# Quenched Limits for Transient, Zero Speed One-Dimensional Random Walk in Random Environment

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April 13, 2007

**Abstract**

We consider a nearest-neighbor, one dimensional random walk $\{X_n\}_{n\geq 0}$ in a random i.i.d. environment, in the regime where the walk is transient but with zero speed, so that $X_n$ is of order $n^s$ for some $s<1$. Under the quenched law (i.e., conditioned on the environment), we show that no limit laws are possible: there exist sequences $\{n_k\}$ and $\{x_k\}$ depending on the environment only, such that $X_{n_k}-x_k=o(\log n_k)^2$ (a *localized regime*). On the other hand, there exist sequences $\{t_m\}$ and $\{s_m\}$ depending on the environment only, such that $\log s_m/\log t_m \to s<1$ and $P_\omega(X_{t_m}/s_m \leq x)\to 1/2$ for all $x>0$ and $\to 0$ for $x\leq 0$ (a *spread out regime*).

Key Words: Random walk, random environment.
AMS (1991) subject classifications: Primary 60K37, Secondary 60F05, 82C41, 82D30.

## 1 Introduction and Statement of Main Results

Let $\Omega=[0,1]^{\mathbb{Z}}$, and let $\mathcal{F}$ be the Borel $\sigma-$algebra on $\Omega$. A random environment is an $\Omega$-valued random variable $\omega=\{\omega_i\}_{i\in\mathbb{Z}}$ with distribution $P$. In this paper we will assume that $P$ is a product measure on $\Omega$.
The *quenched* law $P_\omega^x$ for a random walk $X_n$ in the environment $\omega$ is defined by

$$P_\omega^x(X_0=x)=1, \quad \text{and} \quad P_\omega^x\left(X_{n+1}=j|X_n=i\right)=\begin{cases}\omega_i & \text{if } j=i+1,\\ 1-\omega_i & \text{if } j=i-1.\end{cases}$$

$\mathbb{Z}^{\mathbb{N}}$ is the space for the paths of the random walk $\{X_n\}_{n\in\mathbb{N}}$, and $\mathcal{G}$ denotes the $\sigma-$algebra generated by the cylinder sets. Note that for each $\omega\in\Omega$, $P_\omega$ is a probability measure on $\mathcal{G}$, and for each $G\in\mathcal{G}$, $P_\omega^x(G):(\Omega,\mathcal{F})\to[0,1]$ is a measurable function of $\omega$. Expectations under the law $P_\omega^x$ are denoted $E_\omega^x$.
The *annealed* law for the random walk in random environment $X_n$ is defined by

$$\mathbb{P}^x(F\times G)=\int_F P_\omega^x(G)P(d\omega), \quad F\in\mathcal{F}, G\in\mathcal{G}\,.$$

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‡The research of both authors was partially supported by NSF grant DMS-0503775.

For ease of notation we will use $P_\omega$ and $\mathbb{P}$ in place of $P_\omega^0$ and $\mathbb{P}^0$ respectively. We will also use $\mathbb{P}^x$ to refer to the marginal on the space of paths, i.e. $\mathbb{P}^x(G) = \mathbb{P}^x(\Omega \times G) = E_P\left[P_\omega^x(G)\right]$ for $G \in \mathcal{G}$. Expectations under the law $\mathbb{P}$ will be written $\mathbb{E}$.

A simple criterion for recurrence and a formula for the speed of transience was given by Solomon in [14]. For any integers $i \leq j$ define

$$\rho_i := \frac{1-\omega_i}{\omega_i}, \quad \text{and} \quad \Pi_{i,j} := \prod_{k=i}^{j} \rho_k\,, \tag{1}$$

and for $x \in \mathbb{Z}$ define the hitting times

$$T_x := \min\{n \geq 0 : X_n = x\}\,.$$

Then, $X_n$ is transient to the right (resp. to the left) if $E_P(\log \rho_0) < 0$, (resp. $E_P \log \rho_0 > 0$) and recurrent if $E_P(\log \rho_0) = 0$ (henceforth we will write $\rho$ instead of $\rho_0$ in expectations involving only $\rho_0$). In the case where $E_P \log \rho < 0$ (transience to the right), Solomon established the following law of large numbers

$$v_P := \lim_{n\to\infty} \frac{X_n}{n} = \lim_{n\to\infty} \frac{n}{T_n} = \frac{1}{\mathbb{E}T_1}, \quad \mathbb{P}-a.s.$$

For any integers $i < j$ define

$$W_{i,j} := \sum_{k=i}^{j} \Pi_{k,j}, \quad \text{and} \quad W_j := \sum_{k\leq j} \Pi_{k,j}\,. \tag{2}$$

When $E_P \log \rho < 0$, it was shown in [14],[15] that

$$E_\omega^j T_{j+1} = 1 + 2W_j < \infty, \quad P-a.s., \tag{3}$$

and thus $v_P = 1/(1+2E_P W_0)$. Since $P$ is a product measure, $E_P W_0 = \sum_{k=1}^\infty \left(E_P \rho\right)^k$. In particular, $v_P = 0$ if $E_P \rho \geq 1$.

