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Random Walks in Random Environments

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Abstract

Random walks in random environments (RWRE's) have been a source of surprising phenomena and challenging problems since they began to be studied in the 70's. Hitting times and, more recently, certain regeneration structures, have played a major role in our understanding of RWRE's. We review these and provide some hints on current research directions and challenges.

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1. Introduction

Let S denote the 2d-dimensional simplex, set $\Omega = S^{Z^d}$, and let $\omega(z,\cdot) = \{\omega(z,z+e)\}_{e\in Z^d,|e|=1}$ denote the coordinate of $\omega \in \Omega$ corresponding to $z \in Z^d$. Ω is an "environment" for an inhomogeneous nearest neighbor random walk (RWRE) started at x with *quenched* transition probabilities $P_\omega(X_{n+1} = z+e|X_n = z) = \omega(x,x+e)$ $(e \in \mathbb{Z}^d, |e| = 1)$, whose law is denoted P_ω^x. In the RWRE model, the environment is random, of law P, which is always assumed stationary and ergodic. We also assume here that the environment is *elliptic*, that is there exists an $\epsilon > 0$ such that P-a.s., $\omega(x,x+e) \geq \epsilon$ for all $x, e \in \mathbb{Z}^d, |e| = 1$. Finally, we denote by $\mathbb{P}$ the *annealed* law of the RWRE started at 0, that is the law of $\{X_n\}$ under the measure $P \times P_\omega^0$.

When $d = 1$, we write $\omega_x = \omega(x,x+1)$, $\rho_x = \omega_x/(1-\omega_x)$, and $u = E_P \log \rho_0$. The following theorem reveals the surprising phenomena associated with the RWRE:

Theorem 1.1 (Transience, recurrence, limit speed) *(a) With* $\operatorname{sign}(0) = 1$*, it holds that* $\mathbb{P}$*-a.s.,*

$$\limsup_{n\to\infty} \kappa X_n = \operatorname{sign}(\kappa u)\infty\,, \quad \kappa = \pm 1\,.$$

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Further, there is a v such that

$$\lim_{n\to\infty} \frac{X_n}{n} = v\,, \quad \mathbb{P}-\text{a.s.}\,, \tag{1.2}$$

$v>0$ *if* $\sum_{i=1}^{\infty} E_P(\prod_{j=0}^{i} \rho_{-j}) < \infty$, $v<0$ *if* $\sum_{i=1}^{\infty} E_P(\prod_{j=0}^{i} \rho_{-j}^{-1}) < \infty$, *and* $v=0$ *if both these conditions do not hold.*
(b) If P is a product measure then

$$v = \begin{cases} \frac{1-E_P(\rho_0)}{1+E_P(\rho_0)}\,, & E_P(\rho_0)<1 \\ -\frac{1-E_P(\rho_0^{-1})}{1+E_P(\rho_0^{-1})}\,, & E_P(\rho_0^{-1})<1 \\ 0\,, & \text{else}\,. \end{cases} \tag{1.3}$$

Theorem 1.1 is essentially due to [25], see [29] for a proof in the general ergodic setup. The surprising features of the RWRE model alluded to above can be appreciated if one notes, already for a product measure P, that the RWRE can be transient with zero speed v. Further, if P is a product measure and $v_0(\omega)$ denotes the speed of a (biased) simple random walk with probability of jump to the right equal, at any site, to ω_0, then Jensen's inequality reveals that $|v| \le |E_P(v_0(\omega))|$, with examples of strict inequality readily available.

