$$P_{\lambda,\omega,j}\Big(X_1 = j | X_0 = j\Big) = \frac{\lambda}{(k_j + \lambda)} \tag{4.1}$$

where k_j is the number of children of j). Note that we have now two different measures $P_{\lambda,\omega}$ and $P_{\lambda,\omega,\mathbf{o}}$ for λ-biased random walks on ω, which differ only at visits in $\mathbf{o}$. Define for $j \in \omega$ the hitting time $\widetilde{T}_m(j)$ of level m in $\overline{\omega}^j$ and the time $\tau_j \in \{1, \ldots, \infty\}$ of first return to the root of $\overline{\omega}^j$ of a walk started at the root of $\overline{\omega}^j$. As in (1.1) let

$$P_{\lambda,\mathbf{o}} := \int P_{\lambda,\omega,\mathbf{o}}(\cdot) GW(d\omega)$$

Then, for any $\Delta > 0$ and $\alpha > 1$,

$$P_{\lambda,\omega}\Big(T_{n+\Delta} \leq \alpha(n+\Delta)\Big) \geq P_{\lambda,\omega}(T_n \leq \alpha n) e_\Delta Z_n \tag{4.2}$$

where

$$
\begin{aligned}
Z_n &:= \sum_{j\in D_n} \Big(p(\Delta,j)/e_\Delta\Big) a(n,j) \\
a(n,j) &:= P_{\lambda,\omega}\left(X_{T_n}=j|T_n\le \alpha n\right) \\
p(\Delta,j) &:= P_{\lambda,\omega}\left(\widetilde{T}_\Delta(j)\le \alpha\Delta, \tau_j > \widetilde{T}_\Delta(j)|X_{T_n}=j, T_n\le\alpha n\right) \\
e_\Delta &:= E_\lambda\left(p(\Delta,j)\right)\,,
\end{aligned}
$$

and because of the Markov property,

$$
p(\Delta,j) = P_{\lambda,\omega,j}\left(\widetilde{T}_\Delta(j)\le\alpha\Delta, \tau_j>\widetilde{T}_\Delta(j)\right)\,.
$$

Note that

$$
e_\Delta = E_\lambda\left(p(\Delta,j)\right) = P_{\lambda,\mathbf{o}}(T_\Delta\le\alpha\Delta, \tau_{\mathbf{o}}>T_\Delta)\,,
$$

is independent of j, where $\tau_{\mathbf{o}}$ denote the time of first return to the root. Moreover, the event $\{\tau_{\mathbf{o}}>T_\Delta, T_\Delta\le \alpha\Delta\}$ contains the event of the random walk taking its first Δ steps down from the root, an event whose probability is at least $(\lambda+1)^{-\Delta}$ for any Galton-Watson tree with $p_0=0$. Thus, $e_\Delta\ge(\lambda+1)^{-\Delta}$ for all Δ. Recall that $E(Z_n)=1$ by the independence of $p(\Delta,j)$ and $a(n,j)$. Moreover, the random variables $p(\Delta,j)/e_\Delta$ are i.i.d. (but of law depending upon Δ), whereas $\sum_{j\in D_n} a(n,j)=1$ for all n. Let $\mathcal{G}_n$ be the σ-field generated by $\{a(n,j): j\in D_n\}$.

For a ray r emanating from the root, let

$$
N_n(r) = \left\{\#\text{ of vertices on } r\bigcap\{\bigcup_{k=1}^n D_k\} \text{ with more than one child}\right\}
$$

Applying Lemma 2.2 for $A(\omega,j)=\mathbf{1}_{\{k_{\mathbf{o}}\ge 2\}}$ we see that there exists a $\beta_0>0$ such that

$$
\liminf_{n\to\infty}\inf_r\{N_n(r)/n\}>\beta_0\,,\quad GW-a.s. \tag{4.3}
$$

Fix $c>1/\beta_0$ and let $\mathcal{A}_n=\{\omega:\max_{j\in D_n}a(n,j)\le c/n\}$ which is measurable on $\mathcal{G}_n$. Then, for all $\theta>0,\delta>0,n$,

$$
\begin{aligned}
P_\lambda(Z_n\le\delta,\mathcal{A}_n)\le e^{\theta\delta}E_\lambda\left(e^{-\theta Z_n}\mathbf{1}_{\mathcal{A}_n}\right) &= e^{\theta\delta}E_\lambda\left(\mathbf{1}_{\mathcal{A}_n}E_\lambda(e^{-\theta Z_n}|\mathcal{G}_n)\right) \\
&= e^{\theta\delta}E_\lambda\left(\mathbf{1}_{\mathcal{A}_n}e^{\sum_{j\in D_n}\phi_\Delta(\theta a(n,j))}\right)
\end{aligned}
$$

where

$$
\phi_\Delta(\eta):=\log E_\lambda\left(e^{-\eta p(\Delta,j)/e_\Delta}\right)\le\log\Big(1-(1+\lambda)^{-\Delta}+(1+\lambda)^{-\Delta}e^{-\eta(1+\lambda)^\Delta}\Big):=\psi(\eta)
$$

since $0\le p(\Delta,j)/e_\Delta\le(1+\lambda)^\Delta$, $E_\lambda(p(\Delta,j)/e_\Delta)=1$, and we used the inequality $e^{-ab}\le 1-b+be^{-a}$ for $b\le1$ with $a=\eta(1+\lambda)^\Delta$ and $b=p(\Delta,j)/(e_\Delta(1+\lambda)^\Delta)$. Hence,

$$
P_\lambda(Z_n\le\delta,\mathcal{A}_n)\le e^{\theta\delta}e^{J_n(\theta,c)}
$$

where

$$
J_n(\theta,c)=\sup\Big(\sum_{j=1}^\infty\psi(\eta_j)\;:\;0\le\eta_j\le\frac{\theta c}{n},\sum_{j=1}^\infty\eta_j=\theta\Big)\,.
$$

Note that $\eta \mapsto \psi(\eta)$ is convex, hence $J_n(\theta, c)$, being the supremum of a convex function of $\{\eta_j\}$, subject to a convex constraint set, is obtained at one of the extremal points of the constraint set. Thus, $J_n(\theta, c) = (n/c)\psi(\theta c/n)$, and optimizing over $\theta \geq 0$ yields

$$P_\lambda(Z_n \leq \delta, \mathcal{A}_n) \leq \exp(-\frac{n}{c} H(\delta(1+\lambda)^{-\Delta} | (1+\lambda)^{-\Delta}))\,.$$

We have the following proposition, whose proof is deferred.

Proposition 4.1 *(Uncertainty estimate) Let*

$$B_k = \bigcap_{n \geq k} \mathcal{A}_n = \{\omega : \max_{\substack{j \in D_n \\ n \geq k}} a(n,j) n \leq c\}\,.$$

Then,

$$GW\left(\bigcup_{k=1}^{\infty} B_k\right) = 1\,.$$

Since, for every k,

$$\sum_{n=1}^{\infty} P_\lambda(Z_n \leq \delta, B_k) \leq k + \sum_{n=1}^{\infty} P_\lambda(Z_n \leq \delta, \mathcal{A}_n) < \infty\,,$$

Proposition 4.1 and the Borel-Cantelli lemma imply that for any fixed Δ and δ, $1 \leq \Delta < \infty$, $\delta \in (0,1)$, and a.e. ω, there exists an $n_0(\omega) < \infty$ such that $Z_n \geq \delta$ for any $n \geq n_0(\omega)$.

Note that $P_{\lambda,\omega}(T_n \leq \alpha n) \geq P_{\lambda,\omega}(T_n = n) \geq (1+\lambda)^{-n} > 0$. Hence, for all ω and $n \geq n_0(\omega)$, writing $n = N\Delta + n_1$, $n_1 \in [n_0, n_0 + \Delta)$ and N integer, we have by iterating (4.2)

$$\begin{aligned}
P_{\lambda,\omega}(T_n \leq \alpha n) &\geq P_{\lambda,\omega}(T_{n_1} \leq \alpha n_1) e_\Delta^N \prod_{i=0}^{N-1} Z_{n_1 + i\Delta} \\
&\geq P_{\lambda,\omega}(T_{n_1} \leq \alpha n_1)(\delta e_\Delta)^N
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{n} \log P_{\lambda,\omega}(T_n \leq \alpha n) &\geq \frac{1}{N\Delta + n_1} \log P_{\lambda,\omega}(T_{n_1} \leq \alpha n_1) + \frac{1}{\Delta + n_1/N} \log(\delta e_\Delta) \\
&\geq \frac{1}{N\Delta + n_1} \log((1+\lambda)^{-n_1}) + \frac{1}{\Delta + n_1/N} \log(\delta e_\Delta)
\end{aligned}$$

Taking $n \to \infty$ (hence $N \to \infty$), fixing $\Delta \in [1, \infty)$ we thus get for all $\delta \in (0,1)$ that GW-a.s.,

$$\liminf_{n \to \infty} \frac{1}{n} \log P_{\lambda,\omega}(T_n \leq \alpha n) \geq \frac{1}{\Delta} \log(\delta e_\Delta)\,.$$

Hence, with $\Delta \to \infty$ we have that GW-a.s.,

$$\begin{aligned}
\liminf_{n \to \infty} \frac{1}{n} \log P_{\lambda,\omega}(T_n \leq \alpha n) &\geq \limsup_{\Delta \to \infty} \frac{1}{\Delta} \log e_\Delta \\
&= \limsup_{\Delta \to \infty} \frac{1}{\Delta} \log P_{\lambda,\mathbf{o}}(T_\Delta \leq \alpha\Delta, \tau_{\mathbf{o}} > T_\Delta)
\end{aligned}$$

$$\geq \limsup_{\Delta\to\infty} \frac{1}{\Delta} \log P_\lambda(T_\Delta \leq \alpha\Delta, \tau_{\mathbf{o}} > T_\Delta)\,, \tag{4.4}$$

where the last inequality holds since

$$\begin{aligned} P_{\lambda,\omega,\mathbf{o}}(T_n \leq \alpha n, \tau_{\mathbf{o}} > T_n) &= \frac{k_{\mathbf{o}}}{\lambda + k_{\mathbf{o}}} P_{\lambda,\omega}\left(T_n \leq \alpha n, \tau_{\mathbf{o}} > T_n\right) \\ &\geq \frac{1}{\lambda+1} P_{\lambda,\omega}\left(T_n \leq \alpha n, \tau_{\mathbf{o}} > T_n\right) \end{aligned}$$

and hence

$$P_{\lambda,\mathbf{o}}(T_n \leq \alpha n, \tau_{\mathbf{o}} > T_n) \geq \frac{1}{\lambda+1} P_\lambda\left(T_n \leq \alpha n, \tau_{\mathbf{o}} > T_n\right)\,.$$

We next show that for any ω, all n and α

$$P_{\lambda,\omega}(T_n \leq \alpha n) \leq \alpha n P_{\lambda,\omega}(T_n \leq \alpha n, \tau_{\mathbf{o}} > T_n)\,. \tag{4.5}$$

out of which we have that for all Δ, α,

$$P_\lambda(T_\Delta \leq \alpha\Delta, \tau_{\mathbf{o}} > T_\Delta) \geq \frac{1}{\alpha\Delta} P_\lambda(T_\Delta \leq \alpha\Delta) \tag{4.6}$$

(in fact, whenever $\lambda < d_{\min}$, the factor $1/\alpha\Delta$ in the right hand side of (4.6) can be replaced by a constant, but we will not need it here).

Indeed, let

$$A_k = \{T_n \leq \alpha n, \text{last visit to the root before time } T_n \text{ is at time } k\}\,.$$

Then, $A_0 = \{T_n \leq \alpha n, \tau_{\mathbf{o}} > T_n\}$ while

$$\{T_n \leq \alpha n\} = \bigcup_{k=0}^{[\alpha n]-1} A_k\,.$$

But,

$$P_{\lambda,\omega}(A_k) \leq P_{\lambda,\omega}(T_n \leq \alpha n - k, \tau_{\mathbf{o}} > T_n) \leq P_{\lambda,\omega}(A_0)\,,$$

implying that

$$P_{\lambda,\omega}(T_n \leq \alpha n) \leq \alpha n P_{\lambda,\omega}(A_0)\,,$$

and hence (4.5) and (4.6). Next, (4.6) implies that

$$\limsup_{\Delta\to\infty} \frac{1}{\Delta} \log P_\lambda(T_\Delta \leq \alpha\Delta, \tau_{\mathbf{o}} > T_\Delta) = \limsup_{\Delta\to\infty} \frac{1}{\Delta} \log P_\lambda(T_\Delta \leq \alpha\Delta)$$

Recall (4.4) and conclude that GW-a.s.,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(T_n \leq \alpha n) \geq \limsup_{n\to\infty} \frac{1}{n} \log P_\lambda(T_n \leq \alpha n)\,.$$

Together with Lemma 2.1 and (4.5), this proves that GW-a.s.,

$$-J_\lambda^<(\alpha) := \lim_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(T_n \leq \alpha n) = \lim_{n\to\infty} \frac{1}{n} \log P_\lambda(T_n \leq \alpha n, \tau_{\mathbf{o}} > T_n) = \lim_{n\to\infty} \frac{1}{n} \log P_\lambda(T_n \leq \alpha n)\,. \tag{4.7}$$

Note that since for $\theta \in (0,1)$ and $\alpha = \theta\gamma_1 + (1-\theta)\gamma_2$,

$$P_{\lambda,\omega}(T_n \le \alpha n, \tau_{\mathbf{o}} > T_n) \ge P_{\lambda,\omega}(T_{\theta n} \le \theta\gamma_1 n, \tau_{\mathbf{o}} > T_n, T_n \le \alpha n)$$
$$\ge \sum_{j \in D_{\theta n}} P_{\lambda,\omega}(T_{\theta n} \le \theta\gamma_1 n, X_{T_{\theta n}} = j, \tau_{\mathbf{o}} > T_{\theta n}) P_{\lambda,\omega,j}(\widetilde{T}_{(1-\theta)n}(j) \le (1-\theta)\gamma_2 n, \tau_j > \widetilde{T}_{(1-\theta)n}(j)) \,,$$

it follows, taking expectations and using independence, that

$$J_\lambda^<(\alpha) \le \theta J_\lambda^<(\gamma_1) + (1-\theta) J_\lambda^<(\gamma_2) \,,$$

i.e. $J_\lambda^<(\cdot)$ is convex. We prove below the

Proposition 4.2 *(Exponential decay) $J_\lambda^<(x) > 0$ for $x < 1/v_\lambda$.*

Recall that $|X_n|/n \to v_\lambda$ a.s. Obviously, $T_n \to \infty$, hence $n/T_n = |X_{T_n}|/T_n \to v_\lambda$ as well. Thus, by (4.7), $J_\lambda^<(x) = 0$ for all $x > 1/v_\lambda$. It follows from Proposition 4.2 and the convexity of $J_\lambda^<(\cdot)$ that $J_\lambda^<$ is strictly decreasing on $[1, 1/v_\lambda]$. We now have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in [a,b)\right) &\le \limsup_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(T_{na} \le n) \\ &= a \limsup_{n\to\infty} \frac{1}{na} \log P_{\lambda,\omega}\left(T_{na} \le \frac{na}{a}\right) = -a J_\lambda^<\left(\frac{1}{a}\right)\end{aligned} \tag{4.8}$$

while, for $b > a$ and $a \in (v_\lambda, 1]$, there exists $\varepsilon > 0$ small enough, such that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in [a,b)\right) &\ge \liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}\left(\frac{T_{\lfloor n(a+\varepsilon)\rfloor}}{n} \in (1, 1+\varepsilon]\right) \\ &\ge \liminf_{n\to\infty} \frac{1}{n} \log\left[P_{\lambda,\omega}(T_{\lfloor n(a+\varepsilon)\rfloor} \le n(1+\varepsilon)) - P_{\lambda,\omega}(T_{\lfloor n(a+\varepsilon)\rfloor} \le n)\right] \\ &= -(a+\varepsilon) J_\lambda^<\left(\frac{1+\varepsilon}{a+\varepsilon}\right)\end{aligned} \tag{4.9}$$

where the last equality is due to the fact that $J_\lambda^<$ is *strictly* decreasing and $a > v_\lambda$. Hence, taking the limit as $\varepsilon \to 0$, we conclude from (4.8) and the above that

$$\lim_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in [a,b)\right) = -a J_\lambda^<\left(\frac{1}{a}\right) := -I_\lambda(a) \,. \tag{4.10}$$

The convexity and strict monotonicity of I_λ follow from the corresponding properties of $J_\lambda^<$.

