An overview of tools for the

SPECTRAL THEORY OF RANDOM MATRICES

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Wigner:

$X = \{X_{ij}\}_{1 \le i,j \le N}$, X symmetric

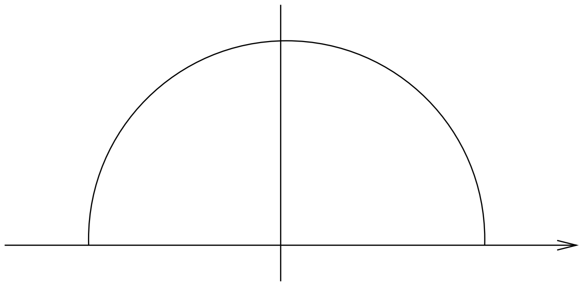
X_{ij} independent (except for symmetry constraint)

$$EX_{ij} = 0, EX_{ij}^2 = \begin{cases} \frac{1}{2N}, & i \neq j \\ \frac{1}{N}\,, & i = j \quad \left(\frac{1}{2N} \text{ also O.K.}\right) \end{cases}$$

$$EX_{ij}^k < \infty, \qquad \forall k$$

$$L_N \stackrel{\triangle}{=} \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$$

Then $L_N \underset{N\to\infty}{\Longrightarrow} \mu_\sigma$ (weak convergence); μ_σ is semicircle law, density $\frac{1}{\pi}\sqrt{2 - x^2}$



(I concentrate here on spectral measure, *not* on spacing).

METHODS OF PROOF

(1) Moments

$E \operatorname{tr} X = 0,\ E(\operatorname{tr} X)^2 = E(\sum X_{ii})^2 = N \cdot \frac{1}{N} = 1$

$$\Rightarrow \frac{1}{N} \operatorname{tr} X = \langle L_N, x \rangle \underset{N \to \infty}{\longrightarrow} 0$$

$$\frac{1}{N} E \operatorname{tr} X^2 = \frac{E \sum X_{ij}^2}{N^2} \underset{N \to \infty}{\longrightarrow} 1/2,$$

$$\operatorname{Var}\left(\frac{1}{N} \operatorname{tr} X^2\right) \underset{N \to \infty}{\longrightarrow} 0$$

More generally:

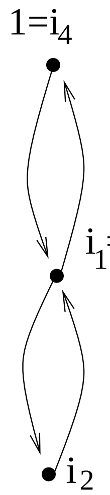
$$\frac{1}{N} E \operatorname{tr} X^k = \sum_{i_1, \cdots i_{k-1}} E\, X_{1i_1} X_{i_1 i_2} \cdots X_{i_{k-1} 1}$$

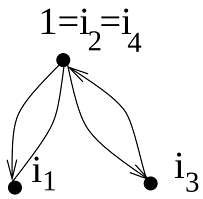
Expectation is zero unless a (X_{ij} or X_{ji}) appears at least twice. For fixed k and $N \to \infty$, dominant terms are:

(X_{ij} or X_{ji}) appears *exactly* twice, and then contribution is $\left(\frac{1}{2N}\right)^{k/2}$.

Number of contributing terms is asymptotic to $N^{k/2}$, with coefficient C_k coming from enumeration:

Example: $k = 4$ ($k/2 = 2$)





Can enumerate recursively to compute C_k and check it yields moments of semicircle! (Wigner ('50s), Komlos-Furedi ('80s), Recently Soshnikov ('98–'99)).

Remarks

1. By computing dominant contribution, easy to see $\operatorname{Var}\left(\frac{1}{N}\operatorname{tr} X^k\right) \underset{N \to \infty}{\longrightarrow} 0$, for all k.