The reason for this behavior is that the RWRE spends a large time in small traps. This is very well understood in the case $d=1$, to which the next section is devoted. We introduce there certain hitting times, show how they yield precise information on the RWRE, and describe the analysis of these hitting times. Understanding the behavior of the RWRE when $d>1$ is a major challenging problem, on which much progress has been done in recent years, but for which many embarrassing open questions remain. We give a glimpse of what is involved in Section 3., where we introduce certain *regeneration* times, and show their usefulness in a variety of situations. Here is a particularly simple setup where law of large numbers (and CLT's, although we do not emphasize that here) are available:

Theorem 1.4 *Assume P is a product measure, $d\ge 6$, and $\omega(x,x+e)=\eta>0$ for $e=\pm e_i, i=1,\ldots,5$. Then there exists a deterministic constant v such that $X_n/n \to v$, $\mathbb{P}$-a.s..*

2. The one-dimensional case

Recursions

Let us begin with a sketch of the proof of Theorem 1.1. The transience and recurrence criterion is proved by noting that conditioned on the environment ω, the Markov chain X_n is reversible. More explicitly, fix an interval $[-m_-, m_+]$ encircling the origin and for z in that interval, define

$$\mathcal{V}_{m_-,m_+,\omega}(z) := P_\omega^z(\{X_n\} \text{ hits } -m_- \text{ before hitting } m_+)\,.$$

Then,

$$\mathcal{V}_{m_-,m_+,\omega}(z)=\frac{\sum_{i=z+1}^{m_+}\prod_{j=z+1}^{i-1}\rho_j}{\sum_{i=z+1}^{m_+}\prod_{j=z+1}^{i-1}\rho_j+\sum_{i=-m_-+1}^{z}\left(\prod_{j=i}^{z}\rho_j^{-1}\right)}, \tag{2.1}$$

from which the conclusion follows. The proof of the LLN is more instructive: define the hitting times $T_n=\min\{t>0: X_t=T_n\}$, and set $\tau_i=T_{i+1}-T_i$. Suppose that $\limsup_{n\to\infty}X_n/n=\infty$. One checks that τ_i is an ergodic sequence, hence $T_n/n\to\mathbb{E}(\tau_0)$ $\mathbb{P}$-a.s., which in turns implies that $X_n/n\to 1/\mathbb{E}(\tau_0)$, $\mathbb{P}$-a.s.. But,

$$\tau_0=\mathbf{1}_{\{X_1=1\}}+\mathbf{1}_{\{X_1=-1\}}(1+\tau'_{-1}+\tau'_0)\,,$$

where τ'_{-1} (τ'_0) denote the first hitting time of 0 (1) for the random walk X_n after it hits -1. Hence, taking P^0_ω expectations, and noting that $\{E_{P^i_\omega}(\tau_i)\}_i$ are, P-a.s., either all finite or all infinite,

$$E_{P^0_\omega}(\tau_0)=\frac{1}{\omega_0}+\rho_0E_{P^{-1}_\omega}(\tau_{-1})\,. \tag{2.2}$$

When P is a product measure, ρ_0 and $E_{P^{-1}_\omega}(\tau_{-1})$ are P-independent, and taking expectations results with $\mathbb{E}(\tau_0)=(1+E_P(\rho_0))/(1-E_P(\rho_0))$ if the right hand side is positive and ∞ otherwise, from which (1.3) follows. The ergodic case is obtained by iterating the relation (2.2).

The hitting times T_n are also the beginning of the study of limit laws for X_n. To appreciate this in the case of product measures P with $E_P(\log\rho_0)<0$ (i.e., when the RWRE is transient to $+\infty$), one first observes that from the above recursions,

$$\mathbb{E}(\tau_0^r)<\infty\Longleftrightarrow E_P(\rho_0^r)<1\,.$$

Defining $s=\max\{r: E_P(\rho_0^r)<1\}$, one then expects that (X_n-vn), suitably rescaled, possesses a limit law, with s-dependent scaling. This is indeed the case: for $s>2$, it is not hard to check that one obtains a central limit theorem with scaling $\sqrt{t}$ (this holds true in fact for ergodic environments under appropriate mixing assumptions and with a suitable definition of the parameter s, see [29]). For $s\in(0,1)\cup(1,2)$, one obtains in the i.i.d. environment case a Stable(s) limit law with scaling $t^{1/s}$ (the cases $s=1$ or $s=2$ can also be handled but involve logarithmic factors in the scaling and the deterministic shift). In particular, for $s<2$ the walk is *sub-diffusive*. We omit the details, referring to [16] for the proof, except to say that the extension to ergodic environments of many of these results has recently been carried out, see [23].