We show below that for some $c(\varepsilon) \to_{\varepsilon\to 0} 0$, any $\omega \in \mathcal{T}$, $n' \le (1+\varepsilon)n$ and $j \in D_n(\omega)$,

$$P_{\lambda,\omega}(X_{n'} = j) \le P_{\lambda,\omega}(X_n = j) e^{c(\varepsilon)n} \,. \tag{4.11}$$

Summing over $j \in D_n(\omega)$ and n' then leads to

$$P_{\lambda,\omega}(|X_{n(1+\varepsilon)}| \ge n) \le \sum_{n'=n}^{n(1+\varepsilon)} P_{\lambda,\omega}(|X_{n'}| = n) \le 2n\varepsilon P_{\lambda,\omega}(|X_n| = n) e^{c(\varepsilon)n} \,.$$

Considering the scaled logarithmic limits of both sides, it follows from (4.10) that

$$-(1+\varepsilon) I_\lambda(\frac{1}{1+\varepsilon}) \le -I_\lambda(1) + c(\varepsilon) \,,$$

so taking $\varepsilon \to 0$ we see that the convex function $I_\lambda : [0, v_\lambda] \mapsto \mathbb{R}_+$ is lower semi-continuous at $x = 1$, hence continuous on $[v_\lambda, 1]$. Turning to prove (4.11), fix $\omega \in \mathcal{T}$, $\varepsilon > 0$, $n' \le (1+\varepsilon)n$ and the ray r emanating from the root such that $j \in D_n(\omega)$ is in r. To any path of the λ-biased random walk for which $X_{n'} = j$, there corresponds a vector $\mathbf{l}$ of (even) integers $l_t \ge 2$ with $\sum_{t=1}^k l_t = n' - n$, and a vector $\mathbf{u}$ of integers $0 \le u_1 \le u_2 \le \ldots \le u_k \le n$, such that upon reaching the unique vertex $v_t \in r \cap D_{u_t}$ (at its $u_t + \sum_{s<t} l_s$ step), the path makes an excursion of length l_t within the subtree ω^{v_t} ending with its first return to v_t (in case $u_{t+1} = u_t$ the next such excursion shall be in the same subtree, and so on). Decomposing the event $\{X_{n'} = j\}$ according to the value of the pair $(\mathbf{l}, \mathbf{u})$, it is not hard to verify that

$$P_{\lambda,\omega}(X_{n'} = j) = \sum_{(\mathbf{l},\mathbf{u})} P_{\lambda,\omega}(X_n = j) \prod_{t=1}^k P_{\lambda,\omega,v_t}(\tau_{v_t} = l_t) \le P_{\lambda,\omega}(X_n = j) |\{(\mathbf{l},\mathbf{u})\}| \,. \tag{4.12}$$

By a similar decomposition of the event $\{S_{n'} = n\}$ for SRW on $\mathbb{Z}$, starting at $S_0 = 0$, we see that the pairs $(\mathbf{l}, \mathbf{u})$ map to disjoint collections of SRW's path of length n', all of which start at 0 and terminate at n. For each $n' \in [n, (1+\varepsilon)n]$ the total number of such path is

$$2^{n'} P_{SRW(1/2)}(S_{n'} = n) \le e^{c(\varepsilon)n} \,,$$

where $c(\varepsilon) := (1+\varepsilon)(H(1|1/2) - H(1/(1+\varepsilon)|1/2))$. Consequently, the number of possible choices of $(\mathbf{l}, \mathbf{u})$ is also at most $\exp(c(\varepsilon)n)$, with (4.12) thus leading to (4.11).

To show that

$$\lim_{n\to\infty} \frac{1}{n} \log P_\lambda \left(\frac{|X_n|}{n} \in [a, b) \right) = -I_\lambda(a) \,,$$

we follow the same argument as in (4.8) and (4.9) but with $P_\lambda(\cdot)$ replacing $P_{\lambda,\omega}(\cdot)$ everywhere.

This completes the proof of Theorem 1.1, except for the proofs of the uncertainty estimate (Proposition 4.1) and the exponential decay (Proposition 4.2). $\square$

We next establish the uncertainty estimate.

Proof of Proposition 4.1: In view of (4.3), it suffices to show that for any $j \in D_n$, GW-a.e. ω, and all n, one has that $a(n, j) \le 1/N_n(r)$ for the ray $r(j)$ connecting the root to j. Toward this end, fix v a branch point, that is, a vertex with $k_v \ge 2$, on $r(j)$ and let $u = r(j) \cap D_{|v|+1}$ denote the child of v on the ray $r(j)$. Let $\Theta_v = \{\text{descendants of } v\} \setminus \{\text{descendants of } u\}$. We show below the

Lemma 4.1

$$P_{\lambda,\omega}(X_{T_n} = j, T_n \le \alpha n) \le P_{\lambda,\omega}(X_{T_n} \in \Theta_v, T_n \le \alpha n) \,.$$

Equipped with Lemma 4.1, note that, using the fact that $\Theta_v \cap \Theta_{v'} = \emptyset$ for $v \ne v'$ in the set $B_r(r(j))$ of branch points on the ray $r(j)$,

$$\begin{aligned} P_{\lambda,\omega}(T_n \le \alpha n) &\ge \sum_{v \in B_r(r(j))} P_{\lambda,\omega}(X_{T_n} \in \Theta_v, T_n \le \alpha n) \\ &\ge N_n(r(j)) P_{\lambda,\omega}(X_{T_n} = j, T_n \le \alpha n) \,, \end{aligned}$$

implying that $a(n, j) = P_{\lambda,\omega}(X_{T_n} = j | T_n \le \alpha n) \le 1/N_n(r(j))$, as claimed. $\square$

Proof of Lemma 4.1: Let $\widetilde{M}_t(v)$ denote the number of visits to a vertex v before time t, and write $M_n(v) = \widetilde{M}_{T_n}(v)$. We prove below that for any $k \ge 1$,

$$P_{\lambda,\omega}(X_{T_n} = j, M_n(v) = k, T_n \le \alpha n) \le P_{\lambda,\omega}(X_{T_n} \in \Theta_v, T_n \le \alpha n, M_n(v) = k) \tag{4.13}$$

out of which Lemma 4.1 follows readily.

The proof of (4.13) is obtained by constructing a coupling between two copies $X^{(1)}_\cdot, X^{(2)}_\cdot$ of the random walk on the tree ω, with associated $(T_n^{(i)}, \widetilde{M}_t^{(i)}, M_n^{(i)}), i = 1, 2$, such that each copy has the same law as $P_{\lambda,\omega}(\cdot)$, and such that

$$\left\{X^{(1)}_{T_n^{(1)}} = j, M_n^{(1)}(v) = k, T_n^{(1)} \le \alpha n\right\} \subset \left\{X^{(2)}_{T_n^{(2)}} \in \Theta_v, M_n^{(2)}(v) = k, T_n^{(2)} \le \alpha n\right\} \tag{4.14}$$

which then proves (4.13).

We thus proceed to the description of our coupling, which depends on ω, on j, k and on v. Fix a deterministic permutation π of the (at least 2) children of v, which has no fixed points (i.e. in particular $\pi(u) \neq u$). Further, define $\theta_k^{(i)} = \min\{t : \widetilde{M}_t^{(i)}(v) = k\}$.

1. Let $X^{(1)}_\cdot$ evolve according to the law $P_{\lambda,\omega}$.
2. Let $X_t^{(2)} = X_t^{(1)}$ for all $t \le T_n^{(1)} \wedge \theta_k^{(1)}$. Note that therefore, $T_n^{(1)} \wedge \theta_k^{(1)} = T_n^{(2)} \wedge \theta_k^{(2)}$.
3. If $M_n^{(1)}(v) < k$ then we let $X_t^{(2)} = X_t^{(1)}$ for all t.
4. We consider in all steps below only the case $M_n^{(1)}(v) \ge k$, which implies $\theta_k^{(1)} < T_n^{(1)}$. In this case we have $\theta_k^{(1)} = \theta_k^{(2)} := \theta_k$. Then, define
$$X^{(2)}_{\theta_k^{(1)}+1} = \begin{cases} X^{(1)}_{\theta_k+1}, & \text{if } |X^{(1)}_{\theta_k+1}| < |v| \\ \pi(X^{(1)}_{\theta_k+1}), & \text{otherwise}. \end{cases}$$
5. If $X^{(1)}_{\theta_k+1} \neq u$ then $\{X^{(2)}_{\theta_k+1+t}\}_{t\ge1}$ is drawn independently according to $P_{\lambda,\omega}$.
6. Define the stopping time $\tau := T_n^{(1)} \wedge \theta_{k+1}^{(1)}$. We construct below a random time $\tau' \le \tau$ and the path $\{X_t^{(2)}\}_{t=\theta_k+2}^{\tau'}$. Then, for $t > \tau'$, we let $\{X_t^{(2)}\}$ proceed according to $P_{\lambda,\omega}$, independently of $\{X^{(1)}_\cdot\}$.
7. We are thus left with the definition of the crucial part of our coupling, namely the definition of the random time τ' and the coupling of $\{X_t^{(2)}\}_{t=\theta_k+2}^{\tau'}$ to $\{X_t^{(1)}\}_{t=\theta_k+2}^{\tau}$. With a (serious) abuse of terminology, we denote by "Loops" anything branching out of the ray $r = r(j)$.
   - (I) We shall "stop the clock" of $X^{(2)}_\cdot$ for all the time units which $X^{(1)}_\cdot$ spends in the Loops, till either one of the following happens:
     - (a) $X^{(1)}_\cdot$ returns to v.
     - (b) $X^{(1)}_\cdot$ hits D_n (exceed $T_n^{(1)}$) (in both these cases, $X^{(2)}_\cdot$ continues independently with law $P_{\lambda,\omega}(\cdot)$, see step 6 above).
     - (c) $X^{(2)}_\cdot$ hits D_n (reach $T_n^{(2)}$) – after which $X^{(2)}_\cdot$ continues according to $P_{\lambda,\omega}$, independently of $X^{(1)}_\cdot$.
   - (II) The moves of $X^{(1)}_\cdot$ along the ray r (ignoring the steps into the Loops) are as in λ-biased random walk, that is probability of going up the ray $= \frac{\lambda}{\lambda+1}$, probability of going down the ray $= \frac{1}{\lambda+1}$. Suppose at time s we have $X^{(2)}_\cdot$ somewhere in $\omega^{\pi(u)}$ and $X^{(1)}_\cdot$ just made a move on the ray r. We then make a local move of $X^{(2)}_\cdot$ according to the $P_{\lambda,\omega}$ law in such a way that $X^{(2)}_\cdot$ *moves up* (towards the root) *only if* $X^{(1)}_\cdot$ *moved up*. Since $p_0 = 0$ we have that $P_\omega(X^{(2)}_\cdot$ should move up | position of $X^{(2)}_\cdot) \le \frac{\lambda}{\lambda+1}$ regardless of the position of $X^{(2)}_\cdot$. This implies that such a coupling is always possible.

Note also that if $\{M_n^{(1)}(v) = k, X^{(1)}_{T_n^{(1)}} = j\}$ then $X^{(1)}_\cdot$ does not hit v for $(k+1)$-st time or D_n during a move in the "Loops" before $T_n^{(1)}$. Our coupling guarantees that thus from the k-th visit to v (followed by a move to u) of $X^{(1)}_\cdot$ to $T_n^{(1)}$, $X^{(2)}_\cdot$ is in $\omega^{\pi(u)}$ (the subtree rooted at $\pi(u)$), with $|X_t^{(2)}| \geq |r(X_t^{(1)})|$, where $r(X_t^{(1)})$ is the ancestor of $X_t^{(1)}$ on the ray r with largest distance from the root. It follows that upon this event, $T_n^{(2)} \leq T_n^{(1)}$ and $X^{(2)}_\cdot$ hits level n at one of the vertices in $\omega^{\pi(u)}$, that is, part of Θ_v. This completes the proof of (4.14), and hence of Lemma 4.1. □

We turn now to establish the exponential decay Proposition 4.2.

Proof of Proposition 4.2: We divide the proof into the study of different cases:
i) $\lambda > m > 1$. Due to (4.6), it clearly suffices to show that

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\lambda(\tau_{\mathbf{o}} \geq T_n) < 0\,. \tag{4.15}$$

Toward this end, define, for a vertex $j \in D_n$, the hitting time $T_j = \min\{n : X_n = j\}$ (possibly, $T_j = \infty$). We then have

$$P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n) \leq \sum_{j \in D_n} P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_j)\,.$$

By coupling with simple random walk, and the well known formula for exit probabilities of the latter from a strip, we get that

$$P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_j) \leq P_{SRW(\frac{1}{1+\lambda})}(T_n < \tau_{\mathbf{o}}) = \frac{\lambda - 1}{\lambda^n - 1} \leq \frac{1}{\lambda^{n-1}}\,,$$

and hence $P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n) \leq \lambda |D_n| \lambda^{-n}$. Integrating with respect to GW we get that $P_\lambda(\tau_{\mathbf{o}} > T_n) \leq \lambda(m/\lambda)^n$, implying (4.15).
ii) $\lambda = m$: a "bare-hands" proof may be constructed using Lemma 2.2 applied to points with large expected return time. Instead, fixing $\alpha < \infty$, we remark that [20, Theorem 1] implies that for GW-a.a. ω,

$$\liminf_{n\to\infty} n^{-1} \log P_{\lambda,\omega}(T_{n+1} \leq \alpha n + 1) < 0 \tag{4.16}$$

(which is all we need, by (4.7)). Indeed, note that X_n is a random walk on the weighted graph $\omega \in \mathcal{T}$ with the weight $\lambda^{-\ell}$ for each edge connecting $D_{\ell-1}(\omega)$ to $D_\ell(\omega)$, $\ell = 1, 2, \ldots$. Let z denote the set D_{n+1} of all vertices of distance $n+1$ from $\mathbf{o}$, noting that then $T_{n+1} = T_{\mathbf{o}z}$. Following [20], let the weight of a collection of vertices be the sum of weights over all incident edges, so in particular, $w_{\mathbf{o}} = \lambda^{-1}|D_1|$ and $w_z = \lambda^{-(n+1)}|D_{n+1}| + \lambda^{-(n+2)}|D_{n+2}|$. Set $g_n(\omega) > 1$ to be the (unique) solution of $(g-1)^2 g^{n-2} = 2w_z/w_{\mathbf{o}}$. Since $\lambda = m$ it follows that $g_n(\omega) \to 1$ as $n \to \infty$ for GW-a.a. ω. So, for all n large enough, $\alpha < (g_n + 1)/(g_n - 1)$ and by [20, Theorem 1(a)],

$$\frac{1}{n} \log P_{\lambda,\omega}(T_{n+1} \leq \alpha n + 1) \leq -\alpha H\Big(\frac{\alpha+1}{2\alpha}\Big| \frac{g_n}{1+g_n}\Big) \underset{n\to\infty}{\longrightarrow} -\alpha H\Big(\frac{\alpha+1}{2\alpha}\Big|\frac{1}{2}\Big) < 0\,.$$

iii) In the case $\lambda < m$, an essential role is played by *regeneration points*. These will be useful also in the case of slowdown, c.f. the proof of Lemma 5.2. Given a path $X_0, X_1, \ldots$, call $n > 0$ a *regeneration time* if $X_n \neq X_k$ for all $k < n$ and $X_k \neq X_{n-1}$ for all $k > n$. Call $n > 0$ a *level regeneration time* if $|X_n| \neq |X_k|$ for all $k < n$ and $|X_k| \neq |X_{n-1}|$ for all $k > n$. It is proved in [11] that whenever $\lambda < m$ then there are P_λ-a.s. infinitely many regeneration times $\eta_1, \eta_2, \ldots$, such that $\{\eta_{n+1} - \eta_n\}_{n\geq 1}$ and $\{|X_{\eta_{n+1}}| - |X_{\eta_n}|\}_{n\geq 1}$ are i.i.d. sequences, and

$$E_\lambda(\eta_2 - \eta_1) < \infty \tag{4.17}$$

and

$$\lim_{n\to\infty} \frac{|X_n|}{n} = \frac{E_\lambda(|X_{\eta_2}| - |X_{\eta_1}|)}{E_\lambda(\eta_2 - \eta_1)}, \quad P_\lambda - a.s. \tag{4.18}$$

Let $r_1, r_2, \ldots$ denote successive level regeneration times. We prove below the following tail estimates for level regeneration points and level regeneration times.

Lemma 4.2 *i) If $\lambda < m$ then there is a $\theta > 0$ such that*

$$E_\lambda\left(e^{\theta(|X_{r_2}|-|X_{r_1}|)}\right) < \infty \;, E_\lambda\left(e^{\theta|X_{r_1}|}\right) < \infty \,. \tag{4.19}$$

ii) If further $\lambda < d_{\min}$ then in addition there exists a $\theta > 0$ such that

$$E_\lambda\left(e^{\theta(r_2-r_1)}\right) < \infty \;, E_\lambda\left(e^{\theta r_1}\right) < \infty \,. \tag{4.20}$$

Rerunning the argument in [11, Proposition 3.4], one concludes that there are P_λ-a.s. infinitely many level regeneration times $r_1, r_2, \ldots$, such that $\{r_{n+1} - r_n\}_{n\geq 1}$ and $\{|X_{r_{n+1}}| - |X_{r_n}|\}_{n\geq 1}$ are i.i.d. sequences. Further, by (4.18), $|X_{r_n}|/r_n \to_{n\to\infty} v_\lambda > 0$, P_λ-a.s. Using (4.19) and the independence of the increments $|X_{r_{i+1}}| - |X_{r_i}|$, this forces that $\limsup_{n\to\infty} r_n/n < \infty$, P_λ-a.s., which then implies

$$E_\lambda(r_2 - r_1) < \infty \tag{4.21}$$

and

$$\lim_{n\to\infty} \frac{|X_n|}{n} = \frac{E_\lambda(|X_{r_2}| - |X_{r_1}|)}{E_\lambda(r_2 - r_1)}, \quad P_\lambda - a.s. \tag{4.22}$$

Due to (4.22) and $xv_\lambda < 1$, one can find a $c > 0$ such that $E_\lambda(|X_{r_2}|-|X_{r_1}|) < 1/c$ while $E_\lambda(r_2-r_1) > x/c$. Then,

$$\begin{aligned}
P_\lambda(T_n \leq xn) \;&\leq\; P_\lambda(\text{there are at most } cn \text{ level regeneration times before } T_n) \\
&\quad + P_\lambda\left(\sum_{j=1}^{cn}(r_{j+1} - r_j) \leq xn\right) \qquad\qquad (4.23)\\
&\leq\; P_\lambda\left(\sum_{j=1}^{cn}(|X_{r_{j+1}}| - |X_{r_j}|) + |X_{r_1}| \geq n\right) + P_\lambda\left(\frac{1}{cn}\sum_{j=1}^{cn}(r_{j+1} - r_j) \leq \frac{x}{c}\right).
\end{aligned}$$

Using (4.21), (4.19) and our choice of $c > 0$, we conclude that both terms in the last inequality decay exponentially, and the proof of Proposition 4.2 is completed. □

The next lemma simplifies the proof of Lemma 4.2 as well as the upper bounds in Theorem 1.3, by showing that whenever $\lambda < m$, the upper tails of $|X_{r_2}| - |X_{r_1}|$ and of $r_2 - r_1$ are dominated by those of $|X_{r_1}|$ and r_1, respectively.

