RANDOM WALKS IN RANDOM ENVIRONMENTS:

ergodic properties, homogenization and counter examples

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MWPC, OCTOBER 2006

Model $\mathbb{Z}^d$ Random Walk in Random Environment

Environment: S is $2d$-simplex , $\Omega = S^{\mathbb{Z}^d}$,

$\omega = \{\omega(z,e)\}_{z\in\mathbb{Z}^d}$, $|e|_1 = 1$.

Walk is conditional on ω (quenched), Markov:

$P_\omega(X_{n+1} = x + e | X_n = x) = \omega(x,e) > \delta$

(uniformly elliptic)

Question LLN? CLT?

Simplest model: P, law on Ω, ergodic (i.i.d.)

Walk is **ballistic** if $\dfrac{X_n}{n} \xrightarrow[n\to\infty]{} v, \quad \mathbb{P} - a.s.$

$(\mathbb{P} = P \times P_\omega$ "annealed")

LDP: not today; CGZ '00 $d = 1$. Zerner '98; Varadhan '02; Rassoul-Agha '03; Kosigina Rezakhanlou Varadhan '05

$d=1$, P i.i.d.: non-classical behavior. $\rho_x=\omega(x,-1)/\omega(x,1)$.

'75-'85: Solomon, Kesten, Kozlov, Spitzer, Sinai

- $X_n\to\infty$ iff $E\log\rho_0<0$, recurrent iff $E\log\rho_0=0$.
- If transient: $X_n/n\to v$, $v=0$ iff $s\le 1$, have annealed stable(s) limit laws $s=\sup\{r: E\rho_0^r<1\}$, $v=(1-E\rho_0)/(1+E\rho_0)$.
- Recurrent $\Rightarrow X_n/(\log n)^2$ converges to non-degenerate limit (annealed).
- With d_x the local drift at x, can have $E_P(d_x)<0$ but $X_n\to\infty$ a.s.; However, will then have $v=0$.
- Behavior controlled by **traps**.
- Approach uses reversibility and hitting times

$\tau_i=\min\{n: X_n=i\}\quad(\{\tau_{i+1}-\tau_i\}$ stationary, ergodic...).

A Markov chain with transition $P(x, y)$ is reversible if detailed balance holds: there exists a non-negative measure μ such that

$$\mu(x)P(x, y) = \mu(y)P(y, x)\,.$$

For such chains, can associate electrical network: conductance on edge (x, y) is $\mu(x)P(x, y)$. Walk is transient iff effective resistance from vertex x to infinity is finite.

Fact: any MC on a tree (and hence on $\mathbb{Z}$) is reversible.

For RWRE: set $\rho_x = \omega(x, x-1)/\omega(x, x+1)$. Then, associated network has resistance:

$$\ldots \bullet_{-1} --\rho_0^{-1}-- \bullet_0 --1-- \bullet_1 --\rho_1-- \bullet_2 --\rho_1\rho_2-- \ldots$$

So resistance is determined by $V(x) = \sum_{i=1}^{x} \log \rho_i$: walk behaves as particle in random potential $V(x)$. In particular, $E \log \rho_0 = 0$ is equivalent to recurrent walk.

Note $V(x)$ itself scales like Brownian motion, so scaling limit is related to diffusion in a Brownian potential!

'94-'04:

• Non-standard tail estimates.

• Large deviations quenched and annealed, relations between them.

• Favorite points and sample path behaviour.

• Extensions: Non i.i.d. environments, non nearest-neighbor.

Alili, Bremont, Bolthausen, Comets, Dembo, DenHollander, Gantert, Goldsheid, Greven, Hu, Mayer-Wolf, Peres, Peterson, Popov, Roitershtein, Shi, Yor, Z.....

$d > 1$: Unlike reversible case, arbitrary bad region may occur. But "reasonable" traps are inherently weaker than for $d = 1$.

Heuristics:

$d = 1$: trap of volume $\log n$ inside $[-\epsilon n, \epsilon n] \Rightarrow$ exit time may be of order n.

$d > 1$: trap of volume $\log n$ inside $[-\epsilon n, \epsilon n]^d$ has side-length $(\log n)^{1/d} \Rightarrow$ exit time of order $o(n)$.

$\Rightarrow$ No obvious strategy for walk to remain trapped so much that subdiffusive behaviour occurs. Maybe does not occur?

Embarrassing open problem: 0-1 law.
For $\ell \in S^{d-1}$, set $A_\ell^\pm := \{X_n \cdot \ell \to \pm\infty\}$.
Prove that $\mathbb{P}(A_\ell^+) \in \{0,1\}$.
Known - P i.i.d.:

- $\mathbb{P}(A_\ell^+ \cup A_\ell^-) \in \{0,1\}$ (Kalikow '81).
- 0-1 true for $d = 2$ (Zerner-Merkl '02).
- On A_ℓ, deterministic speed (Sznitman-Zerner '99, Zerner '02)
- 0-1 law $\Rightarrow$ LLN (possibly 0 speed) (Zerner '02)
- At most two possible limiting values of X_n/n (Varadhan '03 by large deviations; Berger '06).
- If two limiting values and $d \geq 5$, one must be zero (Berger '06).

Some of the above have analogues for certain ergodic Gibbsian environments (F. Rassoul-Agha, '04)
(different, LD-based proof).

But: $0-1$ law not true for certain ergodic environments: Zerner-Merkl ('02), $d=2$, *not unif. elliptic*.