Traps

The unusual behavior of one dimensional RWRE is due to the existence of traps in the medium. This is exhibited most dramatically when one tries to evaluate the probability of slowdown of the RWRE. Assume that P is a product measure, X_n is transient to $+\infty$ with positive speed v (this means that $s>1$ by Theorem 1.1), and that $s<\infty$ (which means that $P(\omega_0<1/2)>0$). One then has:

Theorem 2.3 ([8, 11]) *For any $w \in [0,v)$, $\eta > 0$, and $\delta > 0$ small enough,*

$$\lim_{n\to\infty} \frac{\log \mathbb{P}\left(\frac{X_n}{n} \in (w-\delta, w+\delta)\right)}{\log n} = 1 - s\,, \tag{2.4}$$

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s+\eta}} \log P^0\left(\frac{X_n}{n} \in (w-\delta, w+\delta)\right) = 0\,, \quad P-a.s. \tag{2.5}$$

and

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s-\eta}} \log P^0\left(\frac{X_n}{n} \in (w-\delta, w+\delta)\right) = -\infty\,, \quad P-a.s.. \tag{2.6}$$

(Extensions of Theorem 2.3 to the mixing environment setup are presented in [29]. There are also precise asymptotics available in the case $s=\infty$ and $P(\omega_0 = 1/2) > 0$, see [20, 21]).

One immediately notes the difference in scaling between the annealed and quenched slowdown estimates in Theorem 2.3. These are due to the fact that, under the quenched measure, traps are given, whereas under the annealed measure $\mathbb{P}$ one can create, at some cost in probability, larger traps.

To demonstrate the role of traps in the RWRE model, let us exhibit, for $w=0$, a lower bound that captures the correct behavior in the annealed setup, and that forms the basis for the proof of the more general statement. Indeed, $\{X_n \le \delta\} \subset \{T_{n\delta} \ge n\}$. Fixing $R_k = R_k(\omega) := k^{-1}\sum_{i=1}^k \log \rho_i$, it holds that R_k satisfies a large deviation principle with rate function $J(y) = \sup_\lambda(\lambda y - \log E_P(\rho_0^\lambda))$, and it is not hard to check that $s = \min_{y\ge 0} y^{-1}J(y)$. Fixing a y such that $J(y)/y \le s+\eta$, and $k = \log n/y$, one checks that the probability that there exists in $[0,\delta n]$ a point z with $R_k\circ\theta^z\omega \ge y$ is at least $n^{1-s-\eta}$. But, the probability that the RWRE does not cross such a segment by time n is, due to (2.1), bounded away from 0 uniformly in n. This yields the claimed lower bound in the annealed case. In the quenched case, one has to work with traps of size almost $k = \log n/sy$ for which $kR_k \ge y$, which occur with probability 1 eventually, and use (2.1) to compute the probability of an atypical slowdown inside such a trap. The fluctuations in the length of these typical traps is the reason why the slowdown probability is believed, for P-a.e. ω, to fluctuate with n, in the sense that

$$\liminf_{n\to\infty} \frac{1}{n^{1-1/s}} \log P_\omega^0\left(\frac{X_n}{n} \in (-\delta,\delta)\right) = -\infty\,, \quad P-a.s.,$$

while it is known that

$$\limsup_{n\to\infty} \frac{1}{n^{1-1/s}} \log P_\omega^0\left(\frac{X_n}{n} \in (-\delta,\delta)\right) = 0\,, \quad P-a.s..$$

This has been demonstrated rigorously in some particular cases, see [10].