Lemma 4.3 *If $\lambda < m$, then there exists $C_\lambda < \infty$ such that for any $x \geq 1$, $t \geq 1$,*

$$P_\lambda(|X_{r_2}| - |X_{r_1}| = x, \, r_2 - r_1 = t) \leq C_\lambda P_\lambda(|X_{r_1}| = x, \, r_1 = t) \tag{4.24}$$

Proof of Lemma 4.3: For $k = 2, \dots$ and $\ell \leq k-1$, we let

$$A_{\ell,k} := \{\exists s \in (T_\ell, T_k) : X_s \in D_{\ell-1}\}\,.$$

Then, fixing $t \geq x \geq 1$, and using the strong Markov property at T_k in the second equality,

$$\begin{aligned}
&P_{\lambda,\omega}\left(|X_{r_2}| - |X_{r_1}| = x,\, r_2 - r_1 = t\right)\\
=\;& \sum_{k=1}^{\infty}\sum_{v\in D_k} P_{\lambda,\omega}\left(\bigcap_{\ell=1}^{k-1} A_{\ell,k}; X_{T_k} = v; |X_s| \geq k \text{ for } s > T_k; |X_{t+T_k}| - |X_{T_k}| = x, r_2 = t + T_k\right)\\
=\;& \sum_{k=1}^{\infty}\sum_{v\in D_k} P_{\lambda,\omega}\left(\bigcap_{\ell=1}^{k-1} A_{\ell,k}; X_{T_k} = v\right) P_{\lambda,\omega,v}\left(\tau_v^* = \infty, |X_t| = x, r_1 = t\right)\,,
\end{aligned}$$

where the stopping time $\tau_v^* := \min\{t > 0 : X_{t-1} = v, X_t = v\}$ denotes the first visit of the edge connecting v to itself that is added to ω^v in the definition of the measure $P_{\lambda,\omega,v}$. Consequently,

$$\begin{aligned}
&P_\lambda\left(|X_{r_2}| - |X_{r_1}| = x,\, r_2 - r_1 = t\right)\\
=\;& \sum_{k=1}^{\infty} E_\lambda\left(E_\lambda\left(\sum_{v\in D_k} P_{\lambda,\omega}\left(\bigcap_{\ell=1}^{k-1} A_{\ell,k}; X_{T_k} = v\right) P_{\lambda,\omega,v}\left(\tau_v^* = \infty, |X_t| = x, r_1 = t\right)\Big|\mathcal{F}_k\right)\right)\\
=\;& \sum_{k=1}^{\infty} E_\lambda\left(P_{\lambda,\mathbf{o}}(\tau_{\mathbf{o}}^* = \infty, |X_t| = x, r_1 = t)\sum_{v\in D_k} P_{\lambda,\omega}\left(\bigcap_{\ell=1}^{k-1} A_{\ell,k}; X_{T_k} = v\right)\right)\\
=\;& P_{\lambda,\mathbf{o}}(\tau_{\mathbf{o}}^* = \infty, |X_{r_1}| = x, r_1 = t)\sum_{k=1}^{\infty} P_\lambda\left(\bigcap_{\ell=1}^{k-1} A_{\ell,k}\right)\,.
\end{aligned}$$

Summing over $t \geq x \geq 1$, we get that

$$1 = P_{\lambda,\mathbf{o}}(\tau_{\mathbf{o}}^* = \infty)\sum_{k=1}^{\infty} P_\lambda\left(\bigcap_{\ell=1}^{k-1} A_{\ell,k}\right)\,,$$

which substituting in the above yields the identity

$$P_\lambda\left(|X_{r_2}| - |X_{r_1}| = x,\, r_2 - r_1 = t\right) = \frac{P_{\lambda,\mathbf{o}}\left(|X_{r_1}| = x,\, r_1 = t, \tau_{\mathbf{o}}^* = \infty\right)}{P_{\lambda,\mathbf{o}}(\tau_{\mathbf{o}}^* = \infty)} \tag{4.25}$$

The inequality

$$P_{\lambda,\mathbf{o}}\left(|X_{r_1}| = x,\, r_1 = t, \tau_{\mathbf{o}}^* = \infty\right) \leq P_\lambda(|X_{r_1}| = x,\, r_1 = t) \tag{4.26}$$

is evident by noting that the sub-probability measure $P_{\lambda,\omega,\mathbf{o}}(\cdot, \tau_{\mathbf{o}}^* = \infty)$ is dominated by the probability measure $P_{\lambda,\omega}(\cdot)$ (they differ only at steps of the random walk taken at the origin and there the latter measure dominates the former on the event $\tau_{\mathbf{o}}^* = \infty$). Finally, note that $C_\lambda = 1/P_{\lambda,\mathbf{o}}(\tau_{\mathbf{o}}^* = \infty) < \infty$ by transience of the λ-biased random walk whenever $\lambda < m$, hence (4.24) follows by combining (4.25) and (4.26). □

Proof of Lemma 4.2: We begin by considering the easier case $\lambda < d_{\min}$, and proving then (4.20), which implies of course (4.19) for this case. Couple $|X_n|$ with a reflected biased random walk S_n of law $P_{SRW^R(\frac{d_{\min}}{d_{\min}+\lambda})}$, such that each regeneration time of S_n is also a level regeneration time of X_n, see [12] for this construction when $\lambda = 1$, or [2] for a similar coupling in the case of one dimensional RWRE. Since the regeneration times of S_n possess exponential tails, see e.g. [2, Pg. 680] (covering also the first regeneration time r_1, whose law is typically different than that of $r_2 - r_1$), we are done.

We consider next the proof of (4.19), using a pathwise decomposition of the path $\{X_t\}_{t=0}^{r_1}$ due to H. Kesten (see [8] and [12], [19] for similar constructions). We prove below that for some $c > 0$ and all $\ell > 3$,

$$P_\lambda(|X_{r_1}| > \ell) \le e^{-c\ell} . \tag{4.27}$$

Recall the inequality (4.24), by which such exponential tail estimate then applies to $|X_{r_2}| - |X_{r_1}|$ as well.

We describe next the path decomposition alluded to above, by defining a random variable K and an increasing sequence of random variables $\{s_i\}_{i=0}^{\infty}$ (called for $i \le K$ *ladder times*) as follows. Fix $s_0 = 0$. If $\tilde\tau_0 := \tau_{\mathbf{o}} = \infty$ then $K = 0$ and $s_i = \infty$ for $i > 0$. Otherwise, let $M_0 := \max\{|X_n| : 0 \le n \le \tilde\tau_0\}$ and $s_1 = \min\{n : |X_n| > M_0\}$. We proceed recursively: denote by $\tilde\tau_i$ the time of first return of the walk to level $|X_{s_i}| - 1$ (possibly infinite). Then, if $\tilde\tau_i = \infty$ then $K = i$ and $s_n = \infty$ for all $n > i$. Otherwise, $M_i := \max\{|X_n| : 0 \le n \le \tilde\tau_i\}$ and $s_{i+1} = \min\{n : |X_n| > M_i\}$. Note that $s_K = r_1$ if $K > 0$ and $r_1 = 1$ if $K = 0$, and further the times s_i are stopping times. Therefore, for $\ell > 1$,

$$\begin{aligned} P_\lambda(|X_{r_1}| > \ell) &= P_\lambda(|X_{s_K}| > \ell) \\ &= P_\lambda\left(\sum_{i=1}^{K}(|X_{s_i}| - |X_{s_{i-1}}|) > \ell\right) \\ &= \sum_{k=1}^{\infty} P_\lambda\left(\sum_{i=1}^{k}(|X_{s_i}| - |X_{s_{i-1}}|) > \ell, \tilde\tau_i < \infty, i = 1, \ldots, k-1, \tilde\tau_k = \infty\right) . \end{aligned} \tag{4.28}$$

The interest in the definition of the ladder times are their independence properties. To describe them, let $\{(A_i, B_i)\}$ denote a sequence of independent random vectors taking values in $\mathbb{N} \times \{0, 1\}$, such that for any set C,

$$P_\lambda(|X_{s_2}| - |X_{s_1}| \in C, \tilde\tau_1 < \infty) = P(A_i \in C, B_i = 1), \quad P_\lambda(\tilde\tau_1 = \infty) = P(B_i = 0) .$$

We now have the following lemma.

Lemma 4.4 *For any $k \ge 1$ and sets C_i,*

$$\begin{aligned} &P_\lambda(\{|X_{s_i}| - |X_{s_{i-1}}| \in C_i\}_{i=1}^{k}, \{\tilde\tau_i < \infty\}_{i=0}^{k-1}, \tilde\tau_k = \infty) \\ &= P_\lambda(|X_{s_1}| \in C_1, \tilde\tau_0 < \infty) \prod_{i=2}^{k} P(A_i \in C_i, B_i = 1) P_\lambda(\tilde\tau_1 = \infty) . \end{aligned}$$

Proof of Lemma 4.4: Note first that we can also start at a vertex different from the root: in this case, we modify $\tilde\tau_0 := \inf\{j : |X_j| = |X_0| - 1\}$.
Fix ω, $k \ge 1$ and sets C_i, $i = 1, \ldots, k$ of positive integers. For any fixed path $\nu = \{v_n\}$ on ω, such that v_n and v_{n+1} are connected by an edge, one may define the stopping times $\tilde\tau_i(\nu)$ and ladder times $s_i(\nu)$. We denote by V_i the set of path ν for which $s_i(\nu) < \infty$. For any fixed $\nu \in V_k$, due to the Markov property and the fact that the s_i are stopping times, we have that

$$P_{\lambda,\omega}(\tilde\tau_k = \infty, X_n = v_n, n = 0, \ldots, s_k) = P_{\lambda,\omega}^{v_{s_k}}(\tilde\tau_0 = \infty) P_{\lambda,\omega}(X_n = v_n, n = 0, \ldots, s_k) ,$$

where $P_{\lambda,\omega}^{v_{s_k}}$ is the law of the λ-biased random walk on the (original) tree ω, started at v_{s_k}. Thus, for any vector $\Theta = (\theta_1, \ldots, \theta_k)$ of positive integers, and any $v \in D_{\sum_{n=1}^{k} \theta_n}$,

$$\begin{aligned} &P_{\lambda,\omega}(\tilde\tau_k = \infty, s_k < \infty, X_{s_k} = v, |X_{s_n}| - |X_{s_{n-1}}| = \theta_n, n = 1, \ldots, k) \\ = \ &P_{\lambda,\omega}^{v}(\tilde\tau_0 = \infty) P_{\lambda,\omega}(s_k < \infty, X_{s_k} = v, |X_{s_n}| - |X_{s_{n-1}}| = \theta_n, n = 1, \ldots, k) . \end{aligned} \tag{4.29}$$

The probabilities in (4.29) depend on disjoint parts of ω (one determined by the subtree ω^v, while the other determined by ω truncated at level $|v|$). We note that

$$E_\lambda\left(P^v_{\lambda,\omega}(\tilde\tau_0=\infty)\right)=P_\lambda(\tilde\tau_1=\infty),$$

is independent of v (recall that $|v|\geq 1$). Hence, summing (4.29) over v and the relevant values of Θ, then taking the expectation with respect to ω, we deduce that

$$\begin{aligned}&P_\lambda(\tilde\tau_k=\infty,s_k<\infty,|X_{s_i}|-|X_{s_{i-1}}|\in C_i,i=1,\ldots,k)\\ =\;&P_\lambda(\tilde\tau_1=\infty)P_\lambda(s_k<\infty,|X_{s_i}|-|X_{s_{i-1}}|\in C_i,i=1,\ldots,k)\,.\end{aligned}$$

Similarly, one checks that for $\nu\in V_{i-1}$ and any set C of positive integers,

$$\begin{aligned}&P_{\lambda,\omega}(|X_{s_i}|-|X_{s_{i-1}}|\in C,\tilde\tau_{i-1}<\infty,X_n=v_n,n=0,\ldots,s_{i-1})\\ &=P^{v_{s_{i-1}}}_{\lambda,\omega}(|X_{s_1}|\in C,\tilde\tau_0<\infty)P_{\lambda,\omega}(X_n=v_n,n=0,\ldots,s_{i-1})\,,\end{aligned}$$

so for any $\Theta=(\theta_1,\ldots,\theta_{i-1})$ and $v\in D_{\sum_{n=1}^{i-1}\theta_n}$,

$$\begin{aligned}&P_{\lambda,\omega}(|X_{s_i}|-|X_{s_{i-1}}|\in C,\tilde\tau_{i-1}<\infty,X_{s_{i-1}}=v,|X_{s_n}|-|X_{s_{n-1}}|=\theta_n,n=1,\ldots,i-1)\\ &=P^v_{\lambda,\omega}(|X_{s_1}|\in C,\tilde\tau_0<\infty)P_{\lambda,\omega}(s_{i-1}<\infty,X_{s_{i-1}}=v,|X_{s_n}|-|X_{s_{n-1}}|=\theta_n,n=1,\ldots,i-1)\,.\end{aligned}$$

The latter two probabilities depend on disjoint parts of ω, whereas $|v|\geq 1$, implying that

$$E_\lambda\left(P^v_{\lambda,\omega}(|X_{s_1}|\in C,\tilde\tau_0<\infty)\right)=P_\lambda(|X_{s_2}|-|X_{s_1}|\in C,\tilde\tau_1<\infty)=P(A_i\in C,B_i=1)$$

is independent of v. Recall that a.s. $\{s_i<\infty\}=\{\tilde\tau_{i-1}<\infty\}$. Consequently, summing over v and Θ, then taking expectation with respect to ω, we deduce that

$$\begin{aligned}&P_\lambda(s_i<\infty,|X_{s_n}|-|X_{s_{n-1}}|\in C_n,n=1,\ldots,i)\\ =\;&P(A_i\in C_i,B_i=1)P_\lambda(s_{i-1}<\infty,|X_{s_n}|-|X_{s_{n-1}}|\in C_n,n=1,\ldots,i-1)\,.\end{aligned}$$

To complete the proof of Lemma 4.4, simply iterate the last identity for $i=k,\ldots,2$. □

Further, for $\ell>3$,

$$\begin{aligned}&P_{\lambda,\omega}(|X_{s_1}|>\ell,\tau_{\mathbf{o}}<\infty)=\sum_{j\in D_1(\omega)}P_{\lambda,\omega}(X_1=j,|X_{s_1}|>\ell,\tau_{\mathbf{o}}<\infty)\\ &\leq\sum_{j\in D_1(\omega)}P_{\lambda,\omega}(X_1=j)P^j_{\lambda,\omega}(\exists\text{ excursion from } j \text{ of depth } \ell-1 \text{ at least before returning to } j)\end{aligned}$$

Averaging and using independence, one concludes that

$$P_\lambda(|X_{s_1}|>\ell,\tau_{\mathbf{o}}<\infty)\leq P_\lambda(|X_{s_2}|-|X_{s_1}|>\ell-1,\tilde\tau_1<\infty)\,. \tag{4.30}$$

Combining (4.30), Lemma 4.4 and (4.28), to prove (4.27) it is therefore enough to check that the distance between consecutive ladder points possesses exponential tails. To this end,

$$\begin{aligned}P(A_i\geq t,B_i=1)&=P_\lambda(|X_{s_2}|-|X_{s_1}|\geq t,\tilde\tau_1<\infty)\\ &=P_\lambda(M_1-M_0\geq t,\tilde\tau_1<\infty)\leq P_\lambda(M_0\geq t,\tau_{\mathbf{o}}<\infty)\,.\end{aligned}$$

Moreover,

$$P_\lambda(M_0 \geq t, \tau_{\mathbf{o}} < \infty) \leq P_\lambda(|X_m| = t, |X_n| = 0 \text{ for some } n > m \geq t)\,. \tag{4.31}$$

Recall the notations before Lemma 2.2 and define now, for $j \in D_1(\omega)$ and $\delta > 0$,

$$A(\omega, j) = \mathbf{1}_{\{k_{\mathbf{o}} \geq 2\}} \mathbf{1}_{\{P_{\lambda, \omega_j^R}(\tau_{\mathbf{o}} = \infty) \geq \delta\}}\,.$$

