Conditional exponential moments for iterated Wiener integrals

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Abstract We provide sharp exponential moment bounds for (Stratonovich) iterated stochastic integrals under conditioning by certain small balls, including balls in certain Hölder-like norms of exponent greater than 1/3. The proof uses a control of the variation of the Lévy area, under conditioning. The results are applied to the computation of the Onsager-Machlup functional of diffusion processes with constant diffusion matrix.

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1 Introduction

Throughout this paper, $\{X\}$ denotes a canonical, $\mathbb{R}^d$ valued Brownian motion on $C_0\left([0,1];\,\mathbb{R}^d\right)$ and $\mathbb{P}$ denotes Wiener measure. The r-th (Stratonovich) iterated integral of X at time $s \in [0,1]$ is given by

$$I_r(s) = \int_{0<u_1<\cdots<u_r<s} dX_{u_1}\cdots dX_{u_r} \in \mathbb{R}^{(d^r)}\,, \tag{1.1}$$

where all integrals are understood in the Stratonovich sense. We denote also $I_r := I_r(1)$. Our goal in this paper is to consider various limits of exponential moments of I_r, under the conditioning that X possesses small norm.

To state precisely our results, we need to introduce some notations. Let $\{\phi_{n,k}\}_{n=0,1,2,\cdots;k=0,1,2,\cdots,2^n-1}$ denote the Haar system, namely the set of functions

$$\begin{aligned}
\phi_{0,0}(t) &\triangleq 1\,, \\
\phi_{k,n}(t) &\triangleq \begin{cases} \sqrt{2^n}\,, & t \in \left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right) \\ -\sqrt{2^n}\,, & t \in \left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right) \\ 0\,, & \text{otherwise,} \end{cases}
\end{aligned} \tag{1.2}$$

and let

$$\chi_{k,n} = \int_0^1 \phi_{k,n}(t)\,dX_t\,. \tag{1.3}$$

Then $\{\chi_{k,n}\}$ are i.i.d., standard $N(0, I_d)$ random vectors, and furthermore,

$$X_{\frac{(2k+1)}{2^{n+1}}} = \frac{X_{\frac{k}{2^n}} + X_{\frac{k+1}{2^n}}}{2} + 2^{-(n+2)/2}\chi_{k,n}\,. \tag{1.4}$$

Equation (1.4) is the Lévy-Ciesielski construction of Brownian motion, c.f. [8] for sharp results concerning the convergence of interpolation based on (1.4) to the Brownian motion X_t. We let $\mathcal{F}_n$ denote the σ-algebra generated by the random variables $\{X_{j/2^n}\}_{j=0}^{2^n}$.

In what follows, $|\cdot|_p$ denotes the usual L^p norms on $\mathbb{R}^d$, with $|\cdot| := |\cdot|_2$. $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^d$. For any $f \in C_0\left([0,1];\, \mathbb{R}^d\right)$, let

$$\chi^f_{k,n} = \int_0^1 \phi_{k,n}(t)df(t)\,, \quad \Delta^f_{k,n} = f(\frac{k+1}{2^n}) - f(\frac{k}{2^n})\,.$$

We say that a measurable, separable norm $\|\cdot\|$ on $C_0([0,1];\mathbb{R}^d)$ belongs to $\mathcal{N}_p$, $p \in (2,\infty]$ if it satisfies the following properties:

(P1) There exists a constant C_p such that

$$\|f\| \le 1 \Rightarrow \sup_n \max_{k=0,1,\ldots,2^n-1} |\Delta^f_{k,n}| 2^{n/p} \le C_p\,. \tag{1.5}$$

(P2) There exists a constant C such that $\mathbb{P}(\|X\| < \infty) = 1$ and

$$\mathbb{P}(\|X\| < \varepsilon) \triangleq C_\varepsilon \ge \exp\left(-C\varepsilon^{-\frac{2p}{p-2}}\right)\,. \tag{1.6}$$

The assumptions **(P1)** and **(P2)** are quite different in nature: **(P1)** is a regularity assumption on the norm, while the assumption **(P2)** is a lower bound on small ball probabilities, and in particular allows one to condition on the event $\{\|X\| < \varepsilon\}$. It is easy to check that the standard Hölder norm of exponent $1/p$, as well as the variants described in [1], belong to $\mathcal{N}_p$. Note also that another formulation of **(P1)** is that $\|f\|_{1/p} \le C_p ||f||$ for all $f \in C_0([0,1];\mathbb{R}^d)$, where $\|\cdot\|_{1/p}$ denotes the standard Hölder norm of exponent $1/p$.