The role of traps, and the difference they produce between the quenched and annealed regimes, is dramatic in the scale of large deviations. Roughly, the exponential (in n) rate of decay of the probability of atypical events differ between the quenched and annealed regime:

Theorem 2.7 *The random variables X_n/n satisfy, for P-a.e. realization of the environment ω, a large deviations principle (LDP) under P_ω^0 with a deterministic rate function $I_P(\cdot)$. Under the annealed measure $\mathbb{P}$, they satisfy a LDP with rate function*

$$I(w) = \inf_{Q \in \mathcal{M}_1^e} \left(h(Q|P) + I_Q(w) \right), \tag{2.8}$$

where $h(Q|P)$ is the specific entropy of Q with respect to P and $\mathcal{M}_1^e$ denotes the space of stationary ergodic measures on Ω.

Theorem 2.7 means that to create an annealed large deviation, one may first "modify" the environment (at a certain exponential cost) and then apply the quenched LDP in the new environment. We refer to [13] (quenched) and [3, 7] for proofs and generalizations to non i.i.d. environments.

Sinai's recurrent walk and aging

When $E_P(\log \rho_0) = 0$, traps stop being local, and the whole environment becomes a diffused trap. The walk spends most of its time "at the bottom of the trap", and as time evolves it is harder and harder for the RWRE to move. This is the phenomenum of *aging*, captured in the following theorem:

Theorem 2.9 *There exists a random variable B^n, depending on the environment only, such that*

$$\mathbb{P}\left(\left| \frac{X_n}{(\log n)^2} - B^n \right| > \eta \right) \underset{n\to\infty}{\to} 0\,.$$

Further, for $h > 1$,

$$\lim_{\eta\to 0} \lim_{n\to\infty} \mathbb{P}\left(\frac{|X_{n^h} - X_n|}{(\log n)^2} < \eta \right) = \frac{1}{h^2}\left[\frac{5}{3} - \frac{2}{3} e^{-(h-1)} \right]. \tag{2.10}$$

The first part of Theorem 2.9 is due to Sinai [24], with Kesten [15] providing the evaluation of the limiting law of B^n. The second part is implicit in [12], we refer to [5] and [29] for the proof and references.

3. Multi-dimensional RWRE

Homogenization

Two special features simplify the analysis of the RWRE in the one-dimensional case: first, for every realization of the environment, the RWRE is a reversible Markov chain. This gave transience and recurrence criteria. Then, the location of the walk at the hitting times T_n is deterministic, leading to stationarity and mixing properties of the sequence $\{\tau_i\}$ and to a relatively simple analysis of their tail properties. Both these features are lost for $d > 1$.

A (by now standard) approach to homogenization problems is to consider the *environment viewed from the particle.* More precisely, with θ^x denoting the $\mathbb{Z}^d$ shift by x, the process $\omega_n = \theta^{X_n}\omega$ is a Markov chain with state-space Ω. Whenever the invariant measure of this chain is absolutely continuous with respect to P, law of large numbers and CLT's can be deduced, see [17]. For reversible situations, e.g. in the "random conductance model" [19], the invariant measure of the chain $\{\omega_n\}$ is known explicitly. In the non-reversible RWRE model, this approach has had limited consequences: one needs to establish absolute continuity of the invariant measure without knowing it explicitly. This was done in [18] for balanced environments, i.e. whenever $\omega(x, x+e) = \omega(x, x-e)$ P-a.s. for all $e \in \mathbb{Z}^d, |e| = 1$, by developing a-priori estimates on the invariant measure., valid for *every* realization of the environment. Apart from that (and very recent work [22]), this approach has not been very useful in the study of RWRE's.

Regeneration

We focus here on another approach based on analogs of hitting times. Throughout, fix a direction $\ell \in \mathbb{Z}^d$, and consider the process $Z_n = X_n \cdot \ell$. Define the events $A_{\pm\ell} = \{Z_n \to_{n\to\infty} \pm\infty\}$. Then, with P a product measure, one shows that $\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0,1\}$, [14]. We sketch a proof: Call a time t *fresh* if $Z_t > Z_n, \forall n < t$, and for any fresh time t, define the return time $D_t = \min\{n > t : Z_n < Z_t\}$, and say that t is a regeneration time if $D_t = \infty$. Then, $\mathbb{P}(A_\ell) > 0$ implies by the Markov property that $\mathbb{P}(A_\ell \cap \{D_0 = \infty\}) > 0$. Similarly, on A_ℓ, each fresh time has a bounded away from zero probability to be a regeneration time. One deduces that $\mathbb{P}(\exists \text{ a regeneration time}|A_\ell) = 1$. In particular, on $A_{\pm\ell}$, Z_n changes signs only finitely many times. If $\mathbb{P}(A_\ell \cup A_{-\ell}) < 1$ then with positive probability, Z_n visits a finite centered interval infinitely often, and hence it must change signs infinitely many times. But this implies that $\mathbb{P}(A_\ell \cup A_{-\ell}) = 0$.