Note that $GW(A(\omega, j)) \to_{\delta \to 0} (1 - p_1)$ and that $A(\omega, j)$ is R-defined. A vertex $j \in D_n(\omega)$ is called β-*successful* if there exist at least βn vertices v_i on the ray connecting $\mathbf{o}$ and j satisfying $A(\omega^{v_i}, v_i^j) = 1$. Note that upon first visiting such v_i, with probability of at least $\delta(k_{v_i} - 1)/(k_{v_i} + \lambda) \geq \delta/(2 + \lambda)$ the random walk never returns to the ray connecting $\mathbf{o}$ and j. Define the event

$$B_n(\delta, \beta) = \{\text{some } j \in D_n(\omega) \text{ is not } \beta\text{-successful }\}\,.$$

Then, for any $\omega \notin B_t(\delta, \beta)$, by independence,

$$P_{\lambda,\omega}(|X_m| = t, |X_n| = 0 \text{ for some } n > m \geq t) \leq (1 - \delta/(2 + \lambda))^{\beta t}\,.$$

For δ small enough, Lemma 2.2 implies the existence of a $\beta > 0$ such that $GW(B_t(\delta, \beta))$ decays exponentially in t, completing the proof of exponential decay of the right hand side of (4.31). $\square$

5 Proof of Theorem 1.2

Consider first the case $d_{\min} \geq 2$ and $\lambda < d_{\min}$. It turns out it is easier to first treat a somewhat restricted event. Recall the time of first return to the root

$$\tau_{\mathbf{o}} = \inf\{t \geq 1 : X_t = \mathbf{o}\}$$

(possibly, $\tau_{\mathbf{o}} = \infty$) and the measures $P_{\lambda,\omega,j}$, $P_{\lambda,\omega,\mathbf{o}}$. We will now run an argument similar to the one in the proof of Theorem 1.1. Write, as in (4.2),

$$P_{\lambda,\omega}\Big(\tau_{\mathbf{o}} > T_{n+\Delta} \geq \alpha(n + \Delta)\Big) \geq P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n \geq \alpha n)\hat{e}_\Delta \hat{Z}_n \tag{5.1}$$

where

$$\begin{aligned}
\hat{Z}_n &:= \sum_{j \in D_n} \Big(\hat{p}(\Delta, j)/\hat{e}_\Delta\Big)\hat{a}(n, j) \\
\hat{a}(n, j) &:= P_{\lambda,\omega}\left(X_{T_n} = j | \tau_{\mathbf{o}} > T_n \geq \alpha n\right) \\
\hat{p}(\Delta, j) &:= P_{\lambda,\omega,j}\left(\tau_j > \widetilde{T}_\Delta(j) \geq \alpha\Delta\right) \\
\hat{e}_\Delta &:= E_\lambda(\hat{p}(\Delta, j))\,,
\end{aligned}$$

Our main technical tool, replacing Proposition 4.1, is the following uniform bound on the *conditional* exit measure of λ-biased random walks on $\mathcal{T}$. The proof is deferred to the end of this section. Recall the definition of n-slow trees presented in the proof of (1.5), and the fact proved there that

$$GW\left(\bigcup_M \{\omega \text{ is } \ell\text{-slow for every } \ell \geq M\}\right) = 1\,. \tag{5.2}$$

Proposition 5.1 *(Uncertainty estimate) Assume $d_{\min} \geq 2$ and $d_{\min} > \lambda$. Then there exists a constant $\Delta_0 = \Delta_0(M, d_{\min})$ and $c = c(d_{\min}) > 0$ such that if $\omega \in \mathcal{T}$ is l-slow (with respect to $d_{\min}$) for every $l \geq M$ and $\min(k_j : j \in \omega) \geq d_{\min}$ then, for all $n > \Delta_0$,*

$$\max_{j \in D_n(\omega)} P_{\lambda,\omega}\left(X_{T_n} = j | \tau_{\mathbf{o}} > T_n \geq \alpha n\right) \leq e^{-cn} .$$

Fix M and $\Delta > \Delta_0(M, d_{\min})$. Note that $\hat{e}_\Delta = P_{\lambda,\mathbf{o}}(\tau_{\mathbf{o}} > T_\Delta \geq \alpha\Delta)$ is independent of j. Moreover, the event $\{\tau_{\mathbf{o}} > T_\Delta \geq \alpha\Delta\}$ contains the event of the random walk taking its first two steps down from the root and then spending $\alpha\Delta - 2$ time units oscillating between $D_2(\omega)$ and $D_1(\omega)$, an event of positive probability. Thus, $\hat{e}_\Delta > 0$ for all Δ large enough. Denote $\hat{\mathcal{A}}_n = \{\omega : \max_{j \in D_n} \hat{a}(n,j) \leq e^{-cn}\}$, (here, c is as in Proposition 5.1), and use (5.2) to conclude that

$$P_\lambda\left(\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \hat{\mathcal{A}}_n\right) = 1 .$$

Let $\hat{\mathcal{G}}_n$ be the σ-field generated by $\{\hat{a}(n,j) : j \in D_n\}$. We then have for all $\theta > 0$, $\delta > 0$, n,

$$\begin{aligned} P_\lambda(\hat{Z}_n \leq \delta, \hat{\mathcal{A}}_n) \leq e^{\theta\delta} E_\lambda\left(e^{-\theta \hat{Z}_n} \mathbf{1}_{\hat{\mathcal{A}}_n}\right) &= e^{\theta\delta} E_\lambda\left(\mathbf{1}_{\hat{\mathcal{A}}_n} E(e^{-\theta \hat{Z}_n} | \hat{\mathcal{G}}_n)\right) \\ &= e^{\theta\delta} E_\lambda\left(\mathbf{1}_{\hat{\mathcal{A}}_n} e^{\sum_{j \in D_n} \hat{\phi}_\Delta(\theta \hat{a}(n,j))}\right) \end{aligned}$$

where

$$\hat{\phi}_\Delta(\eta) := \log E_\lambda(e^{-\eta \hat{p}(\Delta,j)/\hat{e}_\Delta}) \leq \log\left(1 - \hat{e}_\Delta + \hat{e}_\Delta e^{-\eta/\hat{e}_\Delta}\right) .$$

Proceeding as in the proof of Theorem 1.1, we conclude that for any $\Delta > \Delta_0$ and $\delta \in (0,1)$ there exists an $n_2(\omega)$ such that $\hat{Z}_n \geq \delta$ for all $n > n_2(\omega)$. This yields, iterating (5.1) and taking first $n \to \infty$ and then $\Delta \to \infty$, c.f. the proof of Theorem 1.1, that

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(T_n \geq \alpha n) &\geq \liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n \geq \alpha n) \\ &\geq \limsup_{\Delta\to\infty} \frac{1}{\Delta} \log P_{\lambda,\mathbf{o}}(\tau_{\mathbf{o}} > T_\Delta \geq \alpha\Delta) \quad GW-\text{a.s.} \\ &\geq \limsup_{\Delta\to\infty} \frac{1}{\Delta} \log P_\lambda(\tau_{\mathbf{o}} > T_\Delta \geq \alpha\Delta) . \end{aligned}$$

Hence, applying Lemma 2.1, we conclude that

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(T_n \geq \alpha n) &\geq \lim_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n \geq \alpha n) \\ &= \lim_{n\to\infty} \frac{1}{n} \log P_\lambda(\tau_{\mathbf{o}} > T_n \geq \alpha n) , \quad GW-\text{a.s.}. \end{aligned}$$

We later prove the following lemma.

Lemma 5.1 *Assume $m < \infty$ and $\lambda < d_{\min}$. Then, for any $\varepsilon > 0$, there exists a $k = k(\varepsilon)$ and a constant $c_\varepsilon < \infty$ such that for all $n > k$, and any non-random $B_n \subseteq \{1, \ldots, b_n\}$,*

$$P_\lambda(T_n \in B_n) \leq c_\varepsilon (1+\varepsilon)^{b_n} P_\lambda(T_n \in B_n, \tau_{\mathbf{o}} > T_n) .$$

It implies that for any $b < \infty$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\lambda(bn \geq T_n \geq \alpha n) \leq \lim_{n\to\infty} \frac{1}{n} \log P_\lambda(\tau_{\mathbf{o}} > T_n \geq \alpha n)\,.$$

By the coupling argument of (3.3) and (3.4), we know that

$$P_\lambda(T_n > bn) \leq P_{\lambda,\omega_{\min}}(T_n > bn) \leq P\left(S^{R,p}_{bn} \leq n\right) \leq P\left(S^p_{bn} \leq n\right)\ ,$$

for $p := d_{\min}/(d_{\min} + \lambda) < 1/2$, so that

$$\lim_{b\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log P_\lambda(T_n > bn) = -\infty.$$

Consequently, considering $b \to \infty$, we see that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log P_\lambda(T_n \geq \alpha n) &= \lim_{n\to\infty} \frac{1}{n} \log P_\lambda(\tau_{\mathbf{o}} > T_n \geq \alpha n) \\
&= \lim_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n \geq \alpha n) \\
&\leq \liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(T_n \geq \alpha n), \quad GW - \text{a.s.}.
\end{aligned}$$

An application of Lemma 2.1 now yields

$$-J^{>}_\lambda(\alpha) := \lim_{n\to\infty} \frac{1}{n} \log P_\lambda\left(T_n \geq \alpha n\right) = \lim_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}(T_n \geq \alpha n)\,, \quad GW - a.s. \tag{5.3}$$

As in the case of Theorem 1.1, one easily checks that $J^{>}_\lambda(\cdot)$ is convex, with $J^{>}_\lambda(x) = 0$ when $x < 1/v_\lambda$. Similarly, we again have exponential decay, as following.

Proposition 5.2 *(Exponential Decay) If $\lambda < d_{\min}$ then $J^{>}_\lambda(x) > 0$ for $x > 1/v_\lambda$,*

Note that for all $l \geq 0$, $v \in D_{an}(\omega)$ and GW-a.a. ω,

$$P^v_{\lambda,\omega}(|X_l| \leq an) \leq P_{\lambda,\omega_{\min}}(|X_l| \leq an \,\big|\, |X_0| = an) \leq e^{-lH(\frac{1}{2}|p)}\,,$$

where $p := d_{\min}/(d_{\min} + \lambda)$ (c.f. the coupling leading to (3.3) and the derivation of (3.4)). Consequently,

$$\begin{aligned}
P_{\lambda,\omega}(|X_n| \leq an) &= \sum_{k=an}^{n} \sum_{v\in D_{an}(\omega)} P_{\lambda,\omega}(X_{T_{an}} = v, T_{an} = k) P^v_{\lambda,\omega}(|X_{n-k}| \leq an) \\
&\leq \sum_{k=an}^{n} P_{\lambda,\omega}(T_{an} \geq k) e^{-(n-k)H(\frac{1}{2}|p)}\,.
\end{aligned}$$

Considering the expectation with respect to ω, we thus have by (5.3) that for any $\delta > 0$ and all $n \geq n_0(\delta)$,

$$\begin{aligned}
P_\lambda(|X_n| \leq an) &\leq n \sup_{0\leq x\leq 1} \left\{P_\lambda(T_{an} \geq xn) \exp(-n(1-x)H(\frac{1}{2}|p))\right\} \\
&\leq n \exp\Big[-n(\inf_{0\leq x\leq 1}\{aJ^{>}_\lambda((x-\delta)/a) + (1-x)H(\frac{1}{2}|p)\} - \delta)\Big]\,.
\end{aligned}$$

Comparing (3.9) and (5.3) we see that $H(\frac{1}{2}|p) \geq \varepsilon J_\lambda^>(1/\varepsilon)$ for all $\varepsilon > 0$. Thus, taking $\varepsilon \to 0$ followed by $\delta \to 0$, the convexity of $J_\lambda^>(\cdot)$ and its continuity at $1/a > 1/v_\lambda$ imply that,

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\lambda(|X_n| \leq an) \leq -aJ_\lambda^>(1/a)\,. \tag{5.4}$$

Note next that for any $0 \leq b < a < v_\lambda$, there exists $\varepsilon > 0$ small enough, such that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in [b,a)\right) &\geq \liminf_{n\to\infty} \frac{1}{n} \log\left[P_{\lambda,\omega}(T_{\lfloor n(a-\varepsilon)\rfloor} > n(1-\varepsilon)) - P_{\lambda,\omega}(T_{\lfloor n(a-\varepsilon)\rfloor} \geq n)\right] \\ &= -(a-\varepsilon)J_\lambda^>\left(\frac{1-\varepsilon}{a-\varepsilon}\right) \end{aligned} \tag{5.5}$$

where the last equality is due to the fact that $J_\lambda^>$ is strictly increasing on $(1/v_\lambda, \infty)$. Taking the limit as $\varepsilon \to 0$, we have from (5.4), (5.5) and Lemma 2.1 that (1.4) holds for the convex, strictly decreasing $I_\lambda(a) := aJ_\lambda^>(1/a) > 0$. This concludes the proof of Theorem 1.2 in the case $d_{\min} \geq 2, \lambda < d_{\min}$, modulo the proof of Propositions 5.1, 5.2, and Lemma 5.1.

It thus remains only to treat the case $m > \lambda \geq d_{\min}$. Fix $0 < a < v_\lambda$ and $\varepsilon > 0$ such that $a - \varepsilon > 0$ and $a + \varepsilon < v_\lambda$. Note that there exists a d_0 with $\lambda < d_0 < \infty$ and $p_{d_0} > 0$; in particular, λ-biased walk on the rooted tree with d_0 children at each vertex is transient. Fix $C := 4/\log(d_0 d_{\min})$ and define h_n to be the even integer nearest to $C \log\log n$. Define next an n-trap as a (finite) rooted tree $\bar{\omega}_n$ of depth h_n such that the vertices in the first $h_n/2$ levels (including the root) possess each d_0 children and the vertices at levels $h_n/2, \ldots, h_n - 1$ possess each $d_{\min}$ children. Note that an n-trap has $L_n = (d_0 d_{\min})^{h_n/2}$ leaves, where our choice of C guarantees that $L_n = O((\log n)^2)$. Let n_1 be the even integer nearest to $(1 - a/v_\lambda)n$ and define the event

$$\mathcal{C}_n := \{\tau_{\mathbf{o}} > T_{h_n} = n_1\}\,.$$

By coupling with a biased random walk on $[0, \ldots, h_n - 1]$ with positive drift (on $[0, \ldots, h_n/2 - 1]$) and negative (or neutral) drift (on $[h_n/2, \ldots, h_n - 1]$), one checks that

$$\limsup_{n\to\infty} \frac{1}{n} \log P_{\lambda,\bar{\omega}_n}(\mathcal{C}_n) = 0\,. \tag{5.6}$$

In similarity with the definition of an n-slow tree in Section 3, a tree $\mathbf{T} \in \mathcal{T}$ is called *n-traplike* if there exists a vertex $j \in D_{b_n}(\mathbf{T})$ (for $b_n = \lfloor \log n \rfloor^3$) such that:

H1. The finite subtree consisting of the first h_n levels of $\mathbf{T}^j$ is an n-trap.

H2. All ancestors l of j satisfy $k_l \leq e^{(\log n)^4}$.

Exactly as in the case of n-slow trees, it is easy to check that for GW-a.e. ω there exists an $n_0(\omega)$ such that ω is n-traplike for each $n > n_0(\omega)$. By symmetry, upon hitting an n-trap at some $j \in D_{b_n}(\omega)$ and conditional upon the event $\mathcal{C}_n$ for this n-trap, the hitting measure of $D_{h_n}(\omega^j)$ is uniform. Let $n' := n - n_1 - b_n$, noting that for all $n > n_0(\omega)$,

$$\begin{aligned} P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in (a-\varepsilon, a+\varepsilon)\right) &\geq P_{\lambda,\omega}\left(\{X_n\} \text{ hits an } n\text{-trap at } j \in D_{b_n}(\omega) \text{ in } b_n \text{ steps}\right) \\ &\quad \cdot P_{\lambda,\bar{\omega}_n}(\mathcal{C}_n) L_n^{-1} \sum_{v \in D_{h_n}(\omega^j)} P_{\lambda,\omega^v}\left(n^{-1}|X_{n'}| \in (a-\varepsilon, a+\varepsilon),\, \tau_{\mathbf{o}} = \infty\right) \\ &\geq (e^{(\log n)^4} + \lambda)^{-b_n} P_{\lambda,\bar{\omega}_n}(\mathcal{C}_n) L_n^{-1} \sum_{v \in D_{h_n}(\omega^j)} W_v\,, \end{aligned} \tag{5.7}$$

where the L_n random variables

$$W_v := P_{\lambda,\omega^v}\left(n^{-1}|X_{n'}| \in (a-\varepsilon, a+\varepsilon),\, \tau_{\mathbf{o}} = \infty\right)$$

are independent of each other and of identical distribution. Since $n'/n \to a/v_\lambda$ as $n \to \infty$, it follows from (1.2) that then $E(W_v) \to P_\lambda(\tau_{\mathbf{o}} = \infty)$. The transience of $\{X_n\}$ implies that $q := P_\lambda(\tau_{\mathbf{o}} = \infty) > 0$, so applying Cramér's Theorem for the i.i.d. $W_v \in [0,1]$, we see that

$$\limsup_{n\to\infty} \frac{1}{L_n} \log P_\lambda(L_n^{-1} \sum_{v\in D_{h_n}(\omega^j)} W_v \leq q/2) < 0\,.$$

Since $L_n = O((\log n)^2)$, it follows by the Borel-Cantelli lemma that there exists a finite $n_0'(\omega) > n_0(\omega)$ such that $L_n^{-1}\sum_{v\in D_{h_n}(\omega^j)} W_v > q/2$ for all $n \geq n_0'(\omega)$. Consequently, by (5.6) and (5.7), one obtains that

$$\liminf_{n\to\infty} \frac{1}{n}\, \log P_{\lambda,\omega}\left(\frac{|X_n|}{n} \in (a-\varepsilon, a+\varepsilon)\right) = 0 \qquad GW-a.s.$$

The proof is then completed by an application of Lemma 2.1. □

Remark: Applying an argument similar to that of (5.7), this time with $n' = an - h_n - b_n$ and

$$W_v := P_{\lambda,\omega^v}\left(v_\lambda T_{n'}/n' \in (1-\varepsilon, 1+\varepsilon),\, \tau_{\mathbf{o}} = \infty\right),$$

one shows that if $m > \lambda \geq d_{\min}$, then for any $0 < a < v_\lambda$,

$$\liminf_{n\to\infty} \frac{1}{n}\, \log P_{\lambda,\omega}\left(\frac{T_{an}}{n} \in (1-2\varepsilon, 1+2\varepsilon)\right) = 0 \qquad GW-a.s.$$

We move on to establish the uncertainty estimate.