Kesten, Kozlov, and Spitzer [10] determined the annealed limiting distribution of a RWRE with $E_P \log \rho < 0$, i.e. transient to the right. They derived the limiting distributions for the walk by first establishing a stable limit law of index $s$ for $T_n$, where $s$ is defined by the equation

$$E_P \rho^s = 1\,.$$

In particular, they showed that when $s < 1$ there exists a $b > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n}{n^{1/s}} \leq x\right) = L_{s,b}(x)\,,$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n}{n^s} \leq x\right) = 1 - L_{s,b}(x^{-1/s}), \tag{4}$$

where $L_{s,b}$ is the distribution function for a stable random variable with characteristic function

$$\hat{L}_{s,b}(t) = \exp\left\{-b|t|^s\left(1 - i\frac{t}{|t|}\tan(\pi s/2)\right)\right\}. \tag{5}$$

The value of $b$ was recently identified [4]. While the annealed limiting distributions for transient one-dimensional RWRE have been known for quite a while, the corresponding quenched limiting distributions have remained largely unstudied until recently. Goldsheid [7] and Peterson [13] independently proved that when $s > 2$, a quenched CLT holds with a random (depending on the environment) centering. Previously, in [12] and [15] it was shown that the limiting statement for the quenched CLT with random centering holds in probability rather than almost surely. No other results of quenched limiting distributions are known when $s \leq 2$.

In this paper, we analyze the quenched limiting distributions of a one-dimensional transient RWRE in the case $s < 1$. One could expect that the quenched limiting distributions are of the same type as the annealed limiting distributions since annealed probabilities are averages of quenched probabilities. However, this turns out not to be the case. In fact, a consequence of our main results, Theorems 1.1, 1.2, and 1.3 below, is that the annealed stable behavior of $T_n$ comes from fluctuations in the environment.

Throughout the paper, we will make the following assumptions:

**Assumption 1.** *$P$ is a product measure on $\Omega$ such that*

$$E_P \log \rho < 0 \quad \text{and} \quad E_P \rho^s = 1 \text{ for some } s > 0. \tag{6}$$

**Assumption 2.** *There exists $\rho_{\max} < \infty$ such that $P(\rho < \rho_{\max}) = 1$, and the distribution of $\log \rho$ is non-lattice under $P$.*

**Note:** Since $E_P \rho^\gamma$ is a convex function of $\gamma$, the two statements in (6) give that $E_P \rho^\gamma < 1$ for all $\gamma < s$ and $E_P \rho^\gamma > 1$ for all $\gamma > s$. Assumption 1 contains the essential assumption necessary for the walk to be transient. The main results of this paper are for $s < 1$ (the zero-speed regime). The technical conditions contained in Assumption 2 simplify our argument; we recall that the non-lattice assumption was also invoked in [10].

Define the "ladder locations" $\nu_i$ of the environment by

$$\nu_0 = 0, \quad \text{and} \quad \nu_i = \begin{cases} \inf\{n > \nu_{i-1} : \Pi_{\nu_{i-1}, n-1} < 1\}, & i \geq 1, \\ \sup\{j < \nu_{i+1} : \Pi_{k,j-1} < 1, \quad \forall k < j\}, & i \leq -1\,. \end{cases} \tag{7}$$

Throughout the remainder of the paper we will let $\nu = \nu_1$. We will sometimes refer to sections of the environment between $\nu_{i-1}$ and $\nu_i - 1$ as "blocks" of the environment. Note that the block between $\nu_{-1}$ and $\nu_0 - 1$ is different from all the other blocks between consecutive ladder locations. Define the measure $Q$ on environments by $Q(\cdot) := P(\cdot|\mathcal{R})$, where the event

$$\mathcal{R} := \{\omega \in \Omega : \Pi_{-k,-1} < 1, \quad \forall k \geq 1\}.$$

Note that $P(\mathcal{R}) > 0$ since $E_P \log \rho < 0$. $Q$ is defined so that the blocks of the environment between ladder locations are i.i.d. under $Q$, all with distribution the same as that of the block from 0 to $\nu - 1$ under $P$. In Section 3 we prove the following annealed theorem:

**Theorem 1.1.** *Let Assumptions 1 and 2 hold, and let $s < 1$. Then there exists a $b' > 0$ such that*

$$\lim_{n\to\infty} Q\left(\frac{E_\omega T_{\nu_n}}{n^{1/s}} \leq x\right) = L_{s,b'}(x).$$

We then use Theorem 1.1 to prove the following two theorems which show that $P - a.s.$ there exist two different random sequences of times (depending on the environment) where the random walk has different limiting behavior. These are the main results of the paper.

**Theorem 1.2.** *Let Assumptions 1 and 2 hold, and let $s < 1$. Then $P$-a.s. there exist random subsequences $t_m = t_m(\omega)$ and $u_m = u_m(\omega)$, such that for any $\delta > 0$,*

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m} - u_m}{(\log t_m)^2} \in [-\delta, \delta]\right) = 1.$$

**Theorem 1.3.** *Let Assumptions 1 and 2 hold, and let $s < 1$. Then $P$-a.s. there exists a random subsequence $n_{k_m} = n_{k_m}(\omega)$ of $n_k = 2^{2^k}$ and a random sequence $t_m = t_m(\omega)$, such that*

$$\lim_{m\to\infty} \frac{\log t_m}{\log n_{k_m}} = \frac{1}{s},$$

*and*

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m}}{n_{k_m}} \le x\right) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{2} & \text{if } 0 < x < \infty \end{cases}.$$

Note that Theorems 1.2 and 1.3 preclude the possiblity of a quenched analogue of the annealed statement (4). It should be noted that in [6], Gantert and Shi prove that when $s \le 1$, there exists a random sequence of times $t_m$ at which the local time of the random walk at a single site is a positive fraction of $t_m$. This is related to the statement of Theorem 1.2, but we do not see a simple argument which directly implies Theorem 1.2 from the results of [6].