The proof above can be extended to non-product P-s having good mixing properties using, due to the uniform ellipticity, a coupling with simple nearest neighbor random walk. This is done as follows: Set $W = \{0\} \cup \{\pm e_i\}_{i=1}^d$. Define the measure

$$\overline{\mathbb{P}} = P \otimes Q_\epsilon \otimes \overline{P}^0_{\omega,\varepsilon} \quad \text{on} \quad \left(\Omega \times W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}}\right)$$

in the following way: Q_ϵ is a product measure, such that with $\varepsilon = (\epsilon_1, \epsilon_2, \dots)$ denoting an element of $W^{\mathbb{N}}$, $Q_\epsilon(\epsilon_1 = \pm e_i) = \epsilon/2, i = 1, \cdots, d$, $Q_\epsilon(\epsilon_1 = 0) = 1 - \epsilon d$. For each fixed ω, ε, $\overline{P}^0_{\omega,\varepsilon}$ is the law of the Markov chain $\{X_n\}$ with state space $\mathbb{Z}^d$, such that $X_0 = 0$ and, for each $e \in W$, $e \neq 0$,

$$\overline{P}^0_{\omega,\varepsilon}(X_{n+1} = z + e | X_n = z) = \mathbf{1}_{\{\epsilon_{n+1} = e\}} + \frac{\mathbf{1}_{\{\epsilon_{n+1}=0\}}}{1 - d\epsilon}[\omega(z, z+e) - \epsilon/2] \,.$$

It is not hard to check that the law of $\{X_n\}$ under $\overline{\mathbb{P}}$ coincides with its law under $\mathbb{P}$, while its law under $Q_\epsilon \otimes \overline{P}^0_{\omega,\varepsilon}$ coincides with its law under P^0_ω. Now, one modifies the definition of D_t by requiring that after the fresh time t, the "ε" coin was used for L steps in the direction ℓ: more precisely, requiring that $\epsilon_{t+i} = u_i, i = 1, \dots, L$ for some fixed sequence $u_i \in \mathbb{Z}^d, |u_i| = 1, u_i \cdot \ell > 0$ such that $\sum_{i=1}^L u_i \cdot \ell \geq L/2$. This, for large L, introduces enough decoupling to carry through the proof, see [29, Section 3.1]. We can now state the:

Embarrassing Problem 1: Prove that $\mathbb{P}(A_\ell) \in \{0,1\}$.

For $d = 2$, and P i.i.d., this was shown in [31], where counter examples using non uniformly elliptic, ergodic P's are also provided. The case $d > 2$, even for P i.i.d., remains open.

Embarrassing Problem 2: Find transience and recurrence criteria for the RWRE under $\mathbb{P}$.

The most promising approach so far toward Problem 2 uses regeneration times. Write $0 \leq d_1 < d_2 < \ldots$ for the ordered sequence of regeneration times, assuming that $\mathbb{P}(A_\ell) = 1$. The name regeneration time is justified by the following property, which for simplicity we state in the case $\ell = e_1$:

Theorem 3.1 ([28]) *For P a product measure, the sequence*

$$\{\{\omega_z\}_{z\cdot\ell\in[Z_{d_i},Z_{(d_{i+1}-1)})}, \{X_t\}_{t\in[d_i,d_{i+1})}\}_{i=2,3,\ldots}$$

is i.i.d..