2. Easy extension: Y rectangular iid, $X = YY^*$ (Wishart); complex entries.

3. Easy conclusion (without actual enumeration): it is enough to handle Gaussian case.

(2) Gaussian case

Law on matrix elements: $C_N \exp(-N \sum X_{ij}^2/2)$

Change of coordinates: $X = \Theta\Lambda\Theta^*$

Jacobian: $J = F(\theta) \prod_{i<j} (\lambda_i - \lambda_j)$

But $\sum X_{ij}^2 = \sum \lambda_i^2$, hence integrate over θ to get

$$p(\lambda_1, \cdots \lambda_N) = Z_N \prod_{i<j} |\lambda_i - \lambda_j| \exp(-N \sum \lambda_i^2/2)$$

Formally:

$$(1)\quad p(\lambda_1, \cdots \lambda_N) = Z_N \exp\left(\frac{1}{2}\sum_{i\neq j} \log|\lambda_i - \lambda_j| - N \sum \lambda_i^2/2\right)$$

$$\simeq Z_N \exp(-N^2(\langle L_N, x^2/2\rangle - \frac{1}{2}\text{“}\langle L_N \log|x-y|, L_N\rangle\text{”}))$$

Hence, L_N must converge to minimizer of

$$I(\mu) = -\frac{1}{2}\iint \log|x-y|\mu(dx)\mu(dy) + \frac{1}{2}\int x^2\mu(dx) + C_2$$

(which is semicircle).

$$(1) \quad p(\lambda_1, \cdots \lambda_N) = Z_N \exp\left(\frac{1}{2}\sum_{i\neq j}\log|\lambda_i - \lambda_j| - N\sum \lambda_i^2/2\right)$$

is starting point for analysis in Mehta's book, and yields starting point also for analysis of spacings

(From Dyson via Mehta to Tracy & Widom!)

But (1) also gives LDP (speed N^2)

(Ben-Arous + Guionnet '97)

and handle on fluctuations

(Johannson '98; Guionnet '00)

Extensions:

Other ensembles: Hiai-Petz (see their new book!),

Ben-Arous - Z. (non self-adjoint).

Remark:

Scale N^2 is not a fluke of Gaussian case.

Guionnet - Z. ('00) show that this speed is consequence of general concentration of measure arguments of Talagrand, and applies e.g. when entries have common bdd support.

(3) Matrix inversion lemma & resolvent:

Recall $\operatorname{tr} A^{-1} = \sum_{k=1}^{N} \frac{1}{a_{kk} - \alpha_k^T A_k^{-1} \alpha_k}$

where α_k is k-th row of A, A_k is k-th minor of A.

Now, use $A = (zI_N - X_N)$, and assume $E\frac{1}{N} \operatorname{tr} A^{-1} \to m(z)$ (which it does), $m(z) = \int \frac{1}{z-\lambda}\mu(d\lambda)$ is Stieljes transform of $\mu(\cdot)$. Then, because α_k and A_k are independent,

$$\alpha_k^T A_k^{-1} \alpha_k \simeq \frac{1}{2N} \operatorname{tr} A_k^{-1} \sim m(z)/2$$

$$a_{kk} \sim z$$

and we get $m(z) \sim \frac{1}{z-m(z)/2}$

$$\Rightarrow m^2(z)/2 \;-\; zm(z) + 1 = 0$$
$$\Rightarrow m(z) \;=\; z \pm \sqrt{z^2 - 2}$$

Turns out need to take $-$ sign, and that this $m(z)$, is Stieljes-transform of semi-circle law.

(Pastur + Marchenko, late 60's)

(Silverstein, Bai & Silverstein)

Extensions: Wishart, non Gaussian entries,

$$X = YDY^*.$$

(4) Free probability
(interpreted for matrices)

$\mathcal{A}$ – Von Neuman algebra: algebra of operators on a Hilbert space, with $\text{tr}: \mathcal{A} \to \mathbb{C}$ and unit vector 1 s.t.
$\text{tr}\, A = \langle 1, A1 \rangle$ ("1" is ground state).
$+$ closure properties wrt. weak norm top.

Example: $\mathcal{A} =$ algebra of $N \times N$ real matrices,

$$\langle 1, A1 \rangle = \frac{1}{N} E \,\text{tr}\, A$$

Classical prob: commutative elements of $\mathcal{A}$ are L^∞ functions g such that

$$Af = g(\omega) \cdot f(\omega)$$

$$\text{tr}\, A = \int 1 \cdot g(\omega) 1 \mu(d\omega)$$

(unital) sub-algebras of $\mathcal{A}$ (say $\mathcal{A}_1, \mathcal{A}_2$) are *freely* independent if, whenever $A_j \in \mathcal{A}_1, B_j \in \mathcal{A}_2, \text{tr}\, A_j = \text{tr}\, B_j = 0$, it holds that $\text{tr}\, A_1 B_1 A_2 B_2 \cdots A_k B_k = 0$.