Proof of Proposition 5.1: Fix $\varepsilon > 0$ and a tree ω that is ℓ-slow (with respect to $d_{\min}$) for all $\ell \geq M$. Let $p = d_{\min}/(d_{\min}+\lambda) > 1/2$ and $\xi_\ell = b_\ell + h_\ell$. It follows from the construction of (3.6)–(3.8) that for some $\ell = \ell(M,\varepsilon)$ and all $n \geq n_0(\ell)$,

$$P_{\lambda,\omega}(\tau_{\mathbf{o}} \wedge T_{\xi_\ell} > \alpha n - n + \xi_\ell) \geq \exp(-n((\alpha-1)H(\frac{1}{2}|p) + \varepsilon))\,.$$

With $v = X_{\alpha n - n + \xi_\ell}$ we note that

$$P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n \geq \alpha n | \tau_{\mathbf{o}} \wedge T_{\xi_\ell} > \alpha n - n + \xi_\ell) \geq \inf_{v\in\omega, 0<|v|<\xi_\ell} P_{\lambda,\omega,v}(\tau_v = \infty) \tag{5.8}$$

and since $\min(k_j : j \in \omega) \geq d_{\min} > \lambda$, the infimum in (5.8) is at least $P_{\lambda,\omega_{\min},\mathbf{o}}(\tau_{\mathbf{o}} = \infty) \geq \exp(-\varepsilon n)$ for all $n \geq n_1(\varepsilon)$. Consequently, for all $n \geq n_0 \vee n_1$,

$$P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n \geq \alpha n) \geq \exp(-n((\alpha-1)H(\frac{1}{2}|p) + 2\varepsilon)). \tag{5.9}$$

To compute an upper bound on $P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n \geq \alpha n, X_{T_n} = j)$, mark a subtree $\omega_{\min} \subset \omega$ rooted at $\mathbf{o}$ such that each vertex has $d_{\min}$ children and $j \in \omega_{\min}$ (we enumerate the children of vertices $v \in \omega$ in such a way that if $v \in \omega_{\min}$ then $\{v_i\}_{i=1}^{d_{\min}}$ are all in $\omega_{\min}$). Extend $\omega_{\min}$ to a $(d_{\min}+1)$-regular tree denoted $\bar{\omega}_{\min}$ by attaching to the root $\mathbf{o}$, using one edge, a copy of $\omega_{\min}$, denoted $\omega'_{\min}$ (we denote the root of $\omega'_{\min}$ by $\mathbf{o}'$). We pick in $\omega'_{\min}$ an arbitrary ray r' emanating from $\mathbf{o}'$. The relations between generations in $\bar{\omega}_{\min}$ are now defined with respect to r': For each $\ell \in \omega_{\min} \setminus \mathbf{o}$ (or $\ell \in \omega'_{\min} \setminus r'$), the parent ℓ^* of ℓ is the parent in $\omega_{\min}$ (respectively, $\omega'_{\min}$), and a similar definition applies to the children of ℓ. For $\ell = \mathbf{o}$ we have $\ell^* = \mathbf{o}'$ whereas the children of $\mathbf{o}$ are its children in $\omega_{\min}$. For $\ell \in r'$ of distance $|\ell|$ from $\mathbf{o}'$, we have ℓ^* as the unique vertex in r' at distance $|\ell|+1$ from $\mathbf{o}'$, while the children of such ℓ are its children in $\omega'_{\min} \setminus r'$ together with the unique vertex in r' at distance $|\ell|-1$ from $\mathbf{o}'$ ($\mathbf{o}$, in case $\ell = \mathbf{o}'$). Note that each vertex in $\bar{\omega}_{\min}$ has now $d_{\min}$ children (each identified with an integer $1, \ldots, d_{\min}$) and one parent. Let $\{S_n^p\}$ denote the biased simple random walk on the integers, for $p = d_{\min}/(d_{\min}+\lambda)$ as above. We now introduce the (modified) λ-biased random walk Z_n on $\bar{\omega}_{\min}$ as the Markov chain, starting at $\mathbf{o}$, which at time 1 jumps to one of its children with equal probability and thereafter jumps to its parent with probability $(1-p)$ and to any one of its $d_{\min}$ children with probability $p/d_{\min}$. Assigning negative levels to the vertices on r', starting at $\mathbf{o}'$ being in level -1, it follows that the sequence of levels $\{|Z_n|\}$ has the law of $\{S_n^p\}$ conditional upon $S_1^p = 1$. We construct below a coupling between the λ-biased random walk X_n on ω and Z_n on $\bar{\omega}_{\min}$ in such a way that until the first time of return of X_t to $\mathbf{o}$, at times in which X_t is on the ray $r = r(j)$ connecting $\mathbf{o}$ and j, we have that Z_t is either on r with $|Z_t| \le |X_t|$ or on r'. Assuming such a coupling, for any integer $t \ge \alpha n$, it follows by the symmetry of Z_t with respect to vertices at the same level in $\omega_{\min}$, that

$$
\begin{aligned}
P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n = t, X_{T_n} = j) &\le \sum_{k=0}^{n} P_{\lambda,\bar{\omega}_{\min}}(|Z_t| = k, Z_t \in r(j)) + P_{\lambda,\bar{\omega}_{\min}}(Z_t \in r') \\
&\le \sum_{k=0}^{n} (d_{\min})^{-k} P(S_t^p = k) + P(S_t^p < 0) \\
&\le \sum_{i=t/2}^{(t+n)/2} \binom{t}{i} (d_{\min})^{-(2i-t)} p^i (1-p)^{t-i} + e^{-tH(\frac{1}{2}|p)} \\
&\le 2^t (p(1-p))^{t/2} (1 + (d_{\min}\lambda)^{-n/2}) + e^{-tH(\frac{1}{2}|p)}
\end{aligned}
$$

where in the last inequality we use the identity $p/(1-p) = d_{\min}/\lambda$ and the estimate

$$
\sum_{i=t/2}^{(t+n)/2} \binom{t}{i} \le 2^t .
$$

With $2\sqrt{p(1-p)} = \exp(-H(1/2|p))$, we thus conclude, summing over $t \ge \alpha n$ and using also (5.9), that for some $c < \infty$ and all $n > n_0 \vee n_1$

$$
\frac{P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n \ge \alpha n, X_{T_n} = j)}{P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_n \ge \alpha n)} \le c e^{2n\varepsilon} e^{-nH(1/2|p)} (2 + (d_{\min}\lambda)^{-n/2}) .
$$

The claim of the lemma follows since $\exp(-H(1/2|p))/\sqrt{d_{\min}\lambda} = 2/(d_{\min}+\lambda) < 1$ due to our assumption $d_{\min} \ge 2$. It thus only remains to construct the coupling announced before. We construct Z_n out of the walk X_n in the following way: If $X_1 \in \omega_{\min}$ then $Z_1 = X_1$ (this is possible because $1/d_{\min} \ge 1/k_{\mathbf{o}}$). Then, whenever $X_{t+1} = (X_t)^*$ then $Z_{t+1} = Z_t^*$ whereas whenever $X_t = \ell \in r \setminus \mathbf{o}$ and $X_{t+1} = \ell_i \in r$ then $Z_{t+1} = (Z_t)_i$. Otherwise, one chooses Z_{t+1} in such a way to possess the correct transition probabilities (this coupling is always possible, up to the first return of X_t to $\mathbf{o}$, because $1/(\lambda + d_{\min}) \ge 1/(\lambda + k_{X_t})$). We now claim that whenever $t < \tau_{\mathbf{o}}$ and X_t is on r, then Z_t is either on r' or on r with $|Z_t| \le |X_t|$. To see this, let n_1 be the first time in which X_t leaves r and m_1 the first time of return to r. Then, $X_{n_1-1} = X_{m_1}$, and hence the number of steps that X has traveled down ω is equal to the number of steps it has traveled up. Since Z_n travels with X_n up to $n_1 - 1$, and each up step of X is an up step of Z, we conclude that Z_{m_1} is above X_{m_1} on the infinite (directed) ray $r \cup r'$. The same arguments apply for later excursions of X_n from the ray r, with the only difference that Z already starts above X. ☐

Proof of Proposition 5.2: The proof of exponential decay for slowdown probabilities in case $\lambda < d_{\min}$ uses a decomposition similar to (4.23). Fix $c > 0$ and $\varepsilon > 0$ such that $E_\lambda(|X_{r_2}| - |X_{r_1}|) > 1/c$ and $E_\lambda(r_2 - r_1) < (x-\varepsilon)/c$ (which is possible since $xv_\lambda > 1$). Note that

$$\{T_n \geq xn\} \subseteq \{r_{cn+1} \geq xn\} \bigcup \{|X_{r_{cn+1}}| \leq n\} \, .$$

Hence,

$$\begin{aligned} P_\lambda(T_n \geq xn) \quad &\leq \quad P_\lambda\left(\sum_{j=1}^{cn} (|X_{r_{j+1}}| - |X_{r_j}|) \leq n\right) \\ &+ \quad P_\lambda\left(\sum_{j=1}^{cn} (r_{j+1} - r_j) \geq (x-\varepsilon)n\right) + P_\lambda\left(r_1 \geq \varepsilon n\right) \, , \end{aligned} \tag{5.10}$$

and the conclusion follows from (4.19) and (4.20). □

The following auxiliary lemma, is needed for the proof of Lemma 5.1.

Lemma 5.2 *Let $\omega_{\min}$ denote the rooted $d_{\min}$-ary tree. Then, there exist $\delta = \delta_k \to_{k\to\infty} 0$, and $c_k < \infty$, such that, for all ℓ of the same parity as k,*

$$\begin{aligned} P_{\lambda,\omega_{\min}}\Big(|X_\ell| \leq k\Big) \quad &\leq \quad c_k(1+\delta_k)^\ell P_{\lambda,\omega_{\min}}\Big(|X_\ell| = k, 1 \leq |X_i| \leq k-1, i = 1,\ldots,\ell-1\Big) \\ &= \quad c_k(1+\delta_k)^\ell P_{\lambda,\omega_{\min}}\Big(\tau_{\mathbf{o}} > T_k = \ell\Big) \, . \end{aligned}$$

Proof: Using the notations introduced in the proof of (1.5), recall that the law of $\{|X_n|\}$ under $P_{\lambda,\omega_{\min}}(\cdot)$ is the same as that of the (biased) reflected random walk $\{S_n^{R,p}\}$ for $p := d_{\min}/(d_{\min}+\lambda) > 1/2$. Bounding the latter in terms of the biased simple random walk $\{S_n^p\}$, it thus suffices to show that

$$P(S_\ell^p \leq k) \leq c_k(1+\delta_k)^\ell P(S_\ell^p = k, 1 \leq S_i^p \leq k-1, i = 1,\ldots,\ell-1).$$

To this end, let

$$\begin{aligned} \mathcal{S}_{k,\ell} \quad &= \quad \Big\{\{s_i\}_{i=0}^\ell : |s_{i+1} - s_i| = 1, s_0 = 0, s_\ell \leq k\Big\} \, , \\ \mathcal{S}_{k,\ell}^+ \quad &= \quad \Big\{\{s_i\}_{i=0}^\ell : |s_{i+1} - s_i| = 1, s_0 = 0, k-1 \geq s_i > 0 \text{ for } i > 0, s_\ell = k\Big\} \subset \mathcal{S}_{k,\ell} \end{aligned}$$

Define $N^+ = \{i : s_{i+1} - s_i > 0\}$. Note that $|N^+| \leq \frac{\ell}{2} + \frac{k}{2}$ on $\{s_i\} \in \mathcal{S}_{k,\ell}$, while $|N^+| = \frac{\ell}{2} + \frac{k}{2} \geq \frac{\ell}{2}$ whenever $\{s_i\} \in \mathcal{S}_{k,\ell}^+$. Consequently,

$$P(S_\ell^p \leq k) = \sum_{\{s_i\}\in\mathcal{S}_{k,\ell}} p^{|N^+|}(1-p)^{\ell-|N^+|} \leq c_k(p(1-p))^{\ell/2}|\mathcal{S}_{k,\ell}| \tag{5.11}$$

where $c_k = (p/(1-p))^{k/2}$, while, similarly,

$$P(S_\ell^p = k, 1 \leq S_i^p \leq k-1, i = 1,\ldots,\ell-1) = \sum_{\{s_i\}\in\mathcal{S}_{k,\ell}^+} p^{|N^+|}(1-p)^{\ell-|N^+|} \geq (p(1-p))^{\ell/2}|\mathcal{S}_{k,\ell}^+| \, . \tag{5.12}$$

Taking into account that

$$|\mathcal{S}^+_{k,\ell}| = 2^\ell P_{SRW(\frac{1}{2})}(0 < S_i < k, i = 1, \dots, \ell-1, S_\ell = k)\,,$$

we find that

$$\lim_{k\to\infty} \liminf_{\ell\to\infty,(\ell=k)_{\mathrm{mod}2}} \frac{1}{\ell}\log|\mathcal{S}^+_{k,\ell}| = \lim_{k\to\infty}\left(\log 2 - \frac{\pi^2}{2k^2}\right) = \log 2\,,$$

while $|\mathcal{S}_{k,\ell}| \le 2^\ell$, which together with (5.11) and (5.12) completes the proof of Lemma 5.2. □

Proof of Lemma 5.1: Fix $\varepsilon > 0$ and choose $k = k(\varepsilon)$ according to Lemma 5.2 such that $\delta_k \le \varepsilon$. Then, for $n > k$, decomposing the path $\{X_t : 0 \le t \le T_n\}$ according to the value ℓ of $\max\{t \le T_n : |X_t| = k\}$ and $v = X_\ell$, it follows that,

$$\begin{aligned} P_{\lambda,\omega}(T_n \in B_n) &= \sum_{(\ell=k)\ \mathrm{mod}\ 2}\ \sum_{v\in D_k(\omega)} P_{\lambda,\omega}(T_n \in B_n, T_n > \ell, X_\ell = v, |X_i| > k, i = \ell+1,\dots,T_n) \\ &= \sum_{(\ell=k)\ \mathrm{mod}\ 2,\ell<b_n}\ \sum_{v\in D_k(\omega)} P_{\lambda,\omega}(X_\ell = v, T_n > \ell)\widehat{W}_v\,, \end{aligned} \tag{5.13}$$

where the random variables

$$\widehat{W}_v := P_{\lambda,\omega,v}(\widetilde{T}_{n-k}(v) \in B_n - \ell, \tau_v > \widetilde{T}_{n-k}(v)),$$

are identically distributed and independent of $\mathcal{F}_k = \sigma(\cup_{i=0}^k D_i)$. Coupling $\{X_n\}$ with the λ-biased random walk on $\omega_{\min}$, we have that

$$\sup_{v\in D_k(\omega)} P_{\lambda,\omega}(X_\ell = v, T_n > \ell) \le P_{\lambda,\omega}(|X_\ell| \le k) \le P_{\lambda,\omega_{\min}}(|X_\ell| \le k)\,. \tag{5.14}$$

Combining (5.13), (5.14) and Lemma 5.2 we obtain that

$$P_{\lambda,\omega}(T_n \in B_n) \le c_k(1+\varepsilon)^{b_n} \sum_{(\ell=k)\ \mathrm{mod}\ 2} P_{\lambda,\omega_{\min}}(\tau_{\mathbf{o}} > T_k = \ell) \sum_{v\in D_k(\omega)} \widehat{W}_v \tag{5.15}$$

(recall that $0 \le \delta_k \le \varepsilon$). Let $\mathcal{T}_k$ denote the set of rooted trees with each vertex in first k levels having $d_{\min}$ children. Note that $|D_k(\omega)| = (d_{\min})^k$ for $\omega \in \mathcal{T}_k$, and by symmetry, the decomposition (5.13) yields,

$$\begin{aligned} &\mathbf{1}_{\{\omega\in\mathcal{T}_k\}} \sum_{(\ell=k)\ \mathrm{mod}\ 2} P_{\lambda,\omega_{\min}}(\tau_{\mathbf{o}} > T_k = \ell)(d_{\min})^{-k} \sum_{v\in D_k(\omega)} \widehat{W}_v \\ = &\ \mathbf{1}_{\{\omega\in\mathcal{T}_k\}} \sum_{(\ell=k)\ \mathrm{mod}\ 2} P_{\lambda,\omega}(\tau_{\mathbf{o}} > T_k = \ell, |X_j| > k, j = \ell+1,\dots,T_n, T_n \in B_n) \\ \le &\ P_{\lambda,\omega}(T_n \in B_n, \tau_{\mathbf{o}} > T_n)\,. \end{aligned} \tag{5.16}$$