From this statement, it is then not hard to deduce that once $\mathbb{E}(d_2 - d_1) < \infty$, a law of large numbers results, with a non-zero limiting velocity. Sufficient conditions for transience put forward in [14] turn out to fall in this class, see [28]. More recently, Sznitman has introduced a condition that ensures both a LLN and a CLT:
Sznitman's T' condition: $\mathbb{P}(A_\ell) = 1$ *and , for some* $c > 0$ *and all* $\gamma < 1$*,*

$$\mathbb{E}(\exp(c \sup_{0\leq n<d_1} |X_n|^\gamma)) < \infty.$$

A remarkable fact about Sznitman's T' condition is that he was able to derive, using renormalization techniques, a (rather complicated) criterion, depending on the restriction of P to finite boxes, to check it. Further, it implies a good control on d_1, and in particular that d_1 possesses all moments, which is the key to the LLN and CLT statements:

$$\mathbb{E}\left(\exp\left(\log d_1\right)^\delta\right) < \infty, \forall\delta < 2d/(d+1)\,.$$

For these, and related, facts see [27]. This leads one to the
Challenging Problem 3: Do there exist non-ballistic RWRE's for $d > 1$ satisfying that $\mathbb{P}(A_\ell) = 1$ for some ℓ?
For $d = 1$, the answer is affirmative, as we saw, as soon as $E_P \log\rho_0 < 0$ but $s < 1$. For $d > 1$, one suspects that the answer is negative, and in fact one may suspect that $\mathbb{P}(A_\ell) = 1$ implies Sznitman's condition T'. The reason for the striking difference is that for $d > 1$, it is much harder to force the walk to visit large traps.

It is worthwhile to note that the modified regeneration times introduced above using the coupling sequence ε can be used to deduce the LLN for a class of mixing environments. We refer to [4] for details. At present, the question of CLT's in such a general set up remains open.

Cut points

Regeneration times are less useful if the walk is not ballistic. Special cases of non-ballistic models have been analyzed in the above mentioned [18], and using a heavy renormalization analysis, in [2] for the case of symmetric, low disorder, i.i.d. P. In both cases, LLN's with zero speed and CLT's are provided. We now introduce, for another special class of models, a different class of times that are not regeneration times but provide enough decoupling to lead to useful consequences.

The setup is similar to that in Theorem 1.4, that is we assume that $d \geq 6$ and that the RWRE, in its first 5 coordinate, performs a deterministic random walk: For $i = 1, \ldots, 5$, $\omega(x, x \pm e_i) = q_{\pm i}$, for some deterministic $q_{\pm i}$, $P - a.s.$. Set $S = \sum_{i=1}^{5}(q_i + q_{-i})$, let $\{R_n\}_{n\in\mathbb{Z}}$ denote a (biased) simple random walk in $\mathbb{Z}^5$ with transition probabilities $q_{\pm i}/S$, and fix a sequence of independent Bernoulli random variable with $P(I_0 = 1) = S$, letting $U_n = \sum_{i=0}^{n-1} I_i$. Denote by X_n^1 the first 5 components of X_n and by X_n^2 the remaining components. Then, for every realization ω, the RWRE X_n can be constructed as the Markov chain with $X_n^1 = R_{U_n}$ and transition probabilities

$$\overline{P}_\omega^0(X_{n+1}^2 = z | X_n) = \begin{cases} 1, & X_n^2 = z, I_n = 1 \\ \omega(X_n, (X_n^1, z))/(1-S), & I_n = 0\,. \end{cases}$$

Introduce now, for the walk R_n, cut times c_i as those times where the past and future of the path R_n do not intersect. More precisely, with $\mathcal{P}_I = \{X_n\}_{n\in I}$,

$$c_1 = \min\{t \geq 0 : \mathcal{P}_{(-\infty,t)} \cap \mathcal{P}_{[t,\infty)} = \emptyset\}\,, c_{i+1} = \min\{t > c_i : \mathcal{P}_{(-\infty,t)} \cap \mathcal{P}_{[t,\infty)} = \emptyset\}\,.$$