As in [10], limiting distributions for $X_n$ arise from first studying limiting distributions for $T_n$. Thus, to prove Theorem 1.3 we first prove that there exists random subsequences $x_m = x_m(\omega)$ and $v_{m,\omega}$ in which

$$\lim_{m\to\infty} P_\omega\left(\frac{T_{x_m} - E_\omega T_{x_m}}{\sqrt{v_{m,\omega}}} \le y\right) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt =: \Phi(y)\,.$$

We actually prove a stronger statement than this in Theorem 5.7 below, where we prove that all $x_m$ "near" a subsequence $n_{k_m}$ of $n_k = 2^{2^k}$ have the same Gaussian behavior (what we mean by "near" the subsequence $n_{k_m}$ is made precise in the statement of the theorem).

The structure of the paper is as follows: In Section 2 we prove some introductory lemmas which will be used throught the paper. Section 3 is devoted to proving Theorem 1.1. In Section 4 we use the latter to prove Theorem 1.2. In Section 5 we prove the existence of random subsequences $\{n_k\}$ where $T_{n_k}$ is approximately gaussian, and use this fact to prove Theorem 1.3. Section 6 contains the proof of the following technical theorem which is used throughout the paper.

**Theorem 1.4.** *Let Assumptions 1 and 2 hold. Then there exists a constant $K_\infty \in (0,\infty)$ such that*

$$Q(E_\omega T_\nu > x) \sim K_\infty x^{-s}$$

The proof of Theorem 1.4 is based on results from [9] and mimics the proof of tail asymptotics in [10].

# 2 Introductory Lemmas

Before proceeding with the proofs of the main theorems we mention a few easy lemmas which will be used throughout the rest of the paper. Recall the definitions of $\Pi_{1,k}$ and $W_i$ in (1) and (2).

**Lemma 2.1.** *For any $c < -E_P \log\rho$, there exist $\delta_c, A_c > 0$ such that*

$$P(\Pi_{1,k} > e^{-ck}) = P\left(\frac{1}{k}\sum_{i=1}^{k} \log\rho_i > -c\right) \le A_c e^{-\delta_c k}. \tag{8}$$

*Also, there exist constant $C_1, C_2 > 0$ such that $P(\nu > x) \le C_1 e^{-C_2 x}$ for all $x \ge 0$.*

*Proof.* First, note that due to Assumption 1, $\log\rho$ has negative mean and finite exponential moments in a neighborhood of zero. If $c < -E_P \log\rho$, Cramér's Theorem [3, Theorem 2.2.3] then yields (8). By the definition of $\nu$ we have $P(\nu > x) \le P(\Pi_{0,\lfloor x\rfloor -1} \ge 1)$, which together with (8) completes the proof of the lemma. ☐

From [9, Theorem 5], there exist constants $K, K_1 > 0$ such that for all $i$

$$P(W_i > x) \sim K x^{-s}, \quad \text{and} \quad P(W_i > x) \le K_1 x^{-s}\,. \tag{9}$$

The tails of $W_{-1}$, however, are different (under the measure $Q$), as the following lemma shows.

**Lemma 2.2.** *There exist constants $C_3, C_4 > 0$ such that $Q(W_{-1} > x) \leq C_3 e^{-C_4 x}$ for all $x \geq 0$.*

*Proof.* Since $\Pi_{i,-1} < 1$, $\quad Q - a.s.$ we have $W_{-1} < k + \sum_{i<-k} \Pi_{i,-1}$ for any $k > 0$. Also, note that from (8) we have $Q(\Pi_{-k,-1} > e^{-ck}) \leq A_c e^{-\delta_c k}/P(\mathcal{R})$. Thus,

$$Q(W_{-1} > x) \leq Q\left(\frac{x}{2} + \sum_{k=\frac{x}{2}}^{\infty} e^{-ck} > x\right) + Q\left(\Pi_{-k,-1} > e^{-ck}, \text{ for some } k \geq \frac{x}{2}\right)$$
$$\leq \mathbf{1}_{\frac{x}{2}+\frac{1}{1-e^{-c}}>x} + \sum_{k=\frac{x}{2}}^{\infty} Q(\Pi_{-k,-1} > e^{-ck}) \leq \mathbf{1}_{\frac{1}{1-e^{-c}}>\frac{x}{2}} + \mathcal{O}\left(e^{-\delta_c x/2}\right).$$

□

We also need a few more definitions that will be used throughout the paper. For any $i \leq k$,

$$R_{i,k} := \sum_{j=i}^{k} \Pi_{i,j}, \quad \text{and} \quad R_i := \sum_{j=i}^{\infty} \Pi_{i,j}. \tag{10}$$

Note that since $P$ is a product measure, $R_{i,k}$ and $R_i$ have the same distributions as $W_{i,k}$ and $W_i$ respectively. In particular with $K, K_1$ the same as in (9),

$$P(R_i > x) \sim K x^{-s}, \quad \text{and} \quad P(R_i > x) \leq K_1 x^{-s}. \tag{11}$$