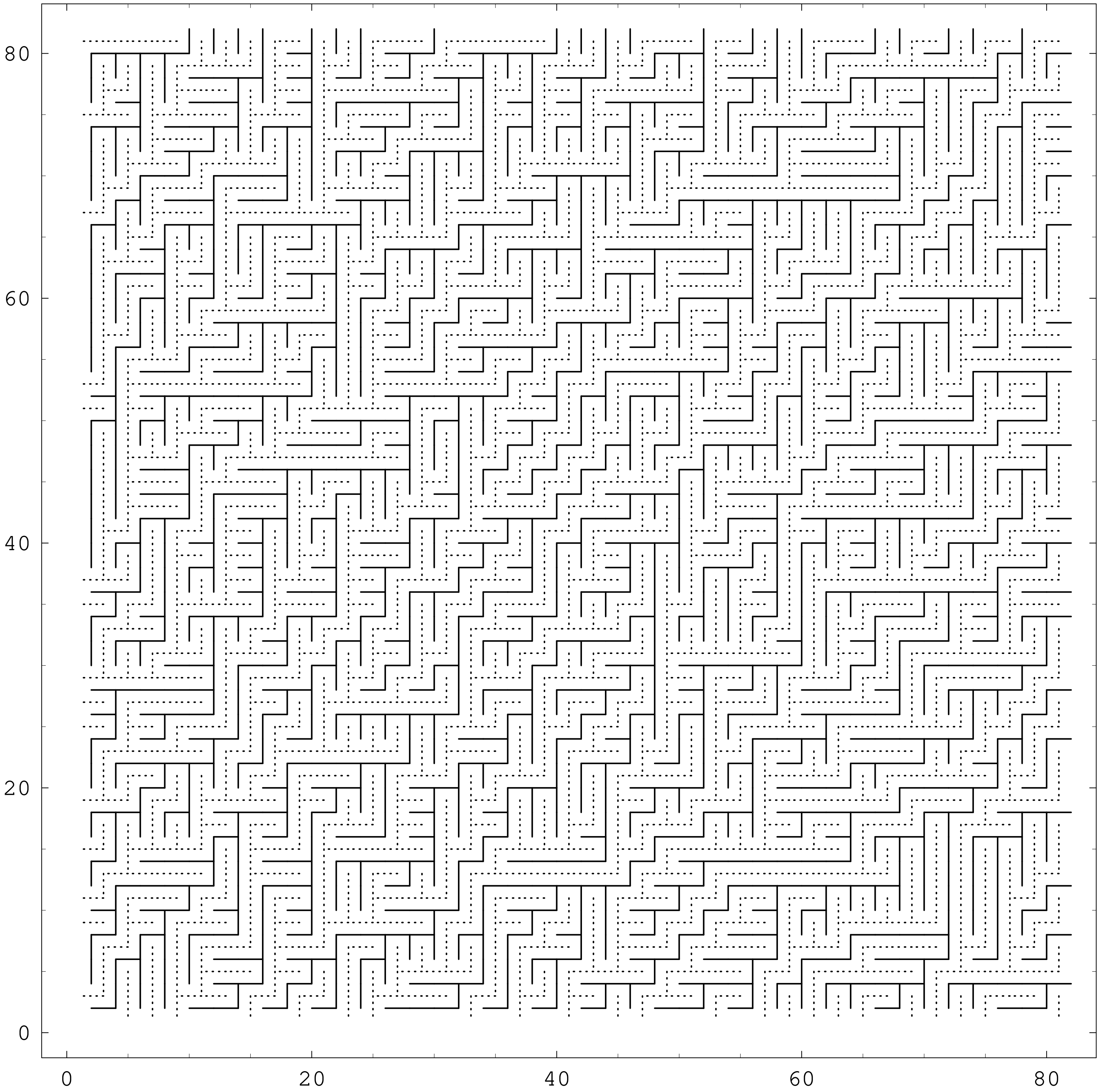
$d \geq 3$, Bramson-Z.-Zerner ('05): example of uniformly elliptic mixing environment

(polynomial decay of correlations, escape in different directions, positive speed).

Zerner-Merkl counter example based on constructing two disjoint spanning trees in $\mathbb{Z}^2$:

On each site of $2\mathbb{Z}^2$, flip a fair coin, and connect the site to its northern or eastern neighbor according to result.

Defines a tree "going in direction (1,1)" and its dual "going in direction -(1,1)". Stationarize (and ergodize).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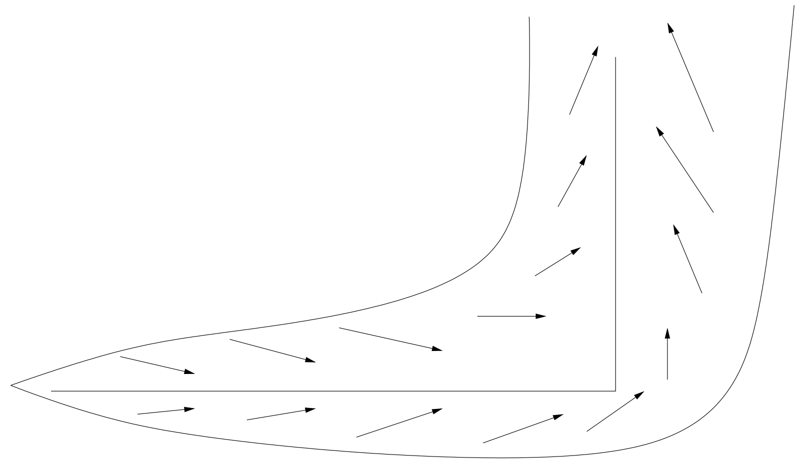
counter example (ct'd) ℓ_z - longest line of descendants of $z \in \mathbb{Z}^2$.
Take

$$P_\omega^z(\text{go toward root}) = 1 - \frac{1}{\ell_z^2}\,.$$

When going toward roots, $\ell_{X_{n+1}} \geq \ell_{X_n} + 1$.
Borel-Cantelli: if on solid tree, then $P(\text{fall off tree}) < 1$.
Hence, $\mathbb{P}(A^+_{(1,1)}) > 0$, and by symmetry, $\mathbb{P}(A^-_{1,1}) > 0$.

Construction drawback: neither uniformly elliptic nor mixing.
Solution: Add "insulation" ($d \geq 3$) (BRAMSON-Z.-Zerner)



Arrows show local drift direction.

Note increasing width of insulation!

To be able to put different insulated trees in $\mathbb{Z}^d$, need that their insulations do not overlap. A simple way to try to do this is to resolve conflicts by eliminating the one with smaller insulation. Can this be done?

A simple way to check: go along a branch of one tree and ask that second tree does not interfere. Roughly, want, with x_i a path in $\mathbb{Z}^d$ corresponding to the first tree,

$$\infty > \sum_{i=1}^{\infty} P(\ell_2(x_i) > \ell_1(x_i))\,.$$

A sufficient condition for that is

$$\sum_{n=1}^{\infty} P(\ell(0) > n) < \infty\,.$$

This does **not** hold for the ZM tree, because tail of $\ell(0)$ in the ZM construction is like that of distance to intersection of 2 independent RW, that is

$$P(\ell(0) > x) \leq C x^{-1/2}\,.$$

Lemma There exists a constant $C = C(d,p) > 0$ such that any stationary forest with $P(0 \text{ in infinite component}) \geq p > 0$ must satisfy $\liminf_{n\to\infty} n^{d-1} P[\ell(0) \geq n] \geq C$.

Proof By stationarity,

$$P(\ell(0) \geq n) = \frac{\sum_{|x|_1 = n} P(\ell(x) \geq n)}{|\partial B_1(0,n)|}.$$

But if x is ancestor of 0 then $\ell(x) \geq |x|_1$. Hence, $P(\ell(x) \geq n) \geq P(x \text{ ancestor of } 0)$. Thus,

$$\sum_{|x|_1 = n} P(\ell(x) \geq n) \geq \sum_{|x|_1 = n} P(x \text{ ancestor of } 0)$$