Note that the cut-points sequence depends on the ordinary random walk R_n only. In particular, because that walk evolves in $\mathbb{Z}^5$, it is not difficult to check, following [9], that there are infinitely many cut points, and moreover that they have a positive density. The crucial observation is that the increments $X_{c_{i+1}}^2 - X_{c_i}^2$ depend on disjoint parts of the environment. Therefore, conditioned on $\{R_n, I_n\}$, they are independent if P is a product measure, and they possess good mixing properties if P has good mixing properties. From here, the statement of Theorem 1.4 is not too far. We refer the reader to [1], where this and CLT statements (with 5 replaced by a larger integer) are proved. An amusing consequence of [1] is that for $d > 5$, one may construct ballistic RWRE's with, in the notations of Section 2., $E_P(v_0(\omega)) = 0$!

Challenging Problem 4: Construct cut points for "true" non-ballistic RWRE's. The challenge here is to construct cut points and prove that their density is positive, without imposing a-priori that certain components of the walk evolve independently of the environment.

Large deviations

We conclude the discussion of multi-dimensional RWRE's by mentioning large deviations for this model. Call a RWRE *nestling* if co supp$Q$, where Q denotes the law of $\sum_{e\in\mathbb{Z}^d:|e|=1} e\omega(0,e)$. In words, an RWRE is nestling if by combining local drifts one can arrange for zero drift. One has then:

Theorem 3.2 ([30]) *Assume P is a product nestling measure. Then, for P-almost every ω, X_n/n satisfies a LDP under P_ω^0 with deterministic rate function.*

The proof of Theorem 3.2 involves hitting times: let T_y denote the first hitting time of $y \in \mathbb{Z}^d$. One then checks, using the subadditive ergodic theorem, that

$$\Lambda(y, \lambda) := \lim_{n\to\infty} n^{-1} \log E_\omega^0(\exp(-\lambda T_{ny}) \mathbf{1}_{\{T_{ny}<\infty\}})$$

exists and is deterministic, for $\lambda \geq 0$. In the nestling regime, where slowdown has sub-exponential decay rate due to the existence of traps much as for $d = 1$, this and concentration of measure estimates are enough to yield the LDP. We state the

Embarrassing Problem 5: Prove the quenched LDP for non-nestling RWRE's. A priori, non nestling walks should have been easier to handle than nestling walks due to good control on the tail of regeneration times!

Challenging Problem 6: Derive an annealed LDP for the RWRE, and relate the rate function to the quenched one.

One does not expect a relation as simple as in Theorem 2.7, because the RWRE can avoid traps by contouring them, and to change the environment in a way that surely modifies the behavior of the walk by time n has probability which seems to decay at an exponential rate faster than n. We mention in this context that in a related model, RWRE's on Galton-Watson trees, the quenched and annealed rate functions coincide [6]. We also note that certain estimates on large deviations for RWRE's, without matching constants, appear in [26].

References

[1] E. Bolthausen, A. S. Sznitman and O. Zeitouni, Cut points and diffusive random walks in random environments, *preprint* (2002). http://www-ee.technion.ac.il/~zeitouni/ps/BSZ4.ps

[2] J. Bricmont and A. Kupiainen, Random walks in asymmetric random environments, *Comm. Math. Phys* 142 (1991), 345–420.

[3] F. Comets, N. Gantert and O. Zeitouni, Quenched, annealed and functional large deviations for one dimensional random walk in random environment, *Prob. Th. Rel. Fields* 118 (2000), 65–114.

[4] F. Comets and O. Zeitouni, A law of large numbers for random walks in random mixing environments, math.PR/0205296 (2002).

[5] A. Dembo, A. Guionnet and O. Zeitouni, Aging properties of Sinai's random walk in random environment, math.PR/0105215 (2001).

[6] A. Dembo, N. Gantert, Y. Peres and O. Zeitouni, Large deviations for random walks on Galton-Watson trees: averaging and uncertainty, *Prob. Th. Rel. Fields* 122 (2002), 241–288.