# 3 Stable Behavior of Expected Crossing Time

Recall from Theorem 1.4 that there exists $K_\infty > 0$ such that $Q(E_\omega T_\nu > x) \sim K_\infty x^{-s}$. Thus $E_\omega T_\nu$ is in the domain of attraction of a stable distribution. Also, from the comments after the definition of $Q$ in the introduction it is evident that under $Q$, the environment $\omega$ is stationary under shifts of the ladder times $\nu_i$. Thus, under $Q$, $\{E_\omega^{\nu_{i-1}} T_{\nu_i}\}_{i\in\mathbb{Z}}$ is a stationary sequence of random variables. Therefore, it is reasonable to expect that $n^{-1/s} E_\omega T_{\nu_n} = n^{-1/s} \sum_{i=1}^{n} E_\omega^{\nu_{i-1}} T_{\nu_i}$ converge in distribution to a stable distribution of index $s$. The main obstacle to proving this is that the random variables $E_\omega^{\nu_{i-1}} T_{\nu_i}$ are not independent. This dependence, however, is rather weak. The strategy of the proof of Theorem 1.1 is to first show that we need only consider the blocks where the expected crossing time $E_\omega^{\nu_{i-1}} T_{\nu_i}$ is relatively large. These blocks will then be separated enough to make the expected crossing times essentially independent.

For every $k \in \mathbb{Z}$, define

$$M_k := \max\{\Pi_{\nu_{k-1},j} : \nu_{k-1} \leq j < \nu_k\}. \tag{12}$$

Theorem 1 in [8] gives that there exists a constant $C_5 > 0$ such that

$$Q(M_1 > x) \sim C_5 x^{-s}. \tag{13}$$

Thus $M_1$ and $E_\omega T_\nu$ have similar tails under $Q$. We will now show that $E_\omega T_\nu$ cannot be too much larger than $M_1$. From (3) we have that

$$E_\omega T_\nu = \nu + 2\sum_{i=0}^{\nu-1} W_j = \nu + 2W_{-1} R_{0,\nu-1} + 2\sum_{i=0}^{\nu-1} R_{i,\nu-1}. \tag{14}$$

From the definitions of $\nu$ and $M_1$ we have that $R_{i,\nu-1} \leq (\nu - i)M_1 \leq \nu M_1$ for any $0 \leq i < \nu$. Therefore, $E_\omega T_\nu \leq \nu + 2W_{-1}\nu M_1 + 2\nu^2 M_1$. Thus, given any $0 < \alpha < \beta$ and $\delta > 0$ we have

$$Q(E_\omega T_\nu > \delta n^\beta, M_1 \leq n^\alpha) \leq Q(\nu + 2W_{-1}\nu n^\alpha + 2\nu^2 n^\alpha > \delta n^\beta) \tag{15}$$
$$\leq Q(W_{-1} > n^{(\beta-\alpha)/2}) + Q\left(\nu^2 > n^{(\beta-\alpha)/2}\right) = o\left(e^{-n^{(\beta-\alpha)/5}}\right),$$

where the second inequality holds for all $n$ large enough and the last equality is a result of Lemmas 2.1 and 2.2. We now show that only the ladder times with $M_k > n^{(1-\varepsilon)/s}$ contribute to the limiting distribution of $n^{-1/s}E_\omega T_{\nu_n}$.

**Lemma 3.1.** *Assume $s<1$. Then for any $\varepsilon>0$ and any $\delta>0$ there exists an $\eta>0$ such that*

$$\lim_{n\to\infty} Q\left(\sum_{i=1}^n (E_\omega^{\nu_{i-1}}T_{\nu_i})\mathbf{1}_{M_i\leq n^{(1-\varepsilon)/s}} > \delta n^{1/s}\right) = o(n^{-\eta})\,.$$

*Proof.* First note that

$$Q\left(\sum_{i=1}^n (E_\omega^{\nu_{i-1}}T_{\nu_i})\mathbf{1}_{M_i\leq n^{(1-\varepsilon)/s}} > \delta n^{1/s}\right) \leq Q\left(\sum_{i=1}^n (E_\omega^{\nu_{i-1}}T_{\nu_i})\mathbf{1}_{E_\omega^{\nu_{i-1}}T_{\nu_i}\leq n^{(1-\frac{\varepsilon}{2})/s}} > \delta n^{1/s}\right) + nQ\left(E_\omega T_\nu > n^{(1-\frac{\varepsilon}{2})/s}, M_1\leq n^{(1-\varepsilon)/s}\right)\,.$$

By (15), the last term above decreases faster than any power of $n$. Thus it is enough to prove that for any $\delta,\varepsilon>0$ there exists an $\eta>0$ such that

$$Q\left(\sum_{i=1}^n (E_\omega^{\nu_{i-1}}T_{\nu_i})\mathbf{1}_{E_\omega^{\nu_{i-1}}T_{\nu_i}\leq n^{(1-\varepsilon)/s}} > \delta n^{1/s}\right) = o(n^{-\eta})\,.$$