$$\geq P(0 \text{ is in infinite component}) > 0$$

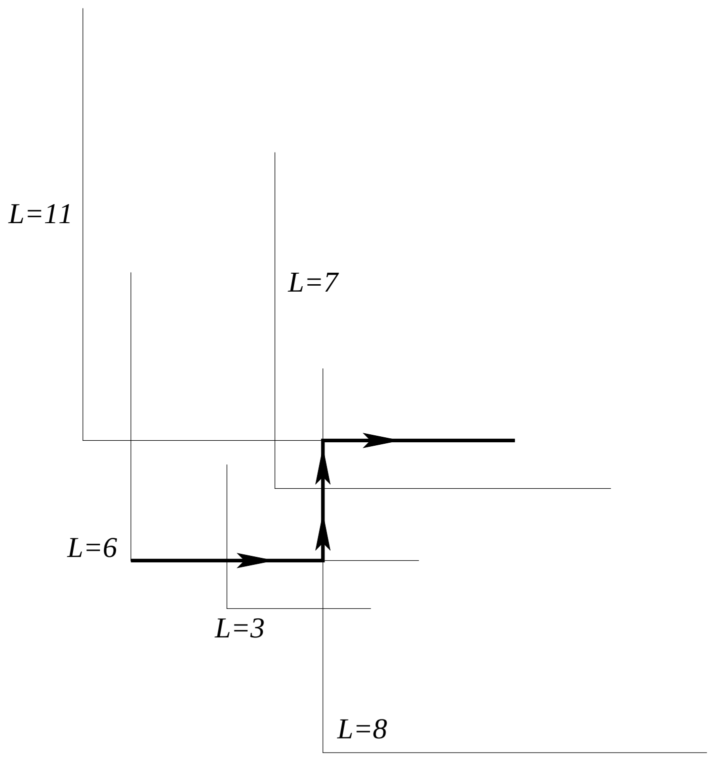
$\square$

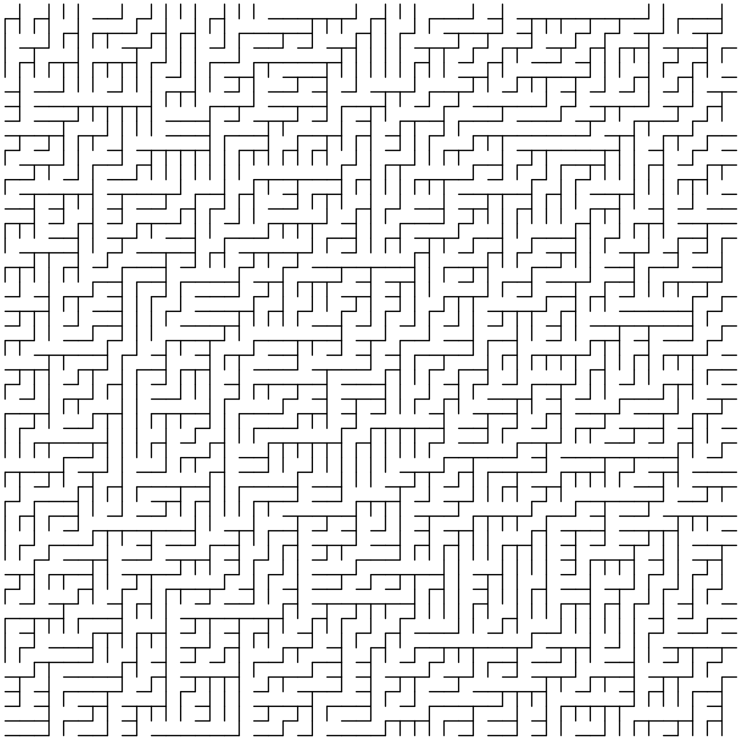
So, we restrict attention to $d \geq 3$, for which $\sum n^{1-d} < \infty$. Can one construct a forest with good mixing properties and tail condition

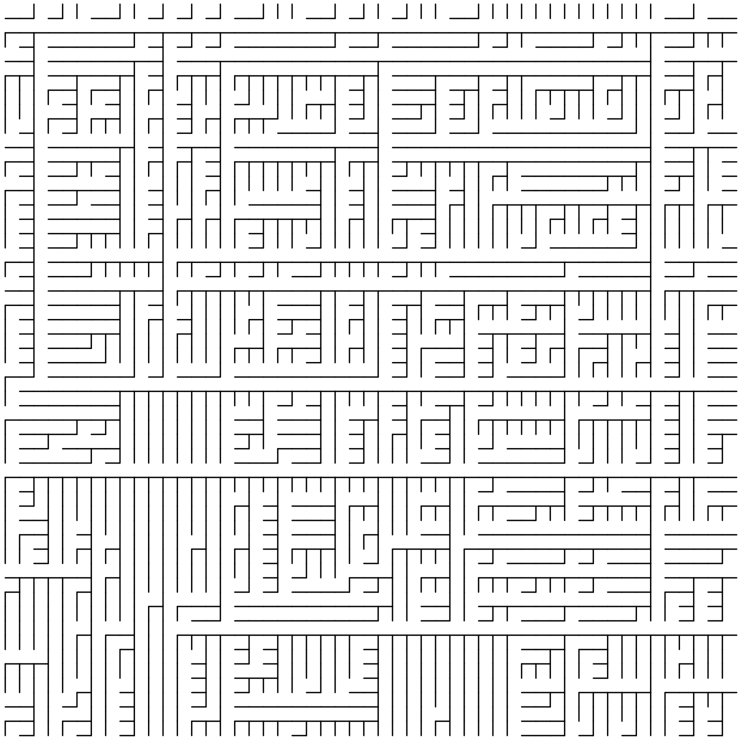
$$(1) \qquad \limsup_{n\to\infty} n^{d-1} P[\ell(0) \geq n] < \infty \,?$$

Theorem (Bramson, Z., Zerner 05) There exists a polynomially mixing law on spanning forests of $\mathbb{Z}^d$ such that (1) holds.

Construction Take L_x i.i.d., support in $(1, \infty)$, atomless, $P(L_0 > t) \sim ct^{-d}$.





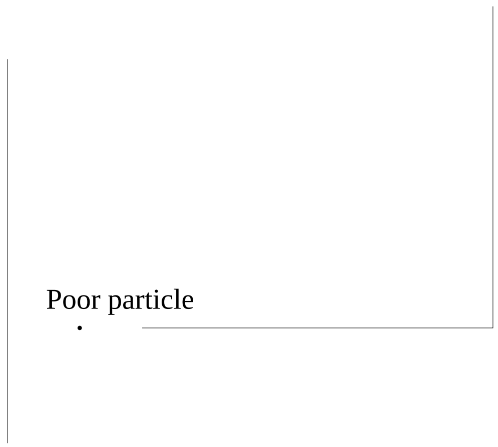


Left figure: Zerner-Merkl tree.

Right figure: part of the BZZ forest. Note the long straight branches.

Back to RWRE: can now introduce a simple “pruning” and “insulation” procedure so as to have two disjoint BZZ forests, pointing in opposite directions, with polynomial size insulation not overlapping. Construction of local drift is not immediate but not too hard.

Problem What should the poor particle do in the following situation:



A similar problem occurs when two rays merge. Resolve by an optimal control point of view.