Our main result is the

Theorem 1.7 *Assume $\|\cdot\| \in \mathcal{N}_p$ with $3 > p > 2$. Let $r < \frac{4(p-1)}{(p-2)}$. Then, for any $C \in \mathbb{R}^{(d^r)}$ and $M < \infty$,*

$$\sup_{\varepsilon < M} \mathbb{E}\left(e^{\langle C, I_r\rangle} \,|\|X\| < \varepsilon\right) < \infty\,. \tag{1.8}$$

In fact, for any C, M as above, any $\alpha < 4(p-1)/(p-2)r$ and any integer r,

$$\sup_{\varepsilon < M} \mathbb{E}\left(e^{|\langle C, I_r\rangle|^\alpha} \,|\|X\| < \varepsilon\right) < \infty\,. \tag{1.9}$$

If $r = 4(p-1)/(p-2)$ then for any $C \in \mathbb{R}^{(d^r)}$ there exists an $M = M(C)$ such that (1.8) remains valid.

Remark 1.10 *Note that for $r \geq 2$, there exists a C such that $\mathbb{E}(e^{\langle C, I_r \rangle}) = \infty$, hence the conditioning plays a definitive role in (1.8) and (1.9).*

Remark 1.11 *It is obvious from the proof that the estimates in Theorem 1.7 hold also for $I_r(s)$, uniformly in $0 \leq s \leq 1$. That is, under the hypotheses and notations of Theorem 1.7,*

$$\sup_{\varepsilon < M} \sup_{s \in [0,1]} \mathbb{E}\left(e^{\langle C, I_r(s) \rangle} \,|\, \|X\| < \varepsilon \right) < \infty\,, \quad \sup_{\varepsilon < M} \sup_{s \in [0,1]} \mathbb{E}\left(e^{|\langle C, I_r(s) \rangle|^\alpha} \,|\, \|X\| < \varepsilon \right) < \infty\,. \tag{1.12}$$

An application, which served as our motivation for deriving Theorem 1.1, is as follows. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be smooth, and denote by $\{Y_t\}$ the solution of the Itô stochastic differential equation

$$dY_t = b(Y_t)\, dt + dX_t, \quad Y_o = 0\,. \tag{1.13}$$

For a deterministic $\psi \in L^2\left([0,1]; \mathbb{R}^d\right)$, let $\psi^I(t) = \int_0^t \psi(s)\, ds$, and define the Onsager Machlup functional of Y at ψ^I as

$$J_{\|\cdot\|}(\psi^I) = -\log \lim_{\varepsilon \to 0} \frac{\mathbb{P}(\|Y - \psi^I\| < \varepsilon)}{\mathbb{P}(\|X\| < \varepsilon)} \tag{1.14}$$

if the limit exists. $J_{\|\cdot\|}(\cdot)$ can serve as a "prior" on path space, and was introduced in the context of Gaussian diffusions by Onsager and Machlup [9]. For the supremum norm $\|f\|_\infty = \sup_{0 \leq t \leq 1} |f(t)|$, and smooth ψ, Stratonovich (cf. the proof in [5]) has shown that

$$J_{\|\cdot\|_\infty}(\psi^I) = \frac{1}{2} \int_0^1 \left| \psi(t) - b(\psi^I(t)) \right|^2 dt + \frac{1}{2} \int_0^1 \operatorname{div} b(\psi^I(t))\, dt\,. \tag{1.15}$$

This result was extended to general $\psi \in L^2([0,1]; \mathbb{R}^d)$ in [11], to more general norms, including Hölder norm up to exponent $1/3$ in [12], and to a wider class of norms (including the Hölder norm up to exponent $1/2$) by [2]. The latter paper imposed on the norm the restriction that it must be invariant w.r.t. rotations in $\mathbb{R}^d$. We show below how this restriction can be avoided by relying on Theorem 1.7.

Recall that a norm is completely convex if for every $i = 1, 2, \ldots, d$, every $\varepsilon > 0$ and every fixed component

$$\left(\tilde{\phi}_i(\cdot), \ldots, \tilde{\phi}_{i-1}(\cdot), \tilde{\phi}_{i+1}(\cdot), \ldots, \tilde{\phi}_d(\cdot) \right) \in C_0\left([0,1]; \mathbb{R}^{d-1} \right)\,,$$

the set

$$B_i = \left\{ \phi(\cdot) \in C_0([0,1];\mathbb{R}) : \| \left(\tilde{\phi}_1(\cdot), \dots, \tilde{\phi}_{i-1}(\cdot), \phi(\cdot), \tilde{\phi}_{i+1}(\cdot), \dots, \tilde{\phi}_d(\cdot) \right) \| < \varepsilon \right\}$$

is symmetric (due to the properties of the norm, it is always convex). Using Theorem 1.7 and a weak convexity requirement, one obtains readily the

Corollary 1.16 *Assume $\| \cdot \| \in \mathcal{N}_p$, $p \in (2,3)$. Further assume that $\| \cdot \|$ is completely convex. Then, $J_{\|\cdot\|}(\psi^I)$ exists and equals $J_{\|\cdot\|_\infty}(\psi^I)$.*

Extension to diffusions on manifolds (c.f. the recent preprint [3]) require different techniques and we do not pursue this direction here.