[7] A. Dembo, N. Gantert and O. Zeitouni, Large deviations for random walk in random environment with holding times, *preprint* (2002).

[8] A. Dembo, Y. Peres and O. Zeitouni, Tail estimates for one-dimensional random walk in random environment, *Comm. Math. Physics* 181 (1996), 667–684.

[9] P. Erdös and S. J. Taylor, Some intersection properties of random walks paths, *Acta Math. Acad. Sci. Hungar.* 11 (1960), 231–248.

[10] N. Gantert, Subexponential tail asymptotics for a random walk with randomly placed one-way nodes, to appear, *Ann. Inst. Henri Poincaré* (2002).

[11] N. Gantert and O. Zeitouni, Quenched sub-exponential tail estimates for one-dimensional random walk in random environment, *Comm. Math. Physics* 194 (1998), 177–190.

[12] A. O. Golosov, On limiting distributions for a random walk in a critical one

dimensional random environment, *Comm. Moscow Math. Soc.* 199 (1985), 199–200.

[13] A. Greven and F. den Hollander, Large deviations for a random walk in random environment, *Annals Probab.* 22 (1994), 1381–1428.

[14] S. A. Kalikow, Generalized random walks in random environment, *Annals Probab.* 9 (1981), 753–768.

[15] H. Kesten, The limit distribution of Sinai's random walk in random environment, *Physica* 138A (1986), 299–309.

[16] H. Kesten, M. V. Kozlov and F. Spitzer, A limit law for random walk in a random environment, *Comp. Math.* 30 (1975), 145–168.

[17] S. M. Kozlov, The method of averaging and walks in inhomogeneous environments, *Russian Math. Surveys* **40** (1985) pp. 73–145.

[18] G.F. Lawler, Weak convergence of a random walk in a random environment, *Comm. Math. Phys.* **87** (1982) pp. 81–87.

[19] A. De Masi, P. A. Ferrari, S. Goldstein and W. D. Wick, An invariance principle for reversible Markov processes. Applications to random motions in random environments, *J. Stat. Phys.* 55 (1989), 787–855.

[20] A. Pisztora and T. Povel, Large deviation principle for random walk in a quenched random environment in the low speed regime, *Annals Probab.* 27 (1999), 1389–1413.

[21] A. Pisztora, T. Povel and O. Zeitouni, Precise large deviation estimates for a one-dimensional random walk in a random environment, *Prob. Th. Rel. Fields* 113 (1999), 191–219.

[22] F. Rassoul-Agha, A law of large numbers for random walks in mixing random environment, *Preprint* (2002).

[23] A. Roitershtein, Ph.D. thesis, *Dept. of Mathematics, Technion* (Forthcoming).

[24] Ya. G. Sinai, The limiting behavior of a one-dimensional random walk in random environment, *Theor. Prob. and Appl.* 27 (1982), 256–268.

[25] F. Solomon, Random walks in random environments, *Annals Probab.* 3 (1975), 1–31.

[26] A. S. Sznitman, Slowdown estimates and central limit theorem for random walks in random environment, *JEMS* **2** (2000), pp. 93–143.

[27] A. S. Sznitman, An effective criterion for ballistic behavior of random walks in random environment, *Probab. Theory Relat. Fields* 122 (2002), 509–544.

[28] A. S. Sznitman and M. Zerner, A law of large numbers for random walks in random environment, *Annals Probab.* 27 (1999), 1851–1869.

[29] O. Zeitouni, Lecture notes on RWRE, notes from the St.-Flour summer school in probability, 2001. Available at http://wwww-ee.technion.ac.il/~zeitouni/ps/notes1.ps

[30] M. P. W. Zerner, Lyapounov exponents and quenched large deviations for multidimensional random walk in random environment, *Annals Probab.* 26 (1998), 1446–1476.

[31] M. P. W. Zerner and F. Merkl, A zero-one law for planar random walks in random environment, *Annals Probab.* **29** (2001), pp. 1716–1732.