Back to RWRE on $\mathbb{Z}^d$, $d \geq 2$: LLN, CLT

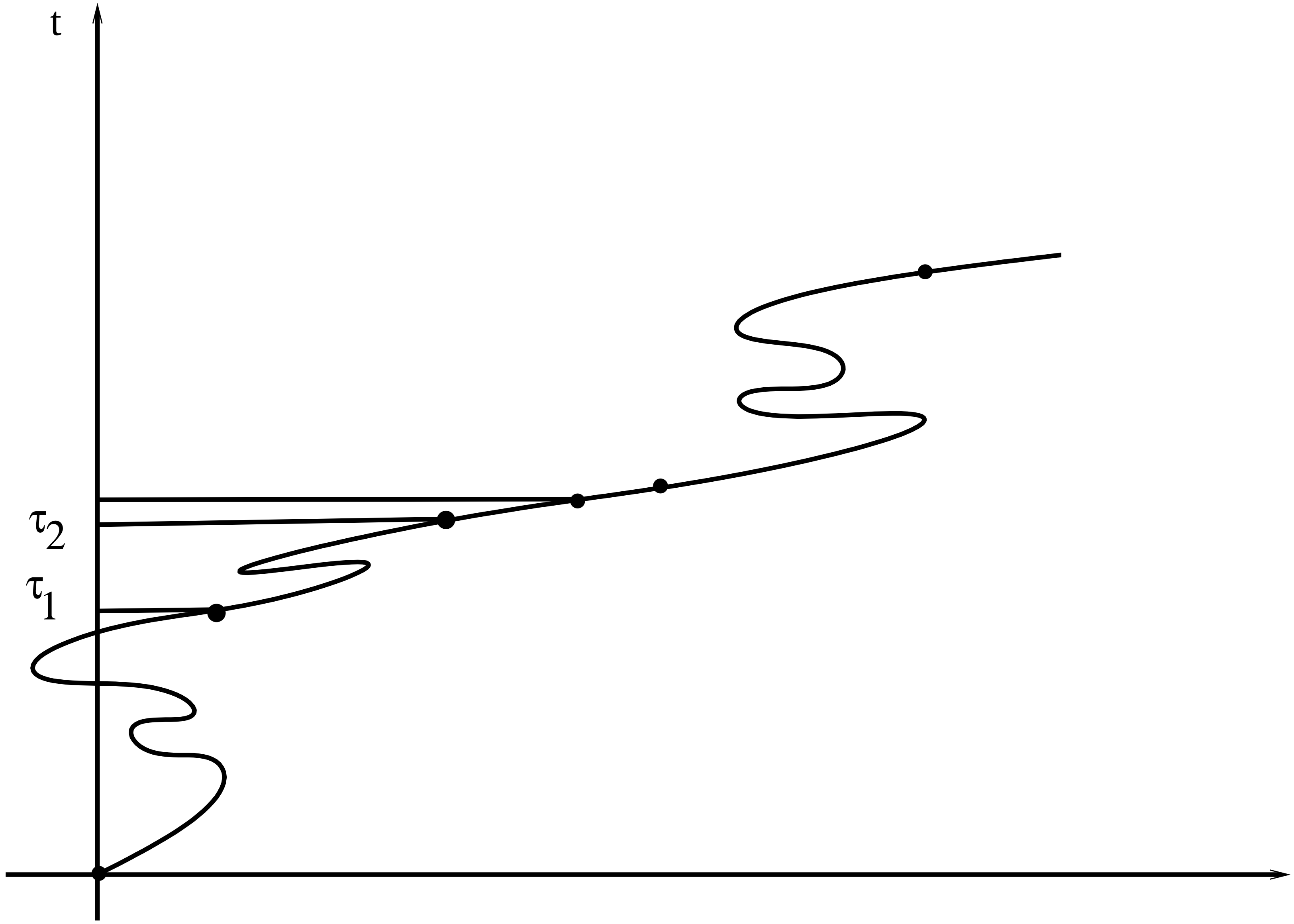
Cannot compute $E_P(\tau_1)$ (not even for $\bar{\tau}_1 = \inf\{n : X_n \cdot e_1 = 1\}$)

Alternative: introduce regeneration times.

For RW on random trees: Kesten-Grimmett ($\sim$ '84), Lyons-Peres-Pemantle ('95), Piau ('98), Dembo-Gantert-Peres-Z. ('01).

For RWRE in $\mathbb{Z}^d$: introduced and exploited by Sznitman-Zerner ('99).

Plot $X_n \cdot e_1$:



Fact: If P is i.i.d.,

$$\Big(\eta_i := (\tau_{i+1} - \tau_i), \Big\{X_{\tau_i+n} - X_{\tau_i}\Big\}_{n=1,\cdots,\eta_i},$$

$$\{\omega(z,\cdot)\}_{X_{\tau_i}\cdot e_1 \le z\cdot e_1 < X_{\tau_{i+1}}\cdot e_1}\Big)_{i\ge 1}$$

are i.i.d. under $\mathbb{P} = P \times P_\omega$ (annealed)

(and, with $\tau_0 = 0$, η_0 has, under conditioning that walk does not backtrack, the same law as η_1).

$\Rightarrow$ (Sznitman-Zerner '99): If $E(\tau_2 - \tau_1) < \infty$ have ballistic LLN.

Results (P i.i.d., Ballistic, $d \geq 2$). Sznitman '99-'03:

There is a condition on the environment (Sznitman's T') such that:

- T' can be checked on finite large boxes.
- $T' \Rightarrow \mathbb{E}(\tau_2 - \tau_1)^k < \infty$ for all k

$\Rightarrow$ LLN, CLT (annealed).

Multi-scale arguments

Open: Ballistic $\rightarrow T'$? Existence of sub-diffusive behavior?

Mixing environments: some LLN's, CLT's: Rassoul-Agha, Comets-Z. Uses modification of regeneration times, which are not i.i.d. anymore.

What about non-ballistic?

- Special case: P i.i.d., $d = d_1 + d_2$,

$$\omega(z, \pm e_i) = \overline{\omega}_{\pm}(i), i = 1, \cdots, d_1$$

(i.e. non random in first d_1 coordinates).

Example: $$\overline{\omega}_{\pm}(i) = \frac{1}{2d}, \quad i = 1, \cdots, d_1$$

Theorem (Bolthausen, Sznitman, Z. '02)

For $d_1 \geq 5$: LLN

$$\left.\begin{array}{l} d_1 \geq 7 \ (P \text{ Symmetric }) \\ \text{or} \\ d_1 \geq 13 \end{array}\right\} \text{CLT}$$

(CLT both quenched and annealed)

Consequence:

There is a P, i.i.d., even small perturbation of SRW! with

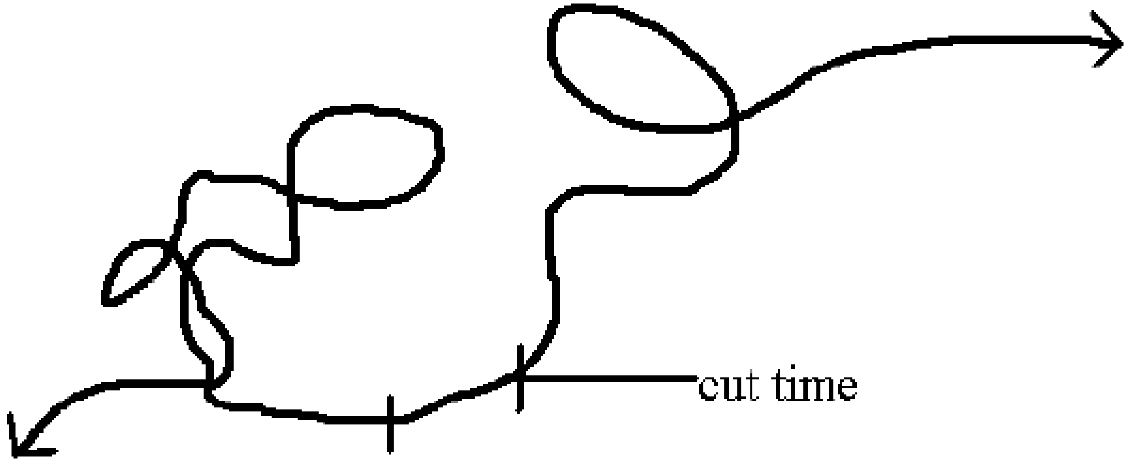
$$E_P(\omega(0,e)\cdot e_d) < 0$$

but

$$\frac{X_n}{n}\cdot e_d \longrightarrow v > 0, \qquad \mathbb{P}-\text{a.s.}$$

(impossible for $d=1$!)

Proof of Theorem based on *cut times* for d_1 random walk, whose density is positive for $d > 4$ by a theorem of Erdös and Taylor. These replace the regeneration times of the transient ballistic setup.