Our proof of Theorem 1.7 avoids the use of correlation inequalities (these are used however in the proof of Corollary 1.16). Rather, our proof relies on the pathwise analysis on rough paths described in [7]. The key idea is to bound the r-th iterated integral by estimates on the Lévy area at different scales, c.f Lemma 2.1. While correlation inequalities can be used at each scale to obtain uniform bounds on the Lévy area, they are not sharp enough, and a multi scale analysis is required. This analysis is reduced to estimates on dyadic partitions by a technique borrowed from [4], and the key estimate is obtained by separately dealing with the "coarse" elements (in which the conditioning is dominant) and the "fine" elements (where conditioning has litle effect and unconditional bounds on the Lévy area are tight enough).

Notations: throughout the paper, $C_1, C_2, \dots$ denote various deterministic constants (which may vary from line to line) and which are independent of ε or n, m, k. Constants $C_a, C_b, \cdots$ denote constants whose value is independent of ε or n, m, k and is kept fixed throughout the paper. For a matrix $A(i,j)$, $|A| = \max_{i,j} |A(i,j)|$.

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2 Proof of Theorem 1.7 and Corollary 1.16

In order to provide the proof of Theorem 1.7, we need to consider conditional estimates for the Lévy area. Define the matrix

$$A_{s,t} = \overbrace{\int_{t<u<v<s} dX_u \, dX_v}$$

where $\frown$ denotes the anti-symmetric part of the integral.

The key to the proof of Theorem 1.7 is the following lemma.

Lemma 2.1 *For any r and $3 > p' > 2$ there exist constants $C_K = C_K(p', r)$, $C_\gamma = C_\gamma(p', r) > 1$ such that*

$$\begin{aligned} |I_r| \quad &\leq \quad C_K \left(\sum_{n=1}^{\infty} n^{C_\gamma} \sum_{k=0}^{2^n-1} \left| A_{\frac{k}{2^n}, \frac{k+1}{2^n}} \right|^{p'/2} \right)^{r/p'} \\ &+ C_K \left(\sum_{n=1}^{\infty} n^{C_\gamma} \sum_{k=0}^{2^n-1} \left| X_{\frac{k}{2^n}} - X_{\frac{k+1}{2^n}} \right|^{p'} \right)^{r/p'} \triangleq C_K \bar{I}^{r/p'} + C_K \hat{I}^{r/p'} \end{aligned} \tag{2.2}$$

Proof: Rewrite, in the obvious way, $I_r = I_r(0,1)$, and more generally $I_r(s,t)$ is the r-th iterated integral over the interval (s,t). Let $\mathcal{D}_{s,t}$ denote all partitions of the interval $[s,t]$, that is if $D \in \mathcal{D}_{s,t}$ then $D = \{[t_{j-1}, t_j]\}_{j=1}^{|D|}$, with $t_0 = s, t_{|D|} = t$ and $t_j < t_{j+1}$. Define

$$\omega(s,t) = \sup_{D \in \mathcal{D}_{s,t}} \sum_D (|X_{t_j} - X_{t_{j-1}}|^{p'} + |A_{t_{j-1},t_j}|^{p'/2}) \, .$$

Recall the definitions for I_r given in (1.1) defined for almost every path via Stratonovich integrals. Then almost surely, the sequence $(I_r)_{r=0}^n$ forms a geometric multiplicative functional in the sense of [7]. Moment estimates (eg., see [10]) can be used to show that for every n and almost surely, this functional has finite p-variation in the sense of [7] for $p > 2$, that is

$$\sup_{D \in \mathcal{D}_{0,1}} \sum_D |I_r(t_j, t_{j-1})|^p < \infty \, , \quad a.s.$$

Now, by Theorem 2.2.1 in [7], for $p' < 3$, and for any multiplicative functional $(I_r)_{r=0}^2$ of finite p'-variation there is a unique multiplicative functional $(I_r)_{r=0}^n$ of finite p'-variation extending $(I_r)_{r=0}^2$.

By the above remarks, in the Brownian case, this extension technique can be applied almost surely to $(I_r)_{r=0}^2$ and by the uniqueness statement coincides with the integral defined via Stratonovich integrals for almost all paths.