Note that $E_{GW}(|D_k|) = m^k$ and that the i.i.d. random variables $\{\widehat{W}_v\}$ are independent of $\mathbf{1}_{\{\omega\in\mathcal{T}_k\}}$ and of $|D_k(\omega)|$. Hence, taking expectations in (5.15) results with

$$P_\lambda(T_n \in B_n) \le c_k(1+\varepsilon)^{b_n} m^k \sum_{(\ell=k)\ \mathrm{mod}\ 2} P_{\lambda,\omega_{\min}}(\tau_{\mathbf{o}} > T_k = \ell) E_\lambda(\widehat{W}_v)\,,$$

while taking expectations in (5.16) gives,

$$GW(\mathcal{T}_k) \sum_{(\ell=k)\ \mathrm{mod}\ 2} P_{\lambda,\omega_{\min}}(\tau_{\mathbf{o}} > T_k = \ell) E_\lambda(\widehat{W}_v) \le P_\lambda(T_n \in B_n, \tau_{\mathbf{o}} > T_n)\,.$$

Consequently,

$$P_\lambda(T_n \in B_n) \le \frac{c_k m^k}{GW(\mathcal{T}_k)}(1+\varepsilon)^{b_n} P_\lambda(T_n \in B_n, \tau_{\mathbf{o}} > T_n)\,,$$

The conclusion of the lemma follows by noting that $c_k m^k / GW(\mathcal{T}_k) < \infty$ is independent of n. $\square$

6 Proof of Theorem 1.3

We can modify Lemma 2.1 to get, for a function $\varphi(n)$ with $\varphi(n)/\log n \to \infty$,

$$\limsup_{n\to\infty} \frac{1}{\varphi(n)} \log P_{\lambda,\omega}(A_n) \le \limsup_{n\to\infty} \frac{1}{\varphi(n)} \log P_\lambda(A_n) \tag{6.1}$$

and

$$\liminf_{n\to\infty} \frac{1}{\varphi(n)} \log P_{\lambda,\omega}(A_n) \le \liminf_{n\to\infty} \frac{1}{\varphi(n)} \log P_\lambda(A_n)\,. \tag{6.2}$$

Taking hereafter $\lambda = d_{\min} < m$ as in the statement of Theorem 1.3, it is therefore enough to provide lower bounds for the quenched probabilities and and upper bounds for the corresponding annealed probabilities.

In particular, to prove part (i) of the theorem, it is thus enough to prove that for $0 < \delta < v < v_1$ and $c_1 = (3/2)(\pi |\log p_1|)^{2/3}$,

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log P_{1,\omega}\left(v - \delta \le \frac{|X_n|}{n} \le v\right) \ge -c_1 \left(1 - \frac{v}{v_1}\right)^{1/3}, \tag{6.3}$$

whereas for some $c_2 > 0$,

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log P_1\left(\frac{|X_n|}{n} \le v\right) \le -c_2 \left(1 - \frac{v}{v_1}\right)^{1/3}. \tag{6.4}$$

We have, with $\varepsilon < \delta/4$, that

$$P_{1,\omega}\left(v - \delta \le \frac{|X_n|}{n} \le v\right) \ge P_{1,\omega}\left(v - 4\varepsilon \le \frac{|X_n|}{n} \le v\right) \ge P_{1,\omega}\left(n(1+\varepsilon) \ge T_{n(v-2\varepsilon)} \ge n(1-\varepsilon)\right) \tag{6.5}$$

Considering $\varepsilon \to 0$ and $n' = n(v - 2\varepsilon)$ we see that to prove (6.3), it suffices to show that for $\beta > u > 1/v_1$, we have

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log P_{1,\omega}(\beta n \ge T_n \ge un) \ge -c_1 \left(u - \frac{1}{v_1}\right)^{1/3}. \tag{6.6}$$

Similarly, since $\{|X_{xn}| \le n\}$ whenever $\{T_n \ge xn\}$, we get (6.4) as soon as we show that for some $c_2 > 0$ and all $x > 1/v_1$,

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log P_1(T_n \ge xn) \le -c_2 \left(x - \frac{1}{v_1}\right)^{1/3}. \tag{6.7}$$

In an analogous manner, part (ii) of Theorem 1.3 is proved by showing that for $c_1 = (\pi \log d_{\min})^2/2$ and all $\beta > u > 1/v_\lambda$, we have

$$\liminf_{n\to\infty} \frac{(\log n)^2}{n} \log P_{\lambda,\omega}(\beta n \ge T_n \ge un) \ge -c_1 \left(u - \frac{1}{v_\lambda}\right) \tag{6.8}$$

whereas for some $c_2 > 0$ and all $x > 1/v_\lambda$,

$$\limsup_{n\to\infty} \frac{(\log n)^2}{n} \log P_\lambda \left(T_n \geq xn\right) \leq -c_2 \left(x - \frac{1}{v_\lambda}\right) . \tag{6.9}$$

In proving (6.6)–(6.9) we let $\mathrm{HIT}^j_{\lambda,\omega}(x) := P^j_{\lambda,\omega}(X_{T_{|x|}} = x)$ denote the probability that the λ-biased random walk on the tree ω hits the $|x|$-th level at x, starting at $X_0 = j$, where $|j| \leq |x|$ and $\mathrm{HIT}_{\lambda,\omega}(x) := \mathrm{HIT}^{\mathbf{o}}_{\lambda,\omega}(x)$.

Proof of (6.6) and (6.8): The finite $d_{\min}$-ary rooted tree of b levels, is called a $d_{\min}$-pipe of length b. We say that $x \in \omega$ starts a $d_{\min}$-pipe of length b if the first b levels of ω^x are a $d_{\min}$-pipe of length b. Fixing $\varepsilon > 0$, a pipe of length b starting at $x \in \omega$ with $|x| \leq (n-b)$ is called *n-good pipe* if its leftmost leaf y satisfies

$$P_{\lambda,\omega,y}(\tau_y = \infty, |n^{-1}\widetilde{T}_{n-|y|}(y) - \frac{1}{v_\lambda}| < \varepsilon) > \varepsilon \tag{6.10}$$

The next lemma is key to the proof of both (6.6) and (6.8).

Lemma 6.1 *Assume $1 \leq \lambda = d_{\min} < m < \infty$. Fix $0 < \varepsilon < P_{\lambda,\mathbf{o}}(\tau_{\mathbf{o}} = \infty)$ and integers $b_n \to \infty$ satisfying $n^{-1}b_n \to 0$. Let $\tilde{q}_n = GW(\mathbf{o}$ starts a $d_{\min}$-pipe of length $b_n)$. For $p > 1$, set $k_n = n^{1/p}$ and assume $(k_n)^{-1}\log \tilde{q}_n \to 0$. Define*

$$H_n(\omega) := P_{\lambda,\omega}(T_{k_n} \leq \varepsilon n, X_{T_{k_n}} \text{ starts an } n-\text{good pipe of length } b_n)$$

Then, there exists $\eta = \eta(\varepsilon) > 0$ such that for GW-a.a. ω and for all except finitely many n,

$$H_n(\omega) \geq \eta \tilde{q}_n \tag{6.11}$$

Proof: Fix ε, b_n and k_n as in the statement of the lemma. In case $\lambda = 1$ it follows from [10, Theorem 9.8] that for some positive constant d', for the sets

$$G_n := \{x \in D_{k_n} : \mathrm{HIT}_{\lambda,\omega}(x) \leq e^{-k_n d'}\}, \tag{6.12}$$

we have for GW-a.a. ω,

$$\mathrm{HIT}_{\lambda,\omega}(G_n) = \sum_{x\in G_n} \mathrm{HIT}_{\lambda,\omega}(x) \underset{n\to\infty}{\longrightarrow} 1 \tag{6.13}$$

In case of $\lambda = d_{\min} > 1$, even $G_n = D_{k_n}$ shall do, since then for all $j, x \in \omega$, such that $|x| = |j| + \ell$

$$\mathrm{HIT}^j_{\lambda,\omega}(x) \leq (d_{\min})^{-\ell} \tag{6.14}$$

(Indeed, mark a $d_{\min}$-ary subtree $\omega_{\min} \subset \omega$ rooted at $\mathbf{o}$ such that $j, x \in \omega_{\min}$. Recording only the part of the path of $\{X_n\}$ within $\omega_{\min}$ results with a λ-biased random walk $\{Z_t\}$ on the latter subtree. The hitting measure of $D_{|x|}(\omega_{\min})$ by $\{Z_t\}$ is uniform on the descendents of the last common ancestor of x and j in ω, and there are at least $(d_{\min})^\ell$ of these. If $X_{T_{|x|}} = x$ then necessarily $\{Z_t\}$ hit $D_{|x|}(\omega_{\min})$ for the first time also at the vertex x. The latter event has probability of at most $(d_{\min})^{-\ell}$, yielding the bound of (6.14)).

Define next for each $x \in D_{k_n}(\omega)$ the random variable

$$\overline{W}_x = \mathbf{1}_{\{x \text{ starts an } n- \text{ good pipe of length } b_n\}},$$

and the modified hitting measure $\widetilde{\mathrm{HIT}}_{\lambda,\omega}(x) := P_{\lambda,\omega}(X_{T_{|x|}} = x, T_{|x|} \le \varepsilon|x|^p)$. Note that

$$H_n(\omega) = \sum_{x \in D_{k_n}(\omega)} \widetilde{\mathrm{HIT}}_{\lambda,\omega}(x)\overline{W}_x \ge \sum_{x \in G_n} \widetilde{\mathrm{HIT}}_{\lambda,\omega}(x)\overline{W}_x =: \widetilde{H}_n(\omega)$$

Let $\mathcal{F}_{k_n} = \sigma(\cup_{i=0}^{k_n} D_i)$ and

$$Y_n(\omega) := P_{\lambda,\omega,\mathbf{o}}\left(\tau_{\mathbf{o}} = \infty, |n^{-1}T_{n-k_n-b_n} - \frac{1}{v_\lambda}| < \varepsilon\right) . \tag{6.15}$$

The random variables $\widetilde{\mathrm{HIT}}_{\lambda,\omega}(x)$ and $\mathbf{1}_{\{x \in G_n\}}$ are measurable on $\mathcal{F}_{k_n}$, whereas given $\mathcal{F}_{k_n}$, $\{\overline{W}_x : x \in G_n\}$ are independent Bernoulli(q_n) random variables, with

$$q_n = \tilde{q}_n GW(\{\omega : Y_n(\omega) > \varepsilon\})$$

(see (6.10)). Thus,

$$E\left(\widetilde{H}_n(\omega)|\mathcal{F}_{k_n}\right) = q_n \sum_{x \in G_n} \widetilde{\mathrm{HIT}}_{\lambda,\omega}(x) = q_n \widetilde{\mathrm{HIT}}_{\lambda,\omega}(G_n) \tag{6.16}$$

and

$$E\left(\widetilde{H}_n(\omega)^2|\mathcal{F}_{k_n}\right) = q_n\left[\sum_{x \in G_n} \widetilde{\mathrm{HIT}}_{\lambda,\omega}(x)^2\right] + q_n^2\left[\sum_{x \in G_n}\sum_{y \in G_n, y \ne x} \widetilde{\mathrm{HIT}}_{\lambda,\omega}(x)\widetilde{\mathrm{HIT}}_{\lambda,\omega}(y)\right]$$

Hence,

$$\begin{aligned} \mathrm{Var}(\widetilde{H}_n(\omega)|\mathcal{F}_{k_n}) &= E\left(\widetilde{H}_n(\omega)^2|\mathcal{F}_{k_n}\right) - [E(\widetilde{H}_n(\omega)|\mathcal{F}_{k_n})]^2 \\ &\le q_n \sum_{x \in G_n} \widetilde{\mathrm{HIT}}_{\lambda,\omega}(x)^2 \le q_n e^{-k_n d'} \widetilde{\mathrm{HIT}}_{\lambda,\omega}(G_n) , \end{aligned} \tag{6.17}$$

relying upon the definition (6.12) of G_n in the second inequality. We have, using first (6.16), then (6.17), that

$$\begin{aligned} & GW\left(\{\omega : \widetilde{H}_n(\omega) < \frac{q_n}{2}\widetilde{\mathrm{HIT}}_{\lambda,\omega}(G_n)\}\Big|\mathcal{F}_{k_n}\right) \\ \le\ & GW\left(\{\omega : |\widetilde{H}_n(\omega) - E[\widetilde{H}_n(\omega)|\mathcal{F}_{k_n}]| \ge \frac{1}{2}E(\widetilde{H}_n(\omega)|\mathcal{F}_{k_n})\}\Big|\mathcal{F}_{k_n}\right) \\ \le\ & \frac{4\mathrm{Var}\left(\widetilde{H}_n(\omega)|\mathcal{F}_{k_n}\right)}{[E(\widetilde{H}_n(\omega)|\mathcal{F}_{k_n})]^2} \le \frac{4e^{-k_n d'}}{q_n \widetilde{\mathrm{HIT}}_{\lambda,\omega}(G_n)} \end{aligned}$$

Recall that $n^{-1}(k_n + b_n) \to 0$ implying that $P_{\lambda,\omega,\mathbf{o}} \times GW$-a.s. $n^{-1}T_{n-k_n-b_n} \to 1/v_\lambda$. Hence, the random variables $Y_n(\omega) \in [0,1]$ of (6.15) are such that $E_{GW}(Y_n) \to P_{\lambda,\mathbf{o}}(\tau_{\mathbf{o}} = \infty) > \varepsilon$. Consequently,

$$q_n \ge 8\eta\tilde{q}_n$$

for some $\eta = \eta(\varepsilon) > 0$ and all n large enough. Let $A_n = \{\omega : \widetilde{H}_n(\omega) < q_n/4\}$ and $B_n = \{\omega : \widetilde{\mathrm{HIT}}_{\lambda,\omega}(G_n) \ge 1/2\}$. Since $(k_n)^{-1}\log\tilde{q}_n \to 0$, we see that

$$\sum_{n=1}^{\infty} P(A_n \cap B_n) \le \sum_{n=1}^{\infty} \frac{8e^{-k_n d'}}{q_n} \le \sum_{n=1}^{\infty} \frac{e^{-k_n d'}}{\eta\tilde{q}_n} < \infty .$$

With $m > \lambda$ and $n^{-1}k_n \to 0$, it follows from [11, Theorem 3.1] that

$$\mathrm{HIT}_{\lambda,\omega}(G_n) - \widetilde{\mathrm{HIT}}_{\lambda,\omega}(G_n) \le P_{\lambda,\omega}(T_{k_n} > \varepsilon n) \underset{n\to\infty}{\longrightarrow} 0$$

for GW-a.a. ω. Consequently, it follows from (6.13) that B_n holds eventually for GW-a.a. ω, and the claim (6.11) follows by a simple modification of the Borel-Cantelli lemma. $\square$

The lower bounds (6.6) and (6.8) on $P_{\lambda,\omega}(\beta n \ge T_n \ge un)$ are established by considering the intersection of the following three events for positive $\varepsilon < (\beta - u)/4$.

$\mathcal{E}_1$: Both $T_{k_n} \le \varepsilon n$ and $X_{T_{k_n}}$ starts an n-good pipe of length b_n.

$\mathcal{E}_2$: Upon T_{k_n}, the random walk X_t stays inside this $d_{\min}$-pipe for at least $(u - 1/v_1 + 2\varepsilon)n$ and at most $(\beta - 1/v_1 - 2\varepsilon)n$ steps, exiting at its leftmost leaf y at level $|y| = k_n + b_n$.

$\mathcal{E}_3$: The random walk X_t proceeds for $t \ge T_{k_n+b_n}$ at the normal speed, such that $T_n - T_{k_n+b_n}$ is at least $(1/v_\lambda - \varepsilon)n$ and at most $(1/v_\lambda + \varepsilon)n$.

Observe that

$$\bigcap_{i=1}^{3} \mathcal{E}_i \subseteq \{\beta n \ge T_n \ge un\}\,, \tag{6.18}$$

and the definition (6.10) of an n-good pipe guarantees that $P_{\lambda,\omega}(\mathcal{E}_3|\mathcal{E}_2,\mathcal{E}_1) > \varepsilon$. Consequently, we concentrate next on lower bounding

$$P_{\lambda,\omega}(\mathcal{E}_1 \cap \mathcal{E}_2) = P_{\lambda,\omega}(\mathcal{E}_1)P_{\lambda,\omega}(\mathcal{E}_2|\mathcal{E}_1) = H_n(\omega)P_{\lambda,\omega}(\mathcal{E}_2|\mathcal{E}_1)\,.$$

To this end, let $t_{b_n} = \inf\{j > 0 : S_j = -1 \text{ or } S_j = b_n\}$. Clearly, the hitting measure of each level in $\bar{\omega}_{\min}$ is uniform. Hence, it follows by the (standard) coupling of $\lambda = d_{\min}$-biased random walk on $\bar{\omega}_{\min}$ and the simple random walk on $\mathbb{Z}$, that

$$P_{\lambda,\omega}(\mathcal{E}_2|\mathcal{E}_1) = (d_{\min})^{-b_n} P_{SRW(\frac{1}{2})}((\beta - 1/v_1 - 2\varepsilon)n \ge t_{b_n} \ge (u - 1/v_1 + 2\varepsilon)n,\ \ S_{t_{b_n}} = b_n) \tag{6.19}$$

(the factor $(d_{\min})^{-b_n}$ represents the condition of exit via the leftmost leaf of the $d_{\min}$-pipe).