Next, pick $C\in\left(1,\frac{1}{s}\right)$ and let $J_{C,\varepsilon,k,n} := \left\{i\leq n : n^{(1-C^k\varepsilon)/s} < E_\omega^{\nu_{i-1}}T_{\nu_i} \leq n^{(1-C^{k-1}\varepsilon)/s}\right\}$. Let $k_0=k_0(C,\varepsilon)$ be the smallest integer such that $(1-C^k\varepsilon)\leq 0$. Then for any $k<k_0$ we have

$$Q\left(\sum_{i\in J_{C,\varepsilon,k,n}} E_\omega^{\nu_{i-1}}T_{\nu_i} > \delta n^{1/s}\right) \leq Q\left(\#J_{C,\varepsilon,k,n} > \delta n^{1/s-(1-C^{k-1}\varepsilon)/s}\right) \leq \frac{nQ(E_\omega T_\nu > n^{(1-C^k\varepsilon)/s})}{\delta n^{C^{k-1}\varepsilon/s}} \sim \frac{K_\infty}{\delta} n^{-C^{k-1}\varepsilon(\frac{1}{s}-C)}\,,$$

where the asymptotics in the last line above is from Theorem 1.4. Letting $\eta = \frac{\varepsilon}{2}\left(\frac{1}{s}-C\right)$ we have for any $k<k_0$ that

$$Q\left(\sum_{i\in J_{C,\varepsilon,k,n}} E_\omega^{\nu_{i-1}}T_{\nu_i} > \delta n^{2/s}\right) = o(n^{-\eta}). \tag{16}$$

Finally, note that

$$Q\left(\sum_{i=1}^n (E_\omega^{\nu_{i-1}}T_{\nu_i})\mathbf{1}_{E_\omega^{\nu_{i-1}}T_{\nu_i}\leq n^{(1-C^{k_0-1}\varepsilon)/s}} \geq \delta n^{1/s}\right) \leq \mathbf{1}_{n^{1+(1-C^{k_0-1}\varepsilon)/s}\geq \delta n^{1/s}}. \tag{17}$$

However, since $C^{k_0}\varepsilon\geq 1 > Cs$ we have $C^{k_0-1}\varepsilon > s$, which implies that the right side of (17) vanishes for all $n$ large enough. Therefore, combining (16) and (17) we have

$$Q\left(\sum_{i=1}^n (E_\omega^{\nu_{i-1}}T_{\nu_i})\mathbf{1}_{E_\omega^{\nu_{i-1}}\leq n^{(1-\varepsilon)/s}} > \delta n^{1/s}\right) \leq \sum_{k=1}^{k_0-1} Q\left(\sum_{i\in I_{C,\varepsilon,k,n}} E_\omega^{\nu_{i-1}}T_{\nu_i} > \frac{\delta}{k_0}n^{1/s}\right) + Q\left(\sum_{i=1}^n (E_\omega^{\nu_{i-1}}T_{\nu_i})\mathbf{1}_{E_\omega^{\nu_{i-1}}T_{\nu_i}\leq n^{(1-C^{k_0-1}\varepsilon)/s}} \geq \frac{\delta}{k_0}n^{1/s}\right) = o(n^{-\eta}).$$

□

In order to make the crossing times of the significant blocks essentially independent, we introduce some reflections to the RWRE. For $n = 1, 2, \ldots$, define

$$b_n := \lfloor \log^2(n) \rfloor. \tag{18}$$

Let $\bar{X}_t^{(n)}$ be the random walk that is the same as $X_t$ with the added condition that after reaching $\nu_k$ the environment is modified by setting $\omega_{\nu_{k-b_n}} = 1$ , i.e. never allow the walk to backtrack more than $\log^2(n)$ ladder times. Denote by $\bar{T}_x^{(n)}$ the corresponding hitting times. The following lemmas show that we can add reflections to the random walk without changing the expected crossing time by very much.

**Lemma 3.2.** *There exist $B, \delta' > 0$ such that for any $x > 0$*

$$Q\left( E_\omega T_\nu - E_\omega \bar{T}_\nu^{(n)} > x \right) \le B(x^{-s} \vee 1) e^{-\delta' b_n}.$$

*Proof.* First, note that for any $n$ the formula for $E_\omega \bar{T}_\nu^{(n)}$ is the same as for $E_\omega T_\nu$ in (14) except with $\rho_{\nu_{-b_n}} = 0$. Thus $E_\omega T_\nu$ can be written as

$$E_\omega T_\nu = E_\omega \bar{T}_\nu^{(n)} + 2(1 + W_{\nu_{-b_n}-1}) \Pi_{\nu_{-b_n},-1} R_{0,\nu-1}. \tag{19}$$

Now, since $\nu_{-b_n} \le -b_n$ we have

$$Q\left( \Pi_{\nu_{-b_n},-1} > e^{-cb_n} \right) \le \sum_{k=b_n}^{\infty} Q\left( \Pi_{-k,-1} > e^{-ck} \right) \le \sum_{k=b_n}^{\infty} \frac{1}{P(\mathcal{R})} P\left( \Pi_{-k,-1} > e^{-ck} \right).$$

Applying (8), we have that for any $0 < c < -E_P \log \rho$ there exist $A', \delta_c > 0$ such that $Q\left( \Pi_{\nu_{-b_n},-1} > e^{-cb_n} \right) \le A' e^{-\delta_c b_n}$. Therefore, for any $x > 0$,

$$\begin{aligned} Q\left( E_\omega T_\nu - E_\omega \bar{T}_\nu^{(n)} > x \right) &\le Q\left( 2(1 + W_{\nu_{-b_n}-1}) \Pi_{\nu_{-b_n},-1} R_{0,\nu-1} > x \right) \\ &\le Q\left( 2(1 + W_{\nu_{-b_n}-1}) R_{0,\nu-1} > x e^{cb_n} \right) + A' e^{-\delta_c b_n} \\ &= Q\left( 2(1 + W_{-1}) R_{0,\nu-1} > x e^{cb_n} \right) + A' e^{-\delta_c b_n}, \end{aligned} \tag{20}$$

where the equality in the second line is due to the fact that the blocks of the environment are i.i.d under $Q$. Also, from (14) and Theorem 1.4 we have

$$Q\left( 2(1 + W_{-1}) R_{0,\nu-1} > x e^{cb_n} \right) \le Q\left( E_\omega T_\nu > x e^{cb_n} \right) \sim K_\infty x^{-s} e^{-csb_n}. \tag{21}$$