In fact, and crucially for us, Theorem 2.2.1 in [7] gives an explicit control over $|I_r(s,t)|$ in terms of the p'-variation of $(I_r)_{r=0}^2$. To apply this result to our setting observe first that the symmetric part of the Stratonovich $I_2(s,t)$ is given by the square of $I_1(s,t)$ and the antisymmetric part is $A_{s,t}$ so the p'-variation of $(I_r)_{r=0}^2$ is controlled by ω defined above. In this case and for some universal constant C_r,

$$|I_r(s,t)| \leq C_r \omega(s,t)^{r/p'} . \tag{2.3}$$

(Note that this is where we use the assumption $p' < 3$: if $p' \geq 3$, one cannot control the higher iterated integrals by means of the p'-variation of the path and the Lévy area only). Lemma 2.1 hence follows if we prove that for some universal constant C_{rr},

$$\begin{aligned} \omega(0,1) \quad &\leq \quad C_{rr} \left(\sum_{n=1}^{\infty} n^{C_\gamma} \sum_{k=0}^{2^n-1} \left| A_{\frac{k}{2^n},\frac{k+1}{2^n}} \right|^{p'/2} \right) \\ &\quad + C_{rr} \left(\sum_{n=1}^{\infty} n^{C_\gamma} \sum_{k=0}^{2^n-1} \left| X_{\frac{k}{2^n}} - X_{\frac{k+1}{2^n}} \right|^{p'} \right) . \end{aligned} \tag{2.4}$$

Fix a partition $D \in \mathcal{D}_{0,1}$. For any real z, let $\lfloor z \rfloor$ ($\lceil z \rceil$) denote the largest integer less or equal to (respectively, the smallest integer greater or equal to) z. Every interval $[t_{j-1}, t_j)$ is a countable union of disjoint intervals $J_j^{i,+}$, $J_j^{i,-}$ of the form

$$J_j^{i,+} := [\lfloor t_j 2^{(i-1)} \rfloor 2^{-(i-1)}, \lfloor t_j 2^i \rfloor 2^{-i}) := [\tau_j^{i,+}, T_j^{i,+}) ,$$

$$J_j^{i,-} := [\lceil t_{j-1} 2^i \rceil 2^{-i}, \lceil t_{j-1} 2^{i-1} \rceil 2^{-(i-1)}) := [\tau_j^{i,-}, T_j^{i,-}) ,$$

with the convention that some of the intervals above can be empty. Denote $DX_j^{i,\pm} := X_{T_j^{i,\pm}} - X_{\tau_j^{i,\pm}}$. Then,

$$\begin{aligned} \sum_D (|X_{t_j} - X_{t_{j-1}}|^{p'}) \quad &= \quad \sum_j \left(\sum_{i=1}^{\infty} (DX_j^{i,+} + DX_j^{i,-}) \right)^{p'} \\ &\leq \quad C_1 \sum_j \sum_{i=1}^{\infty} i^{C_\gamma} (|DX_j^{i,+}|^{p'} + |DX_j^{i,-}|^{p'}) \end{aligned}$$

$$\leq \quad C_2 \sum_{i=1}^{\infty} i^{C_\gamma} \sum_{k=0}^{2^i-1} |X_{\frac{k}{2^i}} - X_{\frac{k+1}{2^i}}|^{p'}\,, \tag{2.5}$$

where the first inequality is due to the reverse Hölder inequality: for any a_i non-negative and $C_\gamma > 0$,

$$(\sum_{i=1}^{\infty} a_i)^{p'} \leq (\sum_{i=1}^{\infty} i^{-C_\gamma/(p'-1)})^{p'-1} \sum_{i=1}^{\infty} i^{C_\gamma} (a_i)^{p'}\,.$$

A similar decomposition holds for the area term: recall that for any $s < t < u$,

$$A_{s,t} + A_{t,u} = A_{s,u} - \frac{[(X_t - X_s) \wedge (X_u - X_t)]}{2}\,,$$

where for vectors a, b in $\mathbb{R}^d$, $(a \wedge b)$ denotes the anti-symmetric matrix with entries $(a \wedge b)_{i,j} = a_i b_j - a_j b_i$. Hence, with the obvious notation for $A_{J_j^{i,\pm}}$, by a computation similar to (2.5),

$$\begin{aligned}
\sum_D |A_{t_{j-1},t_j}|^{p'/2} \quad &\leq \quad \sum_j \left(\sum_{i=1}^{\infty} |A_{J_j^{i,+}}| + |A_{J_j^{i,-}}| + (|DX_j^{i,+}| + |DX_j^{i,-}|)^2 \right)^{p'/2} \\
&\leq \quad C_1 \sum_{i=1}^{\infty} i^{C_\gamma} \sum_{k=0}^{2^i-1} \left(|A_{\frac{k}{2^i},\frac{k+1}{2^i}}|^{p'/2} + |X_{\frac{k}{2^i}} - X_{\frac{k+1}{2^i}}|^{p'} \right)\,. \qquad (2.6)
\end{aligned}$$