Starting with the case of $\lambda = 1$, fix $b > 0$ and set $b_n = bn^{1/3}$, $k_n = \sqrt{n}$ both rounded to the nearest odd integer. Here $\tilde{q}_n = (p_1)^{b_n}$, all the conditions of Lemma 6.1 are satisfied, so it follows from (6.11) that for GW-a.a. ω,

$$\liminf_{n\to\infty} n^{-1/3} \log P_{1,\omega}(\mathcal{E}_1) \ge -b|\log p_1|\,. \tag{6.20}$$

In view of (6.19), it follows by [15, Page 243] and our choice of b_n that

$$\liminf_{n\to\infty} n^{-1/3} \log P_{1,\omega}(\mathcal{E}_2|\mathcal{E}_1) \ge -\frac{\pi^2}{2b^2}(u - 1/v_1 + 2\varepsilon)\,. \tag{6.21}$$

Indeed, this is obvious when starting the simple random walk $\{S_j\}$ at $(b_n - 1)/2$ instead of at 0, while the probability that a simple random walk on $\mathbb{Z}$ starting at 0 visits $(b_n - 1)/2$ before -1 and is doing so within $b_n^{2.5}$ steps, is at least $1/b_n$. Combining (6.18), (6.20) and (6.21) it follows that for GW-a.a. ω,

$$\liminf_{n\to\infty} n^{-1/3} \log P_{1,\omega}(\beta n \ge T_n \ge un) \ge -b|\log p_1| - \frac{\pi^2}{2b^2}(u - 1/v_1 + 2\varepsilon)\,.$$

Optimizing over the constant b, using the fact that $ab + c/b^2 \geq 3(a/2)^{2/3}c^{1/3}$ (where equality holds for $b = (2c/a)^{1/3}$), then taking $\varepsilon \to 0$ yields (6.6).

Turning to the case of $\lambda = d_{\min} > 1$, fix positive $b < 1 - \varepsilon$ and set $b_n = b \log n / \log d_{\min}$, $k_n = n^{b+\varepsilon}$ both rounded to the nearest odd integer. It is not hard to check that now

$$\tilde{q}_n \geq \exp(-c_0 n^b)$$

for some finite c_0 and all large enough n. All conditions of Lemma 6.1 are again satisfied, with (6.11) implying that for GW-a.a. ω,

$$\liminf_{n\to\infty} \frac{(\log n)^2}{n} \log P_{1,\omega}(\mathcal{E}_1) = 0 \, . \tag{6.22}$$

By the same argument as in the derivation of (6.21) out of (6.19), we have for our current choice of b_n that

$$\liminf_{n\to\infty} \frac{(\log n)^2}{n} \log P_{\lambda,\omega}(\mathcal{E}_2|\mathcal{E}_1) \geq -\frac{\pi^2(\log d_{\min})^2}{2b^2}(u - 1/v_1 + 2\varepsilon) \, . \tag{6.23}$$

Combining (6.18), (6.22) and (6.23), then taking $\varepsilon \downarrow 0$ and $b \uparrow 1$, we recover (6.8). □

Proof of (6.7) and (6.9): The key to these upper bounds is the information about the tail of level regeneration times as summarized in the following lemma.

Lemma 6.2 *In case $1 = \lambda = d_{\min} < m < \infty$, there exists $c > 0$ such that for all t large enough,*

$$P_1(r_2 - r_1 \geq t) \leq \exp(-ct^{1/3}) \, , \qquad P_1(r_1 \geq t) \leq \exp(-ct^{1/3}) \, , \tag{6.24}$$

whereas for $1 < \lambda = d_{\min} < m < \infty$,

$$P_\lambda(r_2 - r_1 \geq t) \leq \exp\left(-\frac{ct}{(\log t)^2}\right) \, , \qquad P_\lambda(r_1 \geq t) \leq \exp\left(-\frac{ct}{(\log t)^2}\right) \, . \tag{6.25}$$

Indeed, we use (5.10) for $\varepsilon = (x - 1/v_\lambda)/3 > 0$, which is possible in view of (4.22). By (4.19), the term in (5.10) involving $|X_{r_{cn+1}}|$ is of order $\exp(-Cn)$. Assuming Lemma 6.2 holds, the following well known lemma about partial sums of heavy tailed i.i.d. random variables takes care of the other terms in (5.10), allowing us to deduce both (6.7) and (6.9). Since we did not find in the literature a proof of Lemma 6.3 under our assumptions, we provide such a proof at the end of this section.

Lemma 6.3 *Let $Y_1, Y_2, \ldots$ be an i.i.d. sequence with $E(Y_1^2) < \infty$.*

(i) If $P(Y_1 \geq x) \leq \exp(-cx^\gamma)$ for some $0 < \gamma < 1$, $c > 0$ and all x large enough, then for all $t > E(Y_1)$,

$$\limsup_{n\to\infty} n^{-\gamma} \log P\left(\frac{1}{n}\sum_{j=1}^n Y_j \geq t\right) \leq -c(t - E(Y_1))^\gamma$$

(ii) If $P(Y_1 \geq x) \leq \exp(-cx/(\log x)^2)$ for some $c > 0$ and all x large enough, then for all $t > E(Y_1)$,

$$\limsup_{n\to\infty} \frac{(\log n)^2}{n} \log P\left(\frac{1}{n}\sum_{j=1}^n Y_j \geq t\right) \leq -c(t - E(Y_1))$$

Proof of Lemma 6.2: Recall that by the inequality (4.24) it suffices to prove (6.24) and (6.25) for the random variable r_1. The tail estimates of (6.24) are derived in this case in [12, Theorem 2]. Turning thus to prove (6.25) for r_1, note first that for any $t, \eta > 0$

$$P_\lambda\left(r_1 \geq t\right) \leq P_\lambda\left(|X_{r_1}| \geq \frac{\eta t}{\log t}\right) + P_\lambda\left(|X_t| \leq \frac{\eta t}{\log t}\right) \tag{6.26}$$

It follows from (4.19) that

$$\limsup_{t\to\infty} \frac{\log t}{t} \log P_\lambda\left(|X_{r_1}| \geq \frac{\eta t}{\log t}\right) < 0\,. \tag{6.27}$$

Hence, the tail estimates of (6.25) for r_1 are a direct consequence of the Lemma 6.4 below. $\square$

Lemma 6.4 *Assume* $1 < d_{\min} = \lambda < m$. *For some positive constants* c_0 *and* η,

$$P_\lambda\left(|X_n| \leq \frac{n\eta}{\log n}\right) \leq \exp\left(-\frac{c_0 n}{(\log n)^2}\right)\,.$$

Proof of Lemma 6.4: We need two auxiliary estimates, whose proofs are deferred. We split the n-th level of ω into "fair" and "biased" vertices

$$D_{n,\text{fair}} := \{x \in D_n : k_x = d_{\min}\}\,, \qquad D_{n,\text{biased}} := \{x \in D_n : k_x > d_{\min}\}\,,$$

claiming next that up to a GW "negligible" set of ω, the hitting measure of "biased" vertices is large enough.

Lemma 6.5 *Assume* $1 < d_{\min} = \lambda$. *For* $\delta < (1 - p_{d_{\min}})$ *and* $c > 1 + 1/(\log(1.5))$, *define*

$$\mathcal{A}_n := \{\omega : \mathrm{HIT}^j_{\lambda,\omega}\left(D_{|j|+\ell,\text{biased}}\right) \geq \delta \textit{ for all } |j| \leq n \textit{ and } \lfloor (c-1)\log n \rfloor \leq \ell \leq \lfloor c \log n \rfloor\}. \tag{6.28}$$

Then,

$$\limsup_{n\to\infty} n^{-1} \log GW(\omega \notin \mathcal{A}_n) < 0\,. \tag{6.29}$$

The next lemma shows that, if d is a large enough constant, then the probability $P_{\lambda,\omega}$ that the path X_t meets at least $\delta \log n/2$ "biased" vertices during a time interval $[s, s + d(\log n)^2]$, is uniformly bounded away from zero for $s \leq n$ and $\omega \in \mathcal{A}_n$.

Lemma 6.6 *Assume* $1 < d_{\min} = \lambda$. *Fix* $c > 1 + 1/(\log(1.5))$ *and* $\delta < (1 - p_{d_{\min}})$. *Define* $\mathcal{H}_s = \sigma(X_0, \ldots, X_s)$ *and*

$$B_{s,s+\Delta} := |\{t : k_{X_t} > d_{\min}, s \leq t < s + \Delta\}|\,.$$

Then, there exist finite constants d, n_0, *such that for all* $n \geq n_0$, *any* $s \leq n$ *and any* $\omega \in \mathcal{A}_n$,

$$P_{\lambda,\omega}\left(B_{s,s+\lfloor d(\log n)^2 \rfloor} \geq (\delta/2)\log n \;\middle|\; \mathcal{H}_s\right) \geq \delta^2/8\,. \tag{6.30}$$

We are now ready to prove Lemma 6.4. Fix δ, c as in Lemma 6.5 and d, n_0 as in Lemma 6.6. Set $\xi = \delta^3/(20d)$, and positive η, θ such that $\eta + \theta < \xi/(2d_{\min} + 1)$. Note that for $\lambda = d_{\min}$ the sequence of random variables

$$Z_t = \sum_{s=0}^{t-1} \frac{k_{X_s} - d_{\min}}{k_{X_s} + d_{\min}} - |X_t|\,,$$

is a $P_{\lambda,\omega}$-Martingale with respect to the filtration $\mathcal{H}_s$. Moreover, $Z_0 = 0$ and the martingale differences

$$Z_t - Z_{t-1} = \frac{k_{X_{t-1}} - d_{\min}}{k_{X_{t-1}} + d_{\min}} + (|X_{t-1}| - |X_t|)$$

are such that $|Z_t - Z_{t-1}| \leq 2$. The monotonicity of $k \mapsto (k - d_{\min})/(k + d_{\min})$ on $\mathbb{Z}_+$ implies that

$$Z_n \geq \frac{1}{2d_{\min}+1} \sum_{s=0}^{n-1} \mathbf{1}_{\{k_{X_s} > d_{\min}\}} - |X_n| = \frac{1}{2d_{\min}+1} B_{0,n} - |X_n| \,.$$

Hence, applying the Azuma-Hoeffding inequality we see that

$$P_{\lambda,\omega}\left(B_{0,n} \geq \frac{\xi n}{\log n}, |X_n| \leq \frac{\eta n}{\log n}\right) \leq P_{\lambda,\omega}\left(Z_n \geq \frac{\theta n}{\log n}\right) \leq e^{-\theta^2 n/(8(\log n)^2)} \,, \tag{6.31}$$

for every n and $\omega \in \mathcal{T}$ such that $\min(k_v : v \in \omega) \geq d_{\min}$.

Set $\Delta_n = \lfloor d(\log n)^2 \rfloor$ and $b_n = \lfloor n/\Delta_n \rfloor$. Define the random variables $Y_{i,n} = B_{(i-1)\Delta_n, i\Delta_n}/((\delta/2)\log n)$, $i = 1, \ldots, b_n$. Fixing $\omega \in \mathcal{A}_n$, it follows from (6.30) that for $n \geq n_0$, any $i \leq b_n$ and $\phi > 0$,

$$E_{\lambda,\omega}\left(e^{-\phi Y_{i,n}} \,\middle|\, \mathcal{H}_{(i-1)\Delta_n}\right) \leq 1 - (1 - e^{-\phi}) P_{\lambda,\omega}\left(Y_{i,n} \geq 1 \,\middle|\, \mathcal{H}_{(i-1)\Delta_n}\right) \leq 1 - (1 - e^{-\phi})\delta^2/8 := M(\phi). \tag{6.32}$$

Recall that $\xi = \delta^3/(20d)$ and $Y_{i,n}$ is measurable on $\mathcal{H}_{i\Delta_n}$. Taking $\phi > 0$ for which $e^{\phi\delta^2/10} M(\phi) = \zeta < 1$ (such ϕ exists since $M(\cdot)$ is the moment generating function of Bernoulli($\delta^2/8$) random variables), and combining (6.32) with Chebychev's inequality, we see that for all $\omega \in \mathcal{A}_n$, $n \geq n_0$,

$$P_{\lambda,\omega}\left(B_{0,n} \leq \frac{\xi n}{\log n}\right) \leq P_{\lambda,\omega}\left(\sum_{i=1}^{b_n} Y_{i,n} \leq (\delta^2/10) b_n\right) \leq e^{\phi(\delta^2/10)b_n} E_{\lambda,\omega}\left(\exp(-\phi \sum_{i=1}^{b_n} Y_{i,n})\right) \leq \zeta^{b_n} \,. \tag{6.33}$$

To complete the proof of Lemma 6.4, observe that

$$P_\lambda\left(|X_n| \leq \frac{n\eta}{\log n}\right) \leq GW(\omega \notin \mathcal{A}_n) + \sup_\omega P_{\lambda,\omega}\left(B_{0,n} \geq \frac{\xi n}{\log n}, |X_n| \leq \frac{\eta n}{\log n}\right) + \sup_{\omega \in \mathcal{A}_n} P_{\lambda,\omega}\left(B_{0,n} \leq \frac{\xi n}{\log n}\right)$$

and combine (6.29), (6.31) and (6.33). ☐

Proof of Lemma 6.5: Set $\gamma_\ell = (1.5)^\ell$. Since $\mathrm{HIT}^j_{\lambda,\omega}(D_{k,\text{biased}}) = 1 - \mathrm{HIT}^j_{\lambda,\omega}(D_{k,\text{fair}})$, it suffices to show that,

$$\limsup_{\ell \to \infty} \frac{1}{\gamma_\ell} \log \sup_{|j|} GW\left(\mathrm{HIT}^j_{\lambda,\omega}\left(D_{|j|+\ell,\text{fair}}\right) \geq 1 - \delta\right) \leq p_{d_{\min}} - (1 - \delta) := -2\xi < 0 \,. \tag{6.34}$$

Indeed, using (6.34) and union bounds, for all $n \geq n_0$,

$$\begin{aligned} GW\left(\omega \notin \mathcal{A}_n\right) &\leq GW(|D_n| \geq (m+1)^n) + (m+1)^n (\log n) \exp\left(-\xi(1.5)^{(c-1)\log n}\right) \\ &\leq \left(\frac{m}{m+1}\right)^n + (m+1)^n (\log n) \exp\left(-\xi n^{(c-1)\log(1.5)}\right) \end{aligned}$$

and since $(c-1)\log(1.5) > 1$, it follows immediately that (6.29) is satisfied. Turning to prove (6.34), suppose $Z = \sum_x h_x I_x$ where $\{I_x\}$ are i.i.d. Bernoulli(p) random variables and the non-random $h_x \in [0, \theta]$ are such that $\sum_x h_x = 1$. Then, for any $\gamma > 0$,

$$\log E(e^{\gamma Z}) = \sum_x \log(1 + p(e^{\gamma h_x} - 1)) \leq \sum_x p(e^{\gamma h_x} - 1) \leq \sum_x p h_x \theta^{-1}(e^{\gamma\theta} - 1) = p\theta^{-1}(e^{\gamma\theta} - 1) \,, \tag{6.35}$$

where we used the inequality $\log(1+z) \le z$ and the monotonicity of $f(y) := (e^y - 1)/y$ on $[0,\infty)$.