Why is it important? Suppose want to compute *law* of $(A+B)$. It is enough to compute moments

$$\mathrm{tr}(A+B)^k$$

Suppose $\mathrm{tr}\,A=\mathrm{tr}\,B=0$ (otherwise, look at $(A-(\mathrm{tr}\,A)\cdot I)$. But

$$\begin{aligned}\mathrm{tr}(A+B) &= \mathrm{tr}\,A+\mathrm{tr}\,B\\ \mathrm{tr}(A+B)^2 &= \mathrm{tr}\,A^2+\mathrm{tr}\,B^2+2\,\mathrm{tr}\,AB\\ \mathrm{tr}(A+B)^3 &= \mathrm{tr}\,A^3+3\,\mathrm{tr}(BA^2)\\ &\quad +3\,\mathrm{tr}(AB^2)+\mathrm{tr}\,B^3\\ &= \mathrm{tr}\,A^3+\mathrm{tr}\,B^3\end{aligned}$$

$$\begin{aligned}\mathrm{tr}(A+B)^4=\mathrm{tr}\,A^4 &+ 4\,\mathrm{tr}(A^2B^2)+4\,\mathrm{tr}(AB^3)\\ &+ 4\,\mathrm{tr}(BA^3)+2\,\mathrm{tr}(ABAB)\\ &+ \mathrm{tr}\,B^4\\ =\mathrm{tr}\,A^4 &+ \mathrm{tr}\,B^4+4\,\mathrm{tr}(A^2)\,\mathrm{tr}(B^2)\\ &+4\,\mathrm{tr}((A^2-\mathrm{tr}\,A^2I)(B^2-\mathrm{tr}\,B^2I))\end{aligned}$$

(Exercise: erase all terms which dissappear due to freeness!)

Hence, if A, B are free then $\mathrm{tr}(A+B)^k$ can be read off laws of A and B.

Important fact (Voiculescu):
if A_N, B_N are arbitrary matrices with spectral measures $\longrightarrow \mu_A, \mu_B$, and $A'_N = U_N^* A_N U_N$ (U_N unitary, independent) then A'_N and B_N are asymptotically free!

Hence, independent Gaussian matrices are asymptotically free.

A particularly nice way of coding this:
R transform of μ (formal inverse of S-transform):

Define (for z in $\mathbb{C}_+$!).

$$G_\mu(z) = \int \frac{1}{z-\lambda}\mu(d\lambda) = \sum_0^\infty z^{-n-1}\langle x^n, \mu\rangle$$

Invert formally the power series of $G_\mu(z)$: (formula: find $K_\mu(z)$, s.t.

$$K_\mu(G_\mu(z)) = z$$

and write

$$K_\mu(z) = \frac{1}{z} + \sum_1^\infty C_n(\mu) z^{n-1}$$

$C_n(\mu)$ are free cumulants of μ:

$$C_n(A+B) = C_n(A) + C_n(B)\,.$$

(combinatorial interpretation via enumeration of non crossing partitions).

In particular, since a Gaussian matrix X is the sum of two independent rescalled Gaussian matrices, we must have, using

$$K_{X/\sqrt{2}}(z) = \frac{1}{\sqrt{2}} K_X(z/\sqrt{2})$$

that

$$K_\mu\left(\frac{z}{\sqrt{2}}\right)/\sqrt{2} + K_\mu\left(\frac{z}{\sqrt{2}}\right)/\sqrt{2} = K_\mu(z) + \frac{1}{z}$$

which implies $C_1 = 0$, $C_3 = \cdots = 0$, i.e. $K_\mu(z) = \frac{1}{z} + C_2 z$ and inverting this gives

$$G_\mu(z)\left(\frac{1}{G_\mu(z)} + C_2 G_\mu\right) = zG_\mu(z)$$

$$\Rightarrow \qquad C_2 G_\mu^2(z) + 1 = zG_\mu(z)$$

and one recovers the Pastur-Marchenko result. (Voiculescu; Biane).