The lemma follows by combining (2.5) and (2.6). □

Proof of Theorem 1.7. Choose a fixed $p' > p$ and fix $M < \infty$ (in the case $r = 4(p-1)/(p-2)$m we reduce below the initial choice of M when necessary). Note that under the conditioning,

$$\hat{I} \leq C_1 \sum_{n=1}^{\infty} n^{C_\gamma} \sum_{k=0}^{2^n-1} \varepsilon^{p'} \cdot 2^{-np'/p} = C_1 \sum_{n=1}^{\infty} n^{C_\gamma} \varepsilon^{p'} 2^{-n\left(\frac{p'}{p}-1\right)}.$$

and so, again under the conditioning, $\sup_{\varepsilon \leq M} \hat{I} < C_2 < \infty$. Therefore, Theorem 1.7 will follow if we show that for any $p' > p$, any M, any c, and any $r < \frac{4(p-1)}{(p-2)}$ one has

$$\sup_{\varepsilon \leq M} \mathbb{E}^{\varepsilon} \left(e^{c|\bar{I}^{r/p'}|} \right) < \infty\,, \tag{2.7}$$

whereas for any $p' > p$, any c, and $r = \frac{4(p-1)}{(p-2)}$, there exists an $M = M(c, p, p')$ such that (2.7) still holds.

Let $\tilde{X} = (X_1, X_2) \in C_0([0,1]; \mathbb{R}^2)$ and

$$\tilde{A}_{s,t} = A_{s,t}(1,2) := \frac{1}{2}\left\{\int_s^t (X_1(\theta) - X_1(s))dX_2(\theta) - \int_s^t (X_2(\theta) - X_2(s))dX_1(\theta)\right\} .$$

We let $\tilde{\chi}_{k,n} = \int_0^1 \phi_{k,n}(t) d\tilde{X}_t$.

Fix $C_\delta > C_\gamma > 1$, and note that for some C_g depending on d only, and for all $x > 0$ (we will take below $x > x_0$ for some x_0 large enough),

$$\begin{aligned} \mathbb{P}^\varepsilon(|\bar{I}| > C_g x) &\leq \sum_{n'=1}^{\infty} C_1 \mathbb{P}^\varepsilon\left(\sum_{j=0}^{2^{n'}-1} \left|\tilde{A}_{\frac{j}{2^{n'}},\frac{j+1}{2^{n'}}}\right|^{p'/2} > \frac{x}{(n')^{C_\delta}}\right) \\ &\leq C_1 \sum_{n'=1}^{\infty} 2^{n'} \max_{j=0}^{2^{n'}-1} \mathbb{P}^\varepsilon\left(\left|\tilde{A}_{\frac{j}{2^{n'}},\frac{j+1}{2^{n'}}}\right|^{p'/2} > \frac{x}{2^{n'} {n'}^{C_\delta}}\right) . \end{aligned} \tag{2.8}$$

Define $x_{n'} = (x/{n'}^{C_\delta})^{2/p'} 2^{-2n'/p'}$, $\Delta x_{k,n}^{j,n'} = (\tilde{X}_{\frac{j}{2^{n'}}+\frac{k+1}{2^{n+n'}}} - \tilde{X}_{\frac{j}{2^{n'}}+\frac{k}{2^{n+n'}}})$ and $\Delta h_{k,n}^{j,n'} = 2^{-(n+n'+2)/2} \tilde{\chi}_{2^n j+k, n+n'}$. Then,

$$\hat{A}_{j,n'} \triangleq \tilde{A}_{\frac{j}{2^{n'}},\frac{j+1}{2^{n'}}} = \lim_{m\to\infty} A_{j,n'}^{(m)} ,$$

where

$$A_{j,n'}^{(m)} \triangleq \frac{1}{2} \sum_{n=0}^{m} \sum_{k=0}^{2^n-1} (\Delta x_{k,n}^{j,n'} \wedge \Delta h_{k,n}^{j,n'})_{1,2} .$$

One easily checks that

$$\hat{A}_{j,n'} = A_{j,n'}^{(m-1)} + \sum_{k=0}^{2^m-1} \tilde{A}_{\frac{j}{2^{n'}}+\frac{k}{2^{m+n'}},\frac{j}{2^{n'}}+\frac{k+1}{2^{m+n'}}} . \tag{2.9}$$

On the event $\|X\| \leq \varepsilon$ one has that for some C_e,

$$|A_{j,n'}^{(m-1)}| \leq C_e \varepsilon^2 2^{-2n'/p} 2^{m(1-2/p)} . \tag{2.10}$$

We can next express the Brownian path $\left(\tilde{X}_s\right)_{s\in\left[j/2^{n'}+k/2^{m+n'}, j/2^{n'}+(k+1)/2^{m+n'}\right]}$ as the sum of a linear motion and a Brownian loop $\ell(\cdot)$ independent of the field $\mathcal{F}_{m+n'}$, that is, for