Note that the non-negative $h_x := \mathrm{HIT}^j_{\lambda,\omega}(x)$, $x \in D_{|j|+\ell}$ are measurable with respect to $\mathcal{F}_{|j|+\ell}$, such that $\sum_{x\in D_{|j|+\ell}} h_x = 1$ and by (6.14) we know that $h_x \le 2^{-\ell} := \theta_\ell$. Moreover, $I_x := \mathbf{1}_{\{k_x = d_{\min}\}}$, $x \in D_{|j|+\ell}$ are i.i.d. Bernoulli($p_{d_{\min}}$) random variables that are independent of $\mathcal{F}_{|j|+\ell}$. Hence, applying (6.35) for $\gamma_\ell = (1.5)^\ell$ and

$$Z_\ell := \mathrm{HIT}^j_{\lambda,\omega}\left(D_{|j|+\ell,\mathrm{fair}}\right) = \sum_{x\in D_{|j|+\ell}} \mathrm{HIT}^j_{\lambda,\omega}(x)\mathbf{1}_{\{k_x=d_{\min}\}}$$

with respect to the conditional law $GW(\cdot|\mathcal{F}_{|j|+\ell})$, we see that

$$E_{GW}\left(e^{\gamma_\ell \mathrm{HIT}^j_{\lambda,\omega}(D_{|j|+\ell,\mathrm{fair}})}\right) = E_{GW}\left(E_{GW}\left(e^{\gamma_\ell \mathrm{HIT}^j_{\lambda,\omega}(D_{|j|+\ell,\mathrm{fair}})} \;\Big|\; \mathcal{F}_{|j|+\ell}\right)\right) \le e^{p_{d_{\min}}\gamma_\ell f(\gamma_\ell\theta_\ell)}\,.$$

Since $f(\gamma_\ell\theta_\ell) \to 1$, we now get (6.34) by applying Chebychev's inequality. □

Proof of Lemma 6.6: We first show that there exist finite constants θ and ℓ_0, such that if $\ell \ge \ell_0$ then

$$P^j_{\lambda,\omega}\left(T_{|j|+\ell} \ge \theta\ell^2\right) \le \delta^2/8\,, \tag{6.36}$$

for all $j \in \omega$ and all $\omega \in \mathcal{T}$ for which $\min(k_v : v \in \omega) \ge d_{\min}$. Indeed, for such ω, the probability in (6.36) is maximal if $\omega = \omega_{\min}$, the $d_{\min}$-ary rooted tree. By coupling with a reflected simple random walk on $\mathbb{Z}_+$, we thus see that

$$P^j_{\lambda,\omega}\left(T_{|j|+\ell} \ge \theta\ell^2\right) \le P_{\mathrm{SRW^R}(\frac12)}\left(\tau_{|j|+\ell} \ge \theta\ell^2 \;\Big|\; S_0 = |j|\right) \le P_{\mathrm{SRW}(\frac12)}\left(\tau_\ell \ge \theta\ell^2\right)$$

where $\tau_\ell := \inf\{i : S_i = \ell\}$. By Donsker's invariance principle, with $\{W_t\}$ a standard Brownian motion,

$$\lim_{\ell\to\infty} P_{\mathrm{SRW}(\frac12)}\left(\tau_\ell \ge \theta\ell^2\right) = P\left(\sup_{t\le\theta} W_t \le 1\right)\,.$$

Thus, (6.36) follows by taking $\theta = \theta(\delta) < \infty$ such that $P(\sup_{t\le\theta} W_t \le 1) < \delta^2/10$.

Now set n_0 such that $c\log n_0 > \ell_0$ and $d > 2\theta c^2$. For any $n \ge n_0$ let $\ell_n := \lfloor c\log n\rfloor > \ell_0$ and $\Delta_n := \lfloor d(\log n)^2\rfloor$. Fixing such n, note that by the Markov property of $\{X_t\}$, for all $j \in \omega$,

$$\begin{aligned} P_{\lambda,\omega}\left(B_{s,s+\Delta_n} \ge (\delta/2)\log n \;\Big|\; \mathcal{H}_s, X_s = j\right) &= P^j_{\lambda,\omega}\left(B_{0,\Delta_n} \ge (\delta/2)\log n\right) \\ &\ge P^j_{\lambda,\omega}\left(B^T_{\ell_n} \ge (\delta/2)\log n\right) - P^j_{\lambda,\omega}(T_{|j|+\ell_n} \ge \Delta_n), \end{aligned} \tag{6.37}$$

where

$$B^T_{\ell_n} = |\{i : k_{X_{T_{|j|+i}}} > d_{\min},\ \ell_n - \log n + 1 \le i \le \ell_n\}|\,.$$

Fixing $s \le n$ (hence $|j| \le s \le n$) and $\omega \in \mathcal{A}_n$, it follows from (6.28) that

$$E^j_{\lambda,\omega}(B^T_{\ell_n}) = \sum_{i=\ell_n-\log n+1}^{\ell_n} \mathrm{HIT}^j_{\lambda,\omega}\left(D_{|j|+i,\mathrm{biased}}\right) \ge \delta\log n,$$

and as $B^T_{\ell_n} \in [0,\log n]$, using the inequality

$$P\left(Y \ge \frac{E(Y)}{2}\right) \ge \frac14\frac{E(Y)^2}{E(Y^2)}$$

we see that

$$P^j_{\lambda,\omega}\left(B^T_{\ell_n} \geq (\delta/2)\log n\right) \geq \delta^2/4\,.$$

Recall that $\Delta_n \geq \theta\ell_n^2$, hence together with (6.36) and (6.37), this completes the proof of (6.30). $\square$

Proof of Lemma 6.3: It is clearly enough to prove the lemma under the extra assumption $E(Y_1) = 0$, which we make in the sequel. (i) Fix $\varepsilon > 0$ and $\theta = (c-\varepsilon)t^{\gamma-1}$. Then,

$$\begin{aligned} P\left(\sum_{i=1}^n Y_i \geq nt\right) &\leq nP(Y_1 \geq nt) + P\left(\sum_{i=1}^n Y_i \geq nt; Y_i < nt, i = 1,\dots,n\right) \\ &\leq n\exp(-c(nt)^\gamma) + e^{-\theta n^\gamma t}\Lambda(\theta)^n\,, \end{aligned} \tag{6.38}$$

where

$$\Lambda(\theta) = E\left(e^{\theta Y_1 n^{\gamma-1}}\mathbf{1}_{\{Y_1<nt\}}\right)\,.$$

We now prove a bound on $\Lambda(\theta)$. Using $E(Y_1) = 0$ and the bound $e^u \leq 1+u+u^2$ valid for all $u<1$, we have that for some $C>0$ independent of n,

$$\begin{aligned} E\left(e^{\theta Y_1 n^{\gamma-1}}\mathbf{1}_{\{Y_1<n^{1-\gamma}/\theta\}}\right) &\leq 1+\theta n^{\gamma-1}E(Y_1\mathbf{1}_{\{Y_1<n^{1-\gamma}/\theta\}}) + \theta^2 n^{2\gamma-2}E(Y_1^2) \\ &\leq 1+\theta^2 n^{2\gamma-2}E(Y_1^2) \leq \exp(C\theta^2 n^{2\gamma-2}/2)\,. \end{aligned}$$

On the other hand, for all n large enough,

$$\begin{aligned} E\left(e^{\theta Y_1 n^{\gamma-1}}\mathbf{1}_{\{nt\geq Y_1\geq n^{1-\gamma}/\theta\}}\right) &\leq \int_{n^{1-\gamma}/\theta}^{nt}\exp(\theta u n^{\gamma-1} - cu^\gamma)du \\ &\leq \int_{n^{1-\gamma}/\theta}^{nt} e^{-\varepsilon u^\gamma}du \leq e^{-\varepsilon n^{\gamma(1-\gamma)}/2\theta^\gamma}\,. \end{aligned}$$

Combining the above, we get $\Lambda(\theta) \leq e^{Cn^{2\gamma-2}\theta^2}$, and hence, substituting in (6.38), we obtain, for all n large enough,

$$P\left(\sum_{i=1}^n Y_i \geq nt\right) \leq n\exp(-c(nt)^\gamma) + e^{-(c-2\varepsilon)(nt)^\gamma}\,.$$

$\varepsilon > 0$ being arbitrary, this completes the proof of part (i).

(ii) Fix $L(n) = c/(\log n)^2$, and write

$$P\left(\sum_{i=1}^n Y_i \geq nt\right) \leq nP(Y_1 \geq nt) + e^{-\theta nL(n)t}(\bar\Lambda(\theta))^n\,,$$

now with $\theta = (1-\varepsilon)$ and

$$\begin{aligned} \bar\Lambda(\theta) &= E(e^{\theta L(n)Y_1}\mathbf{1}_{\{Y_1\leq nt\}}) \\ &\leq E(e^{\theta L(n)Y_1}\mathbf{1}_{\{Y_1<1/\theta L(n)\}}) + E(e^{\theta L(n)Y_1}\mathbf{1}_{\{1/(L(n))^2\geq Y_1\geq 1/\theta L(n)\}}) + E(e^{\theta L(n)Y_1}\mathbf{1}_{\{nt\geq Y_1\geq 1/(L(n))^2\}}) \\ &:= \bar\Lambda_1(\theta)+\bar\Lambda_2(\theta)+\bar\Lambda_3(\theta)\,. \end{aligned}$$

Exactly as in part (i), there exists a constant C such that $\bar\Lambda_1(\theta) \leq e^{C\theta^2(L(n))^2}$, while for all n large enough,

$$\bar\Lambda_2(\theta) \leq \int_{1/\theta L(n)}^{1/(L(n))^2} e^{[\theta L(n)-L(u)]u}du \leq \int_{1/\theta L(n)}^{1/(L(n))^2} e^{-c\varepsilon(\log\log n)^{-2}u/20}du \leq e^{-(\log n)^{3/2}}\,,$$

and

$$\bar{\Lambda}_3(\theta) \le \frac{2}{\varepsilon L(n)} e^{-\varepsilon/(2L(n))} .$$

Combining the above yields the claim. $\square$

7 Remarks and open problems

1. Of course, we expect Theorem 1.2 to be true even when $d_{\min} = 1$ and $\lambda < 1$, as soon as $m > 1$. All that is missing here is an uncertainty estimate, similar to that of Proposition 5.1. We note that we can show that (1.5) continues to hold true in this situation.

2. We have stated our main results in terms of the position of the random walk. However, the key to the proofs is the analysis of the hitting times $T_n = \inf\{t : X_t \in D_n\}$. In particular, it follows immediately from our proof that

   Corollary 7.1 Large Deviation Principle - hitting times *Assume $m < \infty$ and either $d_{\min} \ge 2$ or $\lambda \ge 1$. The random variables T_n/n satisfy, under both P_λ and $P_{\lambda,\omega}$, the large deviation principle on $[1,\infty)$ with speed n and the same continuous, convex rate function $J_\lambda(\cdot)$, where $J_\lambda(x) = xI_\lambda(1/x)$.*

   Another immediate corollary concerns the *Lyapounov exponents* associated with the hitting times T_n.

   Corollary 7.2 Lyapounov Exponents *Under the assumptions of Corollary 7.1 there exists a deterministic, finite constant $\gamma_c \ge 0$ such that for any $\gamma < \gamma_c$,*

   $$\mu_\lambda(\gamma) := \lim_{n\to\infty} \frac{1}{n} \log E_\lambda \left(e^{\gamma T_n}\right) = \lim_{n\to\infty} \frac{1}{n} \log E_{\lambda,\omega} \left(e^{\gamma T_n}\right) < \infty\,, \quad GW - a.s.\,,$$

   whereas for $\gamma > \gamma_c$ both limits in the definition above are infinite, GW-a.s.

   The quantity $\mu_\lambda(\gamma)$ is called the Lyapounov exponent associated with $\{X_n\}$. For background on Lyapounov exponents in the context of RWRE, see [17], [21]. The interest in Corollary 7.2 is that it demonstrates the equality of the quenched and annealed Lyapounov exponents.

3. Recall that one may construct an extension of the measure GW on rooted trees to a measure AGW on infinite trees (see [11] for the details). One construction of AGW starts with a Galton-Watson tree and the "leftmost" vertex v in D_n, renaming it as 0 while renaming D_m, $m \ge n$ as $\tilde{D}_{m-n}$, and then taking weak limits, resulting with a measure on infinite trees with a special ray $0 \leftrightarrow -\infty$ marked. One can check that Theorem 1.1, Corollary 1.1, and Theorem 1.2 remain valid when using the measure AGW instead of GW.

4. Still considering the measure AGW on infinite trees, quenched and annealed behavior are different when we consider negative speed, i.e. move along the negative ray. Let $T_{-n} := \inf\{j : X_j = -n\}$ be the first hitting time of the vertex $-n$ on the ray $0 \leftrightarrow -\infty$. Define

   $$\gamma_\lambda := \sup\left\{t : \; E_\lambda\left(e^{tT_{-1}} \mathbf{1}_{\{T_{-1}<\infty\}}\right) < \infty\right\}$$

   and introduce

   $$\gamma_\lambda(\omega) := \sup\left\{t : \; E_{\lambda,\omega}\left(e^{tT_{-1}} \mathbf{1}_{\{T_{-1}<\infty\}}\right) < \infty\right\}$$

One can then prove that $\gamma_\lambda(\omega) = \gamma_\lambda$, AGW-a.s., and if $\mathrm{Var}\,(Z) > 0$ then for $t < \gamma_\lambda$,

$$\lim_{n\to\infty} \frac{1}{n} \log E_{\lambda,\omega}\left(e^{tT_{-n}} \mathbf{1}_{\{T_{-n}<\infty\}}\right) < \lim_{n\to\infty} \frac{1}{n} \log E_\lambda\left(e^{tT_{-n}} \mathbf{1}_{\{T_{-n}<\infty\}}\right) < \infty\,, \quad AGW - \text{a.s.}$$

A discussion of this result in the context of one dimensional random walks in random environment with holding times will appear elsewhere.

5. For $P_{\lambda,\omega}(\cdot)$ we find in part (i) of Theorem 1.3 that $C_1 = \frac{3}{2}(\pi|\log p_1|)^{2/3}(1 - a/v_1)^{1/3}$. We conjecture that this is indeed the GW-a.s. limit of $-n^{-1/3} \log P_{\lambda,\omega}(n^{-1}|X_n| \in (b,a))$. Similarly, in part (ii) we find that $C_1 = \frac{1}{2}(\pi \log d_{\min})^2 (1 - a/v_\lambda)$ and conjecture that this is then the GW-a.s. limit of $-(\log n)^2 n^{-1} \log P_{\lambda,\omega}(n^{-1}|X_n| \in (b,a))$. Proving such statements seems related to the coarse graining analysis in [14].

6. The subexponential regime when $d_{\min} \neq \lambda$ is not well understood. We call asymptotics as in part (i) of Theorem 1.3 (with possibly $1/3$ replaced by another constant $\alpha \in (0,1)$) *stretched exponential*. Using the techniques in this paper one can show that when $1 < d_{\min} < \lambda < m < \infty$, the quenched and annealed slowdown probabilities have stretched exponential upper and lower bounds, however we cannot compute the exponent α, nor show it is the same for the upper and lower bounds or that it is the same in the quenched and annealed situations. When $1 = d_{\min} < \lambda < m < \infty$, the quenched and annealed slowdown asymptotics differ (quenched is stretched exponential; annealed is polynomial) but this is an artifact related to the special structure near the root: under AGW, both possess stretched exponential upper and lower bounds.

References

[1] F. Comets, N. Gantert and O. Zeitouni, "Quenched, Annealed and Functional Large Deviations for One-Dimensional Random Walk in Random Environment", *Probab. Theory Relat. Fields* **118** (2000) pp. 65–114.

[2] A. Dembo, Y. Peres and O. Zeitouni, "Tail estimates for one–dimensional random walk in random environment", *Commun. Math. Phys.* **181** (1996) pp. 667–683.

[3] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, second edition, Springer, New York (1998).

[4] N. Gantert and O. Zeitouni, "Quenched sub-exponential tail estimates for one-dimensional random walk in random environment", *Commun. Math. Phys.* **194** (1998), pp. 177–190.

[5] N. Gantert and O. Zeitouni, "Large deviations for one-dimensional random walk in a random environment - a survey", Proceedings of the conference on random walks, Budapest, 1998. P. Revesz and B. Toth, editors. *Bolyai society mathematical studies* **9** (1999), pp. 127–165.

[6] A. Greven and F. den Hollander, "Large deviations for a random walk in random environment", *Ann. Probab.* **22** (1994) pp. 1381–1428.

[7] G. Grimmett and H. Kesten, "Random electrical networks on complete graphs II: Proofs" unpublished manuscript (1984); available at http://www.arxiv.org/abs/math.PR/0107068.

[8] H. Kesten, "A renewal theorem for random walk in a random environment", *Symposia Pure Math* **31** (1977), pp. 67–77.

[9] R. Lyons, "Random walks, capacity and percolation on trees", *Ann. Probab.* **20** (1992), pp. 2043–2088.

[10] R. Lyons, R. Pemantle and Y. Peres, "Ergodic theory on Galton–Watson trees: speed of random walk and dimension of harmonic measure", *Ergodic Theory Dyn. Systems* **15** (1995) pp. 593–619.

[11] R. Lyons, R. Pemantle and Y. Peres, "Biased random walk on Galton–Watson trees", *Probab. Theory Relat. Fields* **106** (1996) pp. 249–264.

[12] D. Piau, "Théorème central limite fonctionnel pour une marche au hasard en environment aléatoire" *Ann. Probab.* **26** (1998) pp. 1016–1040.

[13] A. Pisztora and T. Povel, "Large deviation principle for random walk in a quenched random environment in the low speed regime", *Ann. Probab.*, **27** (1999) pp. 1389–1413.

[14] A. Pisztora, T. Povel and O. Zeitouni, "Precise large deviation estimates for one-dimensional random walk in random environment", *Probab. Theory Relat. Fields* **113** (1999) pp. 135–170.

[15] F. Spitzer, *Principles of random walk*, Springer, Berlin (1976).

[16] A.-S. Sznitman, "Slowdown and neutral pockets for a random walk in random environment", *Probab. Theory Relat. Fields* **115** (1999) pp. 287–323.

[17] A.-S. Sznitman, "Slowdown estimates and central limit theorem for random walks in random environment", *J. Eur. Math. Soc.* **2** (2000) pp. 93–143.

[18] A.-S. Sznitman, *Brownian motion, obstacles and random media*, Springer, Berlin (1998).

[19] A.-S. Sznitman and M. P. W. Zerner, "A law of large numbers for random walks in random environment", *Ann. Probab.* **27** (1999) pp. 1851–1869.

[20] B. Virág, "Fast graphs for the random walker", preprint (2000).

[21] M. P. W. Zerner, "Lyapounov exponents and quenched large deviations for multi-dimensional random walk in random environment", *Ann. Probab.* **26** (1998), pp. 1446–1476.

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