Summary $\mathbb{Z}^d$ Random Walk in Random Environment

Environment: S is $2d$-simplex , $\Omega = S^{\mathbb{Z}^d}$,

$\omega = \{\omega(z,e)\}_{z\in\mathbb{Z}^d}$, $|e|_1 = 1$.

Walk is conditional on ω (quenched), Markov:

$P_\omega(X_{n+1} = x + e | X_n = x) = \omega(x,e) > \delta$

We saw with $d_x = \sum e\omega(x, x+e)$:

• Can have $Ed_x \cdot \ell < 0$ but $X_n \cdot \ell / n \to v_\ell > 0$ (In fact, for any $\epsilon > 0$ can find $\omega(x,\cdot)$ such that $|\omega(x,\pm e_i) - 1/2d| < \epsilon$ and above holds).

• Role of regeneration/cut times: moment estimates yield LLN and (annealed) CLT. Quenched CLT requires separate argument (Bolthausen-Sznitman '02, BBZ '03, Berger-Z. in progress; Seppalainen-Rassoul Agha '05).

Homogenization techniques

• Balanced ($\omega(0,e) = \omega(0,-e)$):

$$\Longrightarrow \quad \begin{array}{l} X_n \text{ is martingale;} \\ \text{transient } d \geq 3\text{, recurrent } d = 2 \\ + \text{ quenched CLT (Lawler '82)} \end{array}$$

Proof by homogenization techniques: $\omega_n := \theta^{X_n}\omega$ is a Markov chain. X_n is a martingale, and its quadratic variation is an additive functional of the MC $\{\omega_n\}$. Hence, if there exists an invariant measure which is absolutely continuous w.r.t. P, standard ergodic theory will give the CLT.

Existence of such an invariant measure (which must be unique due to ellipticity) is the key to Lawler's proof, and uses discrete analogues of Alexandrov-Bakelman-Pucci estimates.

Two major (related) complications in applying this technique to general RWRE model:

- Existence of local drift.
- No a-priori knowledge on invariant measures.

If know invariant measure, often can address the first point. Illustrate for random walks on random (Galton Watson) trees.

Model Environment $\mathcal{T}$: random tree, Galton Watson, offspring distribution $\{p_k\}$, $p_0 = 0$, $p_1 < 1$, $m = \sum kp_k$ mean offspring, root o. d_v is number of offspring of v.

Weights on $\mathcal{T}$: edge at distance n has weight λ^{-n}.

Start random walk on $\mathcal{T}$, with transition probabilities proportional to weights: if u is an offspring of v then

$$P_{\mathcal{T}}(X_1 = u | X_0 = v) = \frac{1}{\lambda + d_v}.$$

Let $|w|$ denote distance of w from root.

Known (Lyons, Pemantle, and Peres, mid 90's):

• $\lambda > m$ (drift toward root): $\{X_n\}$ positive recurrent, $|X_n|/n \to 0$.

• $\lambda < m$ (drift away from root): $\{X_n\}$ transient, $|X_n|/n \to v > 0$ (ballistic). There is a sequence of regeneration times τ_i, such that $(\tau_{i+1} - \tau_i, |X|_{\tau_{i+1}} - |X|_{\tau_i})$ are i.i.d. (annealed).

•: $\lambda = 1 < m$: an explicit invariant measure for the environment viewed from the point of view of particle is known. Speed $v = \sum p_k(k-1)/(k+1)$.

Piau: CLT, LDP, using regenerations.

• $\lambda = m$: critical case. Walk is null recurrent (Lyons).

CLTs:

Assume $\{p_k\}_{k\geq 1}$ has exponential moments.

Theorem (Peres-Z. '06): $(\lambda = m)$ There is a deterministic $\sigma^2 > 0$ such that, for almost every $\mathcal{T}$,

$$\left\{\frac{|X_{[nt]}|}{\sqrt{\sigma^2 n}}\right\}_{t\geq 0} \to \{|B_t|\}_{t\geq 0}\,.$$

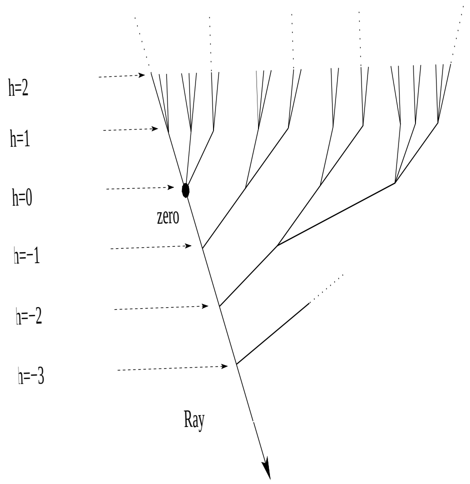
$(\lambda < m)$ (easier):

$$\left\{\frac{|X_{[nt]}| - vt}{\sqrt{\sigma^2 n}}\right\}_{t\geq 0} \to \{B_t\}_{t\geq 0}\,.$$

Quenched CLT!

Basic problem: cannot hope for invariant measure viewed from the point of view of the particle, which is also absolutely continuous with respect to the original GW measure.

Overcome this by modifying initial law and looking at rooted trees with a marked ray.



IGW measure: on Ray, v has a size biased distribution of offspring: $P(d_v = k) = kp_k/m$.

Alternative description: take a rooted GW tree, take a vertex o_n uniformly among the n-th generation, and move vertex there. The sequence of such trees converges weakly to IGW.

To each vertex v, attach the random variable

$$W_v = \lim \frac{|D_n(v)|}{m^n}$$

where $D_n(v)$ is number of descendants of v at distance n. The limit exists by martingale arguments.

Define next martingale: if $X_t = v$,

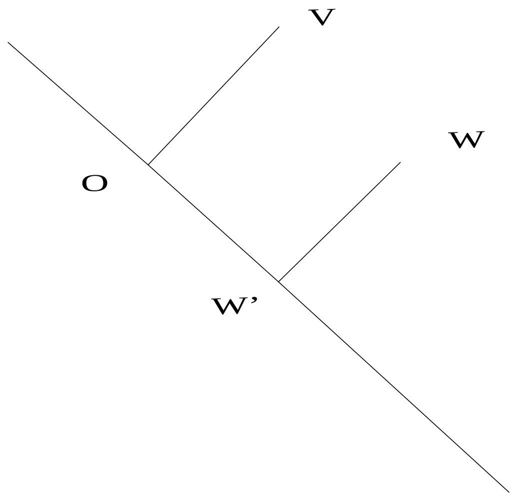
$$M_{t+1} - M_t = \begin{cases} -W_v\,, & X_{t+1} \text{ parent of } v \\ W_y\,, & X_{t+1} = y \text{ descendant of } v \end{cases}$$

Martingale because $W_v = \sum W_{y_i}/\lambda$.