$s \in \left[j/2^{n'} + k/2^{m+n'}, j/2^{n'} + (k+1)/2^{m+n'}\right]$,

$$\tilde{X}_s = \tilde{X}_{j2^{-n'}+k2^{-(n'+m)}} + 2^{m+n'}\Delta x^{j,n'}_{k,m}(s - j2^{-n'} - k2^{-m-n'}) + \ell(s)\,.$$

Denoting

$$\tilde{x}^{j,n'}_{k,m} = 2^{m+n'}\left(-\int_{j/2^{n'}+k/2^{m+n'}}^{j/2^{n'}+(k+1)/2^{m+n'}} \ell_2(s)ds, \int_{j/2^{n'}+k/2^{m+n'}}^{j/2^{n'}+(k+1)/2^{m+n'}} \ell_1(s)ds\right)\,,$$

and denoting the area of the loop by $\bar{A}^{j,n'}_{k,m}$, we may rewrite the above expression as

$$\hat{A}_{j,n'} = A^{(m-1)}_{j,n'} + \sum_{k=0}^{2^m-1}\langle \Delta x^{j,n'}_{k,m}, \tilde{x}^{j,n'}_{k,m}\rangle + \sum_{k=0}^{2^m-1}\bar{A}^{j,n'}_{k,m}\,.$$

Then,

$$\begin{aligned}\mathbb{P}^\varepsilon\left(|\hat{A}_{j,n'}| > x_{n'}\right) &< \mathbb{P}^\varepsilon\left(|A^{(m-1)}_{j,n'}| > x_{n'}/3\right) \\ &+\mathbb{P}^\varepsilon\left(|\sum_{k=0}^{2^m-1}\langle\Delta x^{j,n'}_{k,m}, \tilde{x}^{j,n'}_{k,m}\rangle| > x_{n'}/3\right) \\ &+\mathbb{P}^\varepsilon\left(|\sum_{k=0}^{2^m-1}\bar{A}^{j,n'}_{k,m}| > x_{n'}/3\right)\,. \end{aligned} \tag{2.11}$$

Recalling that $p' > p > 2$, fix next x_0 such that, for all n',

$$\left(\frac{\left(\frac{x_0}{n'^{C_\delta}}\right)^{2/p'} 2^{2n'(1/p-1/p')}}{3C_e M^2}\right)^{p/(p-2)} \geq 2\,, \quad \left(\frac{x_0}{n'^{C_\delta}}\right)^{2/p'}\frac{2^{n'(1-2/p')}}{3} \geq 1\,. \tag{2.12}$$

For $x \geq x_0$, fix

$$m = [\frac{p}{(p-2)\log 2}\log\left(\frac{x^{2/p'}2^{2n'(1/p-1/p')}}{3C_e\varepsilon^2 n'^{2C_\delta/p'}}\right)]\,.$$

(Hence, $2^m \leq \left(\frac{x_{n'}2^{2n'/p}}{3C_e\varepsilon^2}\right)^{\frac{p}{p-2}}$). Therefore, c.f. (2.10), $\mathbb{P}^\varepsilon\left(|A^{(m-1)}_{j,n'}| > x_{n'}/3\right) = 0$. Next,

$$\begin{aligned}&\mathbb{P}^\varepsilon\left(|\sum_{k=0}^{2^m-1}\langle\Delta x^{j,n'}_{k,m}, \tilde{x}^{j,n'}_{k,m}\rangle| > x_{n'}/3\right) \\ &\leq \frac{\mathbb{E}\left(\mathbb{P}\left(|\sum_{k=0}^{2^m-1}\langle\Delta x^{j,n'}_{k,m}, \tilde{x}^{j,n'}_{k,m}\rangle| > x_{n'}/3; \|X\| < \epsilon\,|\mathcal{F}_{m+n'}\right)\right)}{C_\varepsilon}\,.\end{aligned}$$

On the event $\|X\| < \varepsilon$, using (P1) we have that

$$\sum_{k=0}^{2^m-1} |\Delta x_{k,m}^{j,n'}|^2 < C_1\varepsilon^2 2^m 2^{-2(m+n')/p} = C_2 x_{n'} \, ,$$

while the components of $\tilde{x}_{k,m}$ are independent of $\mathcal{F}_{m+n'}$ and i.i.d., Normal zero mean with variance $C_1 2^{-(m+n')}$ and so