Combining (20) and (21) finishes the proof. ☐

**Lemma 3.3.** *For any $x > 0$, $\varepsilon > 0$, and any integer $n \ge 1$,*

$$Q\left( E_\omega \bar{T}_\nu^{(n)} > x n^{1/s}, M_1 > n^{(1-\varepsilon)/s} \right) \sim K_\infty x^{-s} \frac{1}{n}, \quad \textit{as } x \to \infty. \tag{22}$$

*Proof.* Since adding reflections only decreases the crossing times, we can get an upper bound using Theorem 1.4, that is

$$Q\left( E_\omega \bar{T}_\nu^{(n)} > x n^{1/s}, M_1 > n^{(1-\varepsilon)/s} \right) \le Q(E_\omega T_\nu > x n^{1/s}) \sim K_\infty x^{-s} \frac{1}{n}, \quad \text{as } x \to \infty. \tag{23}$$

To get a lower bound we first note that for any $\delta > 0$,

$$\begin{aligned} Q\left( E_\omega T_\nu > (1+\delta) x n^{1/s} \right) &\le Q\left( E_\omega \bar{T}_\nu^{(n)} > x n^{1/s}, M_1 > n^{(1-\varepsilon)/s} \right) + Q\left( E_\omega T_\nu - E_\omega \bar{T}_\nu^{(n)} > \delta x n^{1/s} \right) \\ &\quad + Q\left( E_\omega T_\nu > (1+\delta) x n^{1/s}, M_1 \le n^{(1-\varepsilon)/s} \right) \\ &\le Q\left( E_\omega \bar{T}_\nu^{(n)} > x n^{1/s}, M_1 > n^{(1-\varepsilon)/s} \right) + o(1/n), \end{aligned} \tag{24}$$

where the second inequality is from (15) and Lemma 3.2. The asymptotics in (22) then follow from (23) and (24) by using Theorem 1.4 and then letting $\delta \to 0$. ☐

Our general strategy is to show that the partial sums

$$\frac{1}{n^{1/s}}\sum_{k=1}^{n} E_\omega^{\nu_{k-1}}\bar{T}_{\nu_k}^{(n)}\mathbf{1}_{M_k>n^{(1-\varepsilon)/s}}$$

converge in distribution to a stable law of parameter $s$. To establish this, we will need bounds on the mixing properties of the sequence $E_\omega^{\nu_{k-1}}\bar{T}_{\nu_k}^{(n)}\mathbf{1}_{M_k>n^{(1-\varepsilon)/s}}$. As in [11], we say that an array $\{\xi_{n,k} : k\in\mathbb{Z}, n\in\mathbb{N}\}$ which is stationary in rows is $\alpha-$mixing if $\lim_{k\to\infty}\limsup_{n\to\infty}\alpha_n(k)=0$, where

$$\alpha_n(k) := \sup\left\{|P(A\cap B)-P(A)P(B)| : A\in\sigma\left(\ldots,\xi_{n,-1},\xi_{n,0}\right), B\in\sigma\left(\xi_{n,k},\xi_{n,k+1},\ldots\right)\right\}.$$

**Lemma 3.4.** *For any $0<\varepsilon<\frac{1}{2}$, under the measure $Q$, the array of random variables $\{E_\omega^{\nu_{k-1}}\bar{T}_{\nu_k}^{(n)}\mathbf{1}_{M_k>n^{(1-\varepsilon)/s}}\}_{k\in\mathbb{Z},n\in\mathbb{N}}$ is $\alpha$-mixing, with*

$$\sup_{k\in[1,\log^2 n]}\alpha_n(k) = o(n^{-1+2\epsilon}), \quad \alpha_n(k)=0, \quad \forall k>\log^2 n.$$

*Proof.* Fix $\varepsilon\in(0,\frac{1}{2})$. For ease of notation, define $\xi_{n,k} := E_\omega^{\nu_{k-1}}\bar{T}_{\nu_k}^{(n)}\mathbf{1}_{M_k>n^{(1-\varepsilon)/s}}$. As we mentioned before, under $Q$ the environment is stationary under shifts of the sequence of ladder locations and thus $\xi_{n,k}$ is stationary in rows under $Q$.

If $k>\log^2(n)$, then because of the reflections, $\sigma\left(\ldots,\xi_{n,-1},\xi_{n,0}\right)$ and $\sigma\left(\xi_{n,k},\xi_{n,k+1},\ldots\right)$ are independent and so $\alpha_n(k)=0$. To handle the case when $k\le\log^2(n)$, fix $A\in\sigma\left(\ldots,\xi_{n,-1},\xi_{n,0}\right)$ and $B\in\sigma\left(\xi_{n,k},\xi_{n,k+1},\ldots\right)$, and define the event

$$C_{n,\varepsilon} := \{M_j\le n^{(1-\varepsilon)/s}, \text{for } 1\le j\le b_n\} = \{\xi_{n,j}=0, \text{for } 1\le j\le b_n\}.$$