Set $S_v = \sum_{u \in o \leftrightarrow v} W_u$ if v is descendant of o, and

$$S_w = -\sum_{u \in o \leftrightarrow w'} W_u + \sum_{u \in w' \leftrightarrow w} W_u$$

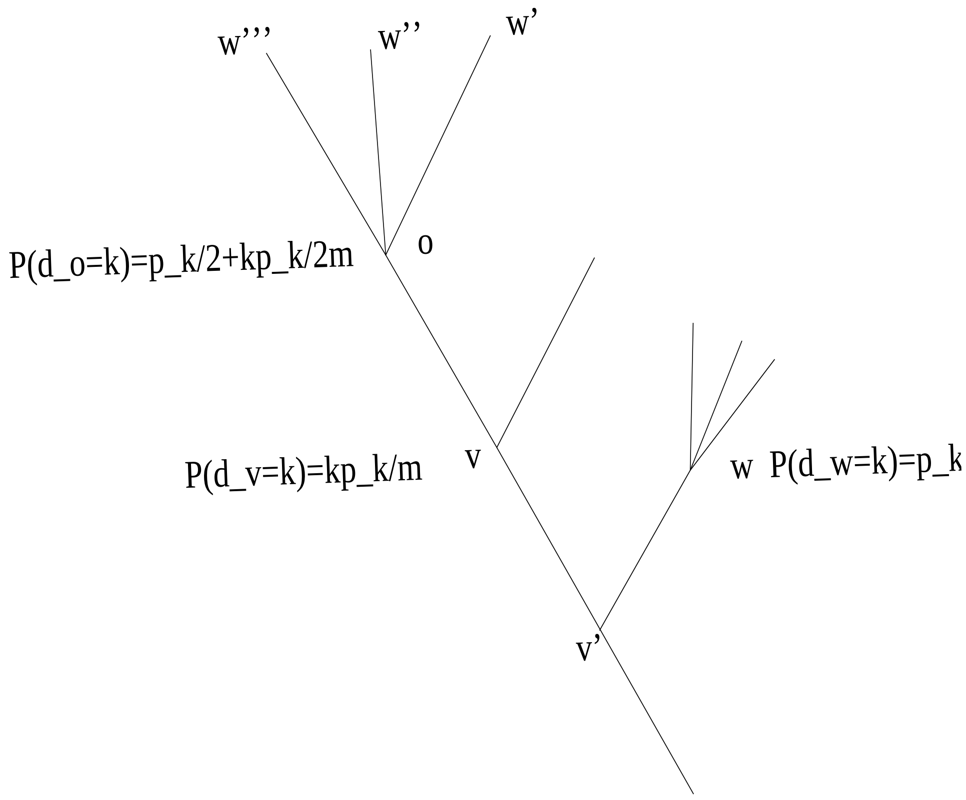
when w' is parent of w on marked Ray.



$M_t = S_{X_t}$, is a martingale, with increasing process of increments

$$\mu_v^2 = \frac{1}{1+d_v} \sum_{i=1}^{d_v} W_{Y_i}^2 + \frac{1}{m(m+d_v} \left(\sum_{i=1}^{d_v} W_{Y_i} \right)^2 .$$

To see a CLT (or invariance principle) for M_t, need to control increasing process, hence need an ergodic argument. But IGW is not an invariant measure for environment viewed from point of view of particle. So, introduce a new one, called IGWR:



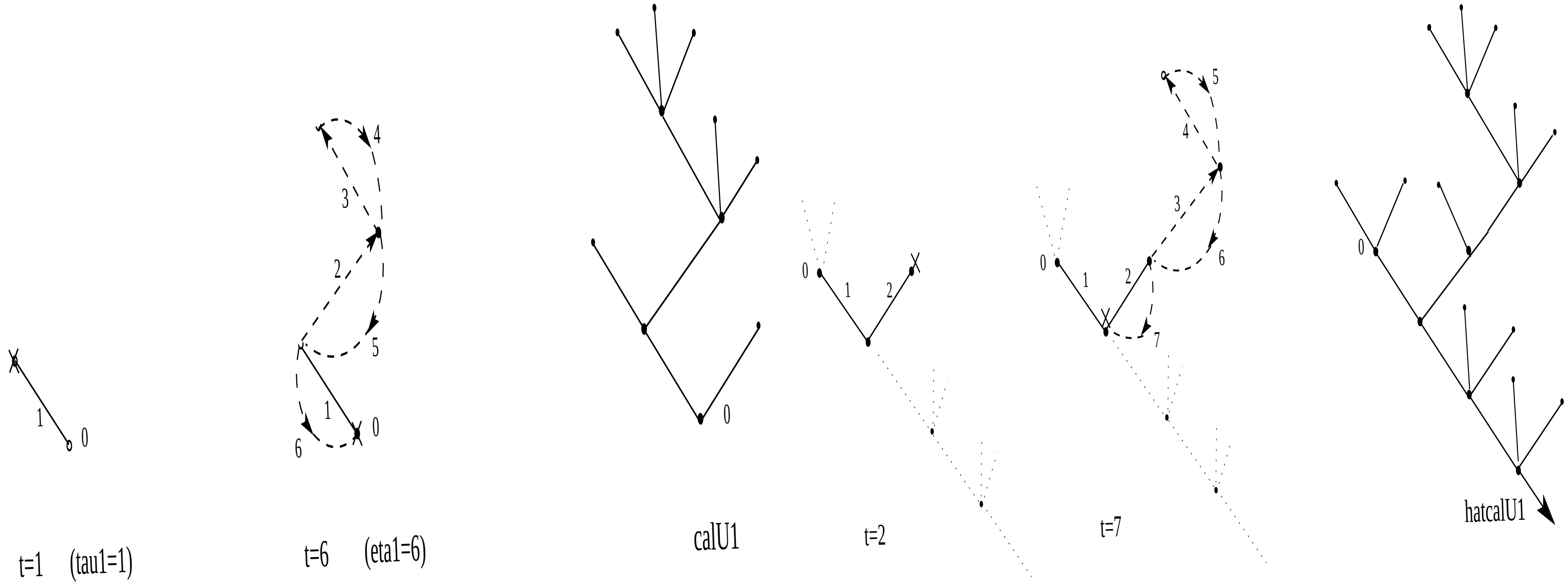
IGWR satisfies $dIGWR/dIGW = 1/2 + m/2d_0$

IGWR turns out to be reversing, and hence invariant, for the environment viewed from point of view of the particle. This measure is of course absolutely continuous w.r.t. IGW, and under this measure, $\langle M_t \rangle / t$ converges (by ergodic theorem). Hence, it also converges under IGW.

$\Rightarrow$ invariance principle for $M_t = S_{X_t}$.

For "most" points v at distance r from o, S_v, being a sum of weakly dependent random variables along the ray connecting o to v, is close to $rE_{IGW}W_o$. For r large, the walk seldom visits the bad points at distance r. Conclude from this a CLT for $h(X_t)$.

Going back to GW: "dirty coupling".



Note garbage time (solid lines).

One checks: "Total garbage time by time t" is $o(t)$.

"Total garbage distance by distance t" is $o(\sqrt{t})$.

Basic idea: look at walk that cuts out dashed parts of path. This is almost a RW - except for deterministic jump at the end of each excursion. Then $S_{\tilde{X}_t}$ is super martingale, and thus cannot go too far too quick. Also, not too many garbage segments.

Back to RWRE on $\mathbb{Z}^d$:

- Small disorder, $d \geq 3$: $\omega(0,e) = \frac{1}{2d} + \varepsilon k(0,e)$

P i.i.d. invariant under $\mathbb{Z}^d$-rotations.

$\Longrightarrow$ Quenched CLT (Bricmont-Kupianien '91)

Proof uses (hard) renormalization techniques, hard to penetrate.

Present now two alternatives to the proof of BK, using different approaches, and hopefully more transparent.