$$\begin{aligned} \mathbb{P}^\varepsilon\left(|\sum_{k=0}^{2^m-1}\langle\Delta x_{k,m}^{j,n'},\tilde{x}_{k,m}^{j,n'}\rangle| > x_{n'}/3\right) &\leq \frac{\exp\left(-C_1 2^{n'p/(p-2)} x_{n'}^{2(p-1)/(p-2)}\varepsilon^{-2p/(p-2)}\right)}{C_\varepsilon} \\ &\leq \exp\left(-C_h \frac{2^{n'(pp'-4(p-1))/(p'(p-2))}}{(n')^{4C_\delta(p-1)/p'(p-2)}} \frac{x^{4(p-1)/p'(p-2)}}{\varepsilon^{2p/(p-2)}}\right) C_\varepsilon^{-1} \, , \end{aligned} \tag{2.13}$$

for some constant C_h independent of ε, j, n', m. Hence, for $x > C_i$, some C_i large enough,

$$\sum_{n'=1}^{\infty} 2^{n'} \max_{j=0}^{2^{n'}-1} \mathbb{P}^\varepsilon\left(|\sum_{k=0}^{2^m-1}\langle\Delta x_{k,m}^{j,n'},\tilde{x}_{k,m}^{j,n'}\rangle| > x_{n'}/3\right) < \exp(-C_j x^{4(p-1)/p'(p-2)}\epsilon^{-2p/(p-2)}) \, , \tag{2.14}$$

where we used the fact that $pp' - 4(p-1) > 0$ and the bound (1.6).

Finally, it remains to treat the term

$$\mathbb{P}^\varepsilon\left(|\sum_{k=0}^{2^m-1} \bar{A}_{k,m}^{j,n'}| > x_{n'}/3\right) \, .$$

For this we may use standard Laplace transform methods. The Laplace transform of $\bar{A}_{k,m}^{j,n'}$ is (c.f [6, Pg. 172])

$$\begin{aligned} \phi_{m,n'}(z) &= \mathbb{E}(\exp(-z\bar{A}_{k,m}^{j,n'})) \\ &= \mathbb{E}\left(\exp\left(-z\tilde{A}_{0,2^{-m-n'}}\right)\Big|\left|\tilde{X}_{2^{-m-n'}} - \tilde{X}_0\right| = 0\right) \\ &= \frac{2^{-(1+m+n')}z}{\sin 2^{-(1+m+n')}z} \end{aligned}$$

for $\pi 2^{m+n'+1} > |z|$. Let ζ be the Legendre transform

$$\zeta(x) = \sup_{z\in[0,2\pi]} \{zx - \log\frac{z}{2\sin z/2}\} \, ,$$

then, by Chebycheff's inequality,

$$\mathbb{P}\left(2^{-m}|\sum_{k=0}^{2^m-1} 2^m \bar{A}_{k,m}^{j,n'}| > x_{n'}/3\right) \le 2\exp -2^m\zeta\left(2^{n'}x_{n'}/3\right) .$$

Since at the very least $\zeta(x) > C_1 x > 0$ for $x > 1$, one concludes that

$$\begin{aligned}
\mathbb{P}\left(2^{-m}|\sum_{k=0}^{2^m-1} 2^m \bar{A}_{k,m}^{j,n'}| > x_{n'}/3\right) &\le 2\exp\left(-C_1\frac{x_{n'}^{2(p-1)/(p-2)}2^{n'p/(p-2)}}{\varepsilon^{2p/(p-2)}}\right) \\
&\le 2\exp\left(-C_1\frac{x^{4(p-1)/p'(p-2)}2^{n'(pp'-4(p-1))/p'(p-2)}}{\varepsilon^{2p/(p-2)}(n')^{4C_\delta(p-1)/p'(p-2)}}\right) . \quad (2.15)
\end{aligned}$$

Hence,

$$\mathbb{P}^\varepsilon\left(|\sum_{k=0}^{2^m-1} \bar{A}_{k,m}^{j,n'}| > x_{n'}/3\right) \le 2C_\varepsilon^{-1}\exp\left(-C_1\frac{x^{4(p-1)/p'(p-2)}2^{n'(pp'-4(p-1))/p'(p-2)}}{\varepsilon^{2p/(p-2)}(n')^{4C_\delta(p-1)/p'(p-2)}}\right) . \quad (2.16)$$

Combining (2.11), (2.8), (2.14), (2.16) and (1.6), one concludes that for $x > C_k$, some constant C_k,

$$\mathbb{P}^\varepsilon(|\bar{I}| > C_g x) \le \exp\left(-C_2 x^{4(p-1)/p'(p-2)}\varepsilon^{-2p/(p-2)}\right) ,$$

from which (2.7) follows in the case $r < 4(p-1)/(p-2)$. To handle the case $r = 4(p-1)/(p-2)$, fix $c > 0$, and reduce M if necessary such that $C_2/M^{2p/(p-2)} > cC_g^{r/p'}$. □