For any $j>b_n$, we have that $\xi_{n,j}$ only depends on the environment to the right of zero. Thus,

$$Q(A\cap B\cap C_{n,\varepsilon}) = Q(A)Q(B\cap C_{n,\varepsilon})$$

since $B\cap C_{n,\varepsilon}\in\sigma(\omega_0,\omega_1,\ldots)$. Also, note that by (13) we have $P(C_{n,\varepsilon}^c)\le b_nQ(M_1>n^{(1-\varepsilon)/s}) = o(n^{-1+2\varepsilon})$. Therefore,

$$\begin{aligned}|Q(A\cap B)-Q(A)Q(B)| \le{}& |Q(A\cap B)-Q(A\cap B\cap C_{n,\varepsilon})| \\ &+ |Q(A\cap B\cap C_{n,\varepsilon})-Q(A)Q(B\cap C_{n,\varepsilon})| \\ &+ Q(A)|Q(B\cap C_{n,\varepsilon})-Q(B)| \le 2Q(C_{n,\varepsilon}^c) = o(n^{-1+2\varepsilon})\end{aligned}$$

□

***Proof of Theorem 1.1.***
First, we show that the partial sums

$$\frac{1}{n^{1/s}}\sum_{k=1}^{n} E_\omega^{\nu_{k-1}}\bar{T}_{\nu_k}^{(n)}\mathbf{1}_{M_k>n^{(1-\varepsilon)/s}}$$

converge in distribution to a stable random variable of parameter $s$. To this end, we will apply [11, Theorem 5.1(III)]. We now verify the conditions of that theorem. The first condition that needs to be satisfied is:

$$\lim_{n\to\infty} nQ\left(n^{-1/s}E_\omega\bar{T}_\nu^{(n)}\mathbf{1}_{M_1>n^{(1-\varepsilon)/s}}>x\right) = K_\infty x^{-s}.$$

However, this is exactly the content of Lemma 3.3.
Secondly, we need a sequence $m_n$ such that $m_n\to\infty$, $m_n=o(n)$ and $n\alpha_n(m_n)\to 0$ and such that for any $\delta>0$,

$$\lim_{n\to\infty}\sum_{k=1}^{m_n} nQ\left(E_\omega\bar{T}_\nu^{(n)}\mathbf{1}_{M_1>n^{(1-\varepsilon)/s}}>\delta n^{1/s}, E_\omega^{\nu_k}\bar{T}_{\nu_{k+1}}^{(n)}\mathbf{1}_{M_{k+1}>n^{(1-\varepsilon)/s}}>\delta n^{1/s}\right)=0. \qquad (25)$$

However, by the independence of $M_1$ and $M_{k+1}$ for any $k \geq 1$, the probability inside the sum is less than $Q(M_1 > n^{(1-\varepsilon)/s})^2$. By (13) this last expression is $\sim C_5 n^{-2+2\varepsilon}$. Thus letting $m_n = n^{1/2-\varepsilon}$ yields (25). (Note that by Lemma 3.4, $n\alpha_n(m_n) = 0$ for all $n$ large enough.)
Finally, we need to show that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} nE_Q \left[ E_\omega \bar{T}_\nu^{(n)} \mathbf{1}_{M_1 > n^{(1-\varepsilon)/s}} \mathbf{1}_{E_\omega \bar{T}_\nu^{(n)} \leq \delta} \right] = 0 \,. \tag{26}$$

Now, by (23) there exists a constant $C_6 > 0$ such that for any $x > 0$,

$$Q\left( E_\omega \bar{T}_\nu^{(n)} > x n^{1/s}, M_1 > n^{(1-\varepsilon)/s} \right) \leq C_6 x^{-s} \frac{1}{n}.$$

Then using this we have

$$\begin{aligned} nE_Q \left[ E_\omega \bar{T}_\nu^{(n)} \mathbf{1}_{M_1 > n^{(1-\varepsilon)/s}} \mathbf{1}_{E_\omega \bar{T}_\nu^{(n)} \leq \delta} \right] &= n \int_0^\delta Q\left( E_\omega \bar{T}_\nu^{(n)} > x n^{1/s}, M_1 > n^{(1-\varepsilon)/s} \right) dx \\ &\leq C_6 \int_0^\delta x^{-s} dx = \frac{C_6 \delta^{1-s}}{1-s} \,, \end{aligned}$$

where the last integral is finite since $s < 1$. (26) follows.

Having checked all its hypotheses, [11, Theorem 5.1(III)] applies and yields that there exists a $b' > 0$ such that

$$Q\left( \frac{1}{n^{1/s}} \sum_{k=1}^n E_\omega^{\nu_{k-1}} \bar{T}_{\nu_k}^{(n)} \mathbf{1}_{M_k > n^{(1-\varepsilon)/s}} \leq x \right) = L_{s,b'}(x) \,, \tag{27}$$

where the characteristic function for the distribution $L_{s,b'}$ is given in (5). To get the limiting distribution of $\frac{1}{n^{1/s}} E_\omega T_{\nu_n}$ we use (19) and re-write this as

$$\begin{aligned} \frac{1}{n^{1/s}} E_\omega T_{\nu_n} &= \frac{1}{n^{1/s}} \sum_{k=1}^n E_\omega^{\nu_{k-1}} \bar{T}_{\nu_k}^{(n)} \mathbf{1}_{M_k > n^{(1-\varepsilon)/s}} && (28) \\ &\quad + \frac{1}{n^{1/s}} \sum_{k=1}^n E_\omega^{\nu_{k-1}} \bar{T}_{\nu_k}^{(n)} \mathbf{1}_{M_k \leq n^{(1-\varepsilon)/s}} && (29) \\ &\quad + \frac{1}{n^{1/s}} \left( E_\omega T_{\nu_n} - E_\omega \bar{T}_{\nu_n}^{(n)} \right) \,. && (30) \end{aligned}$$