First setup: diffusions in random environments.

$$dX_t = b(X_t, \omega)dt + \sigma(X_t, \omega)dW_t\,, X_t \in \mathbb{R}^d$$

$a := \sigma\sigma^T, b$ - finite range dependence, recall BZZ!

stationary, restricted isotropy: for any rotation r preserving the union of coordinates of $\mathbb{R}^d$,

$\left(a(rx,\omega), b(rx,\omega)\right)_{x\in\mathbb{R}^d}$ has the same law as $\left(ra(x,\omega)r^T,\ rb(x,\omega)\right)_{x\in\mathbb{R}^d}$.

$$dX_t = b(X_t, \omega)dt + \sigma(X_t, \omega)dW_t \, , X_t \in \mathbb{R}^d$$

Theorem (A.-S. Sznitman-Z., '04), $d \geq 3$:

There is $\eta_0(d, K, R) > 0$, such that when

$|a(x, \omega) - I| \leq \eta_0$, $|b(x, \omega)| \leq \eta_0$, for all $x \in \mathbb{R}^d$

then for P-a.e. ω,

$\frac{1}{\sqrt{t}} X_{\cdot t}$ converges (quenched) to a Brownian motion with deterministic variance σ^2

In PDE terms: define the operator

$$L_\epsilon = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\epsilon},\omega\right)\partial^2_{x_i x_j} + \sum_{i=1}^{d}\frac{1}{\epsilon}b_i\left(\frac{x}{\epsilon},\omega\right)\partial_{x_i}$$

Set, on $(0,\infty)\times\mathbb{R}^d$

$$\partial_t u_\epsilon = L_\epsilon u_\epsilon + g\,,\, u_\epsilon(0,\cdot) = f\,.$$

(a,b -Lipschitz, dependence on ω as above; in particular, restricted isotropy condition and finite range dependence, and smallness of perturbation condition).

Then

$$u_\epsilon(x,t) = E^{x/\epsilon}\Big[f(\epsilon X_{t/\epsilon^2}) - \int_0^t g(\epsilon X_{u/\epsilon^2})du\Big]$$

$$\partial_t u_\epsilon = L_\epsilon u_\epsilon + g\,, u_\epsilon(0,\cdot) = f\,.$$

Thus, the CLT implies that u_ϵ converges uniformly over compacts to the solution of

$$\partial_t u_0 = \frac{\sigma^2}{2}\Delta u_0 + g,\; u_0(0,t) = f\,.$$

Note: not in divergence form!

With no $1/\epsilon$ before drift, Yurinskii (1980), based on Alexandrov-Bakelman-Pucci L^q estimates, $1 < q < 1 + 1/d$, on invariant measure (also Papanicolau-Varadhan,....).

Basic perturbation expansion and isotropy:
use sequence of scales L_n
P_n - rescaled transition density at time L_n^2
Suppose $P_n = P_n^0 + \Delta_n$, where P_n^0 is Brownian motion transition.

$$P_{n+1}(x,y) = \int \dots \int P_n(x,y_1) P_n(y_1,y_2) \cdots P_n(y_N,y) \prod dy_i$$

where $N = (L_{n+1}/L_n)^2$.
Perturbation expansion: linear term is sum of terms:

$$\sum_{k=0}^{N-1} \int (P_n^0)^k (x,y_1) \Delta_n(y_1,y_2) (P_n^0)^{N-k-1}(y_2,y) \prod dy_i$$

Recall: $\sum_{y_2} \Delta_n(y_1,y_2) = 0$ (because P_n is prob. measure!)

Expand

$$(P_n^0)^{N-k-1}(y_2,y) = (P_n^0)^{N-k-1}(y_1,y)+(y_2-y_1)\partial(P_n^0)^{N-k-1}(y_1,y)+\dots$$

Therefore, linear term in perturbation expansion has as leading term

$$\sum_{k=0}^{N-1}\int (P_n^0)^k(x,y_1)\Delta_n(y_1,y_2)$$

$$[(P_n^0)^{N-k-1}(y_1,y) + (y_2-y_1)\partial(P_n^0)^{N-k-1}(y_1,y)+\dots]\prod dy_i$$

$$= \sum_{k=0}^{N-1}\int dy_1(P_n^0)^k(x,y_1)\Big(\int dy_2\Delta_n(y_1,y_2)(y_2-y_1)\Big)\partial(P_n^0)^{N-k-1}(y_1,y)+\dots$$

Now, the leading term is sum of (essentially independent) random variables, **whose mean is zero due to isotropy assumption**. This is good, because otherwise we would have N terms whereas gain from derivative would be only of spatial order which is $N^{-1/2}$!

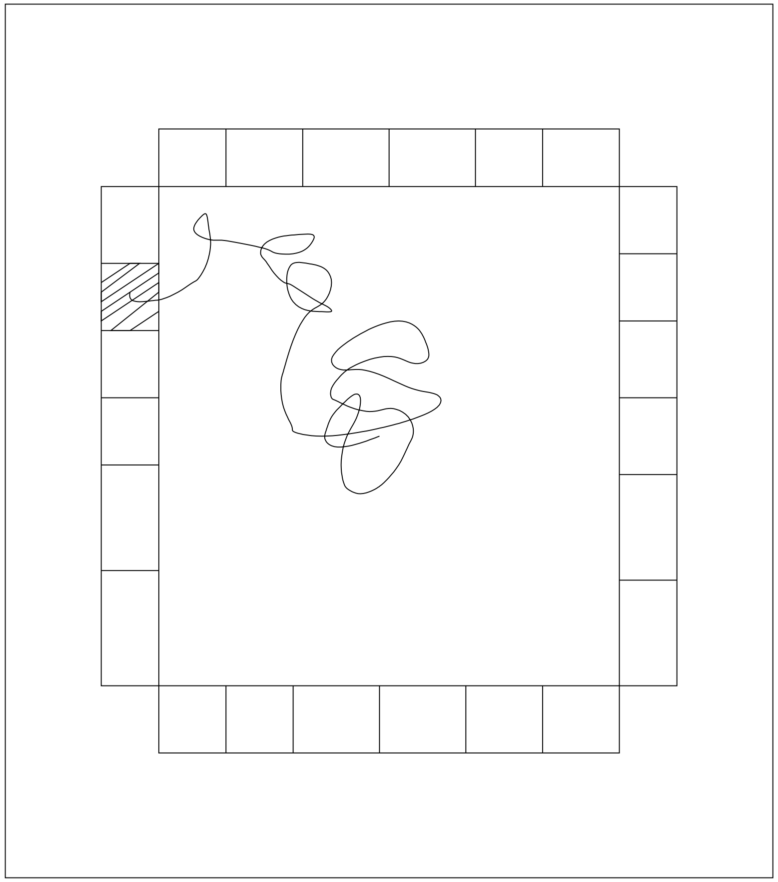
Actual proof is based on inductive scheme:
scales $L_{n+1} = L_n^{1+a}$
$\tilde{R}_n(\omega)$ - truncated transition probability at scale L_n, time roughly L_n^2
$||\cdot||_n$ - rescaled Hölder norm
R_n^0- transition probability of BM, times $\alpha_n L_n^2$.

$$\mathcal{B}_n(\omega) = \{x \in L_n\mathbb{Z}^d : \text{“truncation can be justified”, and } ||\tilde{R}_n - R_n^0||_n \leq L_n^{-\delta}\}\,.$$

Also, introduce “worst case large deviation exponent in L_n boxes”, $J_{n,x}$, $x \in L_n\mathbb{Z}^d$ measuring how unlikely it is to exit an L_n box through a particular part of its boundary.