Proof of Corollary 1.16. We follow the notations and proof of [2], where the general argument leading to the computation of the Onsager–Machlup functional is presented. A norm satisfying the assumption of Corollary 1.16 satisfies also (P1), (P2) of [2]. Following the proof there, it is clear that with $\psi \in L^2([0,1];\mathbb{R}^d)$, all that one needs to show is property (i) in [2, page 196], namely that for any monomial M_t of order $2 \le k \le [p/(p-2] \stackrel{\triangle}{=} k_p$ in X_t, any $c \in \mathbb{R}$, and any deterministic function $\Psi(t)$ with $\dot{\Psi}(t) \in L^2([0,1];\mathbb{R})$, it holds that

$$\limsup_{\varepsilon\to 0}\mathbb{E}^\varepsilon\left(\exp\left(c\int_0^1 \Psi(s)M_s dX_1(s)\right)\right) \le 1 . \quad (2.17)$$

(Strictly speaking, property (i) in [2] is stated in terms of Ito integrals; However, since $k \ge 2$, the Ito to Stratonovich correction terms are all bounded uniformly on the set $\|X\| < \varepsilon$ and converge uniformly to 0 as $\varepsilon \to 0$.)

Integrating by parts, we have that

$$\int_0^1 \Psi(s) M_s dX_1(s) = \Psi(1)V_1 - \int_0^1 \dot{\Psi}(s) V_s ds\,,$$

where $V_t = \int_0^t M_s dX_1(s)$. Therefore,

$$\begin{aligned}
\mathbb{E}^\varepsilon\left(\exp\left(c\int_0^1 \Psi(s)M_s dX_1(s)\right)\right) &\leq \left(\mathbb{E}^\varepsilon\left(\exp 2c\Psi(1)V_1\right)\right)^{1/2}\left(\mathbb{E}^\varepsilon\left(\exp\left(-2c\int_0^1 \dot{\Psi}(s)V_s ds\right)\right)\right)^{1/2} \\
&\leq \left(\mathbb{E}^\varepsilon\left(\exp 2c\Psi(1)V_1\right)\right)^{1/2}\left(\mathbb{E}^\varepsilon\left(\exp C_1\left(\int_0^1 V_s^2\right)^{1/2}\right)\right)^{1/2},
\end{aligned}$$

where C_1 depends on c and on the L^2 norm of $\dot{\Psi}$. Note that V_t can be written as a (finite) linear combination of iterated (Stratonovich) integrals of order bounded by k_p+1. Further, c.f. (2.3), (2.4), and Lemma 2.1, for all $s \in [0,1]$,

$$I_r(s) \leq C_r \omega(0,1)^{r/p'} \leq C_K(\bar{I}^{r/p'} + \hat{I}^{r/p'}).$$

Therefore, using Lemma 1 in [2], and the uniform convergence to 0 with ε of $\hat{I}$ on the set $\|X\| < \varepsilon$, c.f. the first line in the proof of Theorem 1.7, it follows that it is enough to show that for all $r \leq k_p + 1$, $3 > p' > p > 2$, and all C_1,

$$\limsup_{\varepsilon \to 0} \mathbb{E}^\varepsilon\left(\exp C_1 |\bar{I}^{r/p'}|\right) < \infty\,, \tag{2.18}$$

and that for all $K > 0$,

$$\limsup_{\varepsilon \to 0} \mathbb{P}^\varepsilon(\bar{I} > K) = 0\,. \tag{2.19}$$

(2.18) is an immediate consequence of (2.7) because $(k_p+1) \leq 2(p-1)/(p-2) < 4(p-1)/(p-2)$. To see (2.19), it is enough, c.f. (2.2), to show that

$$\sum_{n=1}^{\infty} n^{C_\gamma} \sum_{k=0}^{2^n-1} \mathbb{E}^\varepsilon\left(|A_{\frac{k}{2^n},\frac{k+1}{2^n}}|^2\right)^{p'/4} \to_{\varepsilon \to 0} 0\,. \tag{2.20}$$

Using the complete convexity of the norm, it follows from [12] and **(P1)** that

$$\mathbb{E}^\varepsilon\left(|A_{\frac{k}{2^n},\frac{k+1}{2^n}}|^2\right) \leq C_q \mathbb{E}^\varepsilon\left(\int_{k/2^n}^{(k+1)/2^n} (X_1(s) - X_1(k/2^n))^2 ds\right) \leq C_q C_p^2 \varepsilon^2 2^{-n} 2^{-2n/p}\,,$$

which, because $3 > p' > p > 2$, implies (2.20). $\square$

Remark 2.21 *Note that in order to prove Corollary 1.16, one needs much less than the conclusion of Theorem 1.7. In particular, one needs to consider only certain special functionals of the iterated Stratonovich integrals* I_k.

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