Lemma 3.1 gives that (29) converges in distribution (under Q) to 0. Also, we can use Lemma 3.2 to show that (30) converges in distribution to 0 as well. Indeed, for any $\delta > 0$

$$Q\left( E_\omega T_{\nu_n} - E_\omega \bar{T}_{\nu_n}^{(n)} > \delta n^{1/s} \right) \leq nQ\left( E_\omega T_\nu - E_\omega \bar{T}_\nu^{(n)} > \delta n^{1/s-1} \right) = \mathcal{O}\left( n^s e^{-\delta' b_n} \right) \,.$$

Therefore $n^{-1/s} E_\omega T_{\nu_n}$ has the same limiting distribution (under $Q$) as the right side of (28), which by (27) is an $s$-stable distribution with distribution function $L_{s,b'}$. □

# 4 Localization along a subsequence

The goal of this section is to show when $s < 1$ that $P$-a.s. there exists a subsequence $t_m = t_m(\omega)$ of times such that the RWRE is essentially located in a section of the environment of length $\log^2(t_m)$. This will essentially be done by finding a ladder time whose crossing time is *much* larger than all the other ladder times before it. As a first step in this direction we prove that with strictly positive probability this happens in the first $n$ ladder locations. Recall the definition of $M_k$, c.f. (12).

**Lemma 4.1.** *Assume $s < 1$. Then for any $C > 1$ we have*

$$\liminf_{n\to\infty} Q\left(\exists k \in [1, n/2] : M_k \geq C \sum_{j:k\neq j\leq n} E_\omega^{\nu_{j-1}} \bar{T}_{\nu_j}^{(n)}\right) > 0\,.$$

*Proof.* Recall that $\bar{T}_x^{(n)}$ is the hitting time of $x$ by the RWRE modified so that it never backtracks $b_n = \lfloor \log^2(n) \rfloor$ ladder locations.

To prove the lemma, first note that since $C > 1$ and $E_\omega^{\nu_{k-1}} \bar{T}_{\nu_k}^{(n)} \geq M_k$ there can only be at most one $k \leq n$ with $M_k \geq C \sum_{k\neq j\leq n} E_\omega^{\nu_{j-1}} \bar{T}_{\nu_j}^{(n)}$. Therefore

$$Q\left(\exists k \in [1, n/2] : M_k \geq C \sum_{k\neq j\leq n} E_\omega^{\nu_{j-1}} \bar{T}_{\nu_j}^{(n)}\right) = \sum_{k=1}^{n/2} Q\left(M_k \geq C \sum_{k\neq j\leq n} E_\omega^{\nu_{j-1}} \bar{T}_{\nu_j}^{(n)}\right) \tag{31}$$

Now, define the events

$$F_n := \{\nu_j - \nu_{j-1} \leq b_n, \quad \forall j \in (-b_n, n]\}, \quad G_{k,n,\varepsilon} := \{M_j \leq n^{(1-\varepsilon)/s}, \quad \forall j \in (k, k+b_n]\}. \tag{32}$$

$F_n$ and $G_{k,n,\varepsilon}$ are both *typical* events. Indeed, from Lemma 2.1 $Q(F_n^c) \leq (b_n + n)Q(\nu > b_n) = \mathcal{O}(ne^{-C_2 b_n})$, and from (13) we have $Q(G_{k,n,\varepsilon}^c) \leq b_n Q(M_1 > n^{(1-\varepsilon)/s}) = o(n^{-1+2\varepsilon})$. Now, from (3) adjusted for reflections we have for any $j$ that

$$\begin{aligned}
E_\omega^{\nu_{j-1}} \bar{T}_{\nu_j}^{(n)} &= (\nu_j - \nu_{j-1}) + 2 \sum_{l=\nu_{j-1}}^{\nu_j - 1} W_{\nu_{j-1-b_n}, l} \\
&= (\nu_j - \nu_{j-1}) + 2 \sum_{\nu_{j-1}\leq i\leq l<\nu_j} \Pi_{i,l} + 2 \sum_{\nu_{j-1-b_n} < i < \nu_{j-1} \leq l < \nu_j} \Pi_{i,\nu_{j-1}-1} \Pi_{\nu_{j-1}, l} \\
&\leq (\nu_j - \nu_{j-1}) + 2\left(\nu_j - \nu_{j-1}\right)^2 M_j + 2(\nu_j - \nu_{j-1})(\nu_{j-1} - \nu_{j-1-b_n}) M_j,
\end{aligned}$$

where we used the fact that $\Pi_{i,\nu_{j-1}-1} < 1$ for all $i < \nu_{j-1}$ in the last inequality. Then, on the event $F_n \cap G_{k,n,\varepsilon}$ we have for $k+1 \leq j \leq k+b_n$ that

$$E_\omega^{\nu_{j-1}} \bar{T}_{\nu_j}^{(n)} \leq b_n + 2b_n^2 n^{(1-\varepsilon)/s} + 2b_n^3 n^{(1-\varepsilon)/s} \leq 5b_n^3 n^{(1-\varepsilon)/s},$$

where for the first inequality we used that on the event $F_n \cap G_{k,n,\varepsilon}$ we have $\nu_j - \nu_{j-1} \leq b_n$ and $M_1 \leq n^{(