Large Deviations Exponent:

$$\min P(\text{hit shaded before exterior}) \geq L_n^{-\zeta J}\,.$$



Basic induction: if

$$P[0 \notin \mathcal{B}_n(\omega)] \leq L_n^{-M_0}$$

and have "good control" on the $J_{n,x}$ at different locations, then both controls propagate to level $n+1$.

Control is achieved by smoothing properties of transition kernel, measured in terms on Hölder norms. Control of those is achieved by the use of wavelets!

More on diffusion inductive argument:

If for all $0 \leq n \leq n_0$,

$$P[0 \notin \mathcal{B}_n(\omega)] \leq L_n^{-M_0}\,, \tag{1}$$

and for any finite subset $\mathcal{A} \subset L_n\mathbb{Z}^d$,

$$\begin{aligned} &P\Big[\text{for all } x \in \mathcal{A},\ J_{n,x} \geq u_x\Big] \leq L_n^{-\overline{M}_n \sum_{x\in\mathcal{A}}(u_x+1)}, \text{ with} \\ &\overline{M}_n = M \prod_{0\leq j<n} \Big(1 - \frac{c_2}{\log L_j}\Big)\,, \end{aligned} \tag{2}$$

and if

$$\frac{1}{2} \leq \alpha_n \leq 2,\ |\alpha_{n+1} - \alpha_n| \leq L_n^{-(1+\frac{9}{10})\delta}, \text{ for } 0 \leq n < n_0\,,$$

then the estimates (1), (2) hold with $n_0 + 1$ in place of n_0 and

$$|\alpha_{n_0+1} - \alpha_{n_0}| \leq L_{n_0}^{-(1+\frac{9}{10})\delta}\,.$$

Steps in proof

a) If, at level n, close to BM+not too many strong traps at lower levels, then P(trap at level n+1) is small.
Idea of proof: construct explicit exit strategies for diffusion that avoid the bad traps.

b) Hölder norm controls: set

$$||f||_n = |f|_\infty + \mathsf{sup}\,\frac{L_n^\beta |f(x)-f(y)|}{|x-y|^\beta}$$

By step a, there do not exist strong trap (probability of exceptional event is small compared to target $L_{n+1}^{-M_0}$).

Perform surgery of bad boxes (which are not traps, and by induction hypothesis not too many of them), and replace environment there by BM.

After the surgery, use perturbation expansion: because heat kernel is smoothing for Hölder norm, the nonlinear terms in perturbation expansion are of lower order than the linear term. For latter, use integration by parts (up to 3-rd order, thanks to the restricted isotropy condition!) to show perturbation is small.

Remark: Hölder norm control is by representation via wavelet coefficients, which allows one to deal with supremum over i.i.d. random variables!

c) Final step: compare environment before and after surgery (when no strong traps exist). Use again perturbation expansion, this time knowing one cannot hit too many bad boxes because there are only a few (finitely many) of them.

Bad boxes near origin, which are hit, are regularized by the transition kernel after exits from a neighborhood of them.

Back to RWRE: exit measures

Bolthausen+Z. '06

- Small disorder, $d \geq 3$:

$\omega(0,e) = \frac{1}{2d} + \eta k(0,e)$

P i.i.d. invariant under $\mathbb{Z}^d$-rotations.

Goal: investigate exit measure from ball

$B_R = \{z \in \mathbb{Z}^d : |z|_2 \leq R\}$.

Set

$$\Pi_R(A) = P_\omega(\text{RWRE exits } B_R \text{ through } A)\,.$$

$$\pi_R(A) = P(\text{RW exits } B_R \text{ through } A)\,.$$

Theorem$(d \geq 3)$ For any $\delta > 0$ there is an η_0 such that with perturbations $\eta < \eta_0$ it holds that

$$\limsup_{R\to\infty} \|\Pi_R - \pi_R\|_{T.V.} < \delta$$

and

$$\limsup_{s\to\infty} \limsup_{R\to\infty} \|\Pi_R * \tilde{\pi}_s - \pi_R * \tilde{\pi}_s\|_{T.V.} = 0\,.$$

Further, the RWRE is transient.

Here $\tilde{\pi}_s$ is exit law from a ball of random radius of order s.

Take scales $L_{n+1} = L_n(\log L_n)^3$.

Say a point x is bad at level i, $i = 1, 2, 3$, if

$$\|\Pi_{L_n} - \pi_{L_n}\|_{T.V.} < \delta$$

and

$$(\log L_n)^{-9+\frac{9(i-1)}{4}} \leq \|\Pi_{L_n} * \tilde{\pi}_{L_n} - \pi_{L_n} * \tilde{\pi}_{L_n}\|_{T.V.} < (\log L_n)^{-9+\frac{9i}{4}}\,.$$

It is bad at level 4 if either

$$\|\Pi_{L_n} - \pi_{L_n}\|_{T.V.} < \delta$$

or

$$\|\Pi_{L_n} * \tilde{\pi}_{L_n} - \pi_{L_n} * \tilde{\pi}_{L_n}\|_{T.V.} > (\log L_n)^{-2.25}\,.$$

Basic induction:

$$P(0 \text{ is } i \text{ bad})$$

$$\leq \frac{1}{4} \exp\left[-(1-(4-i)/13)(\log L_n)^2\right] .$$

Theorem is that induction hypothesis on $n \leq n_0$ propagates to $n_0 + 1$.

Propagation of smoothed estimates

Coarse grain walk, keeping coarse graining all the way to the boundary (slightly finer scale near boundary, but still mesoscopic).

- There is only one bad box.
- If no bad box: Perturbation expansion.

$$\Pi_{L_{n+1}} * \tilde{\pi}_{L_{n+1}} - \pi_{L_{n+1}} * \tilde{\pi}_{L_{n+1}}$$

$$= \sum \cdots \sum g_{L_n}(0,y)\Delta^{k_1}(y,y')\pi_{L_n}(y',y_1)\cdots g_{L_n}\Delta^{k_j}\pi_{L_n}\tilde{\pi}_{L_n}$$

where g_L is green function for (coarse grained) SRW, and Δ is difference between (coarse grained) RWRE and SRW.

Linear term ($j=1$) is most delicate.

Smoothed estimates - linear term

$$\sum g(0,y)\Delta^k(y,y')\pi(y',z)\tilde{\pi}_L$$

Separate according to location of y, using everywhere the smoothing $\Delta\pi$ on which induction hypothesis gives information.

• $k=1$, y in bulk: since $\sum g(0,y) = (\log L_n)^6$, must use averaging (and smoothness of Green function) and not brute force summation; Here isotropy hypothesis enters.

• $k=1$, y near boundary: use hitting estimates.

Slightly more delicate at boundary, but cannot go very far there and smooth anyway!

Smoothed estimates - linear term (ctd')

- $k > 1$ handled by $k = 1$+non-smoothed estimates.
- handle the single bad box if present by deriving rough Green function estimates for environment w/out bad boxes.

Propagation of non-smoothed estimates

Use perturbation expansion, refining scale all the way to the boundary.

• bulk errors smoothed by π following it and induction hypothesis on smoothed errors.

• There are many "bad" boundary boxes, however have good hitting estimates of them.

Remaining challenges:

- Add time control to second (simpler) approach.
- Get rid of isotropy condition (possible non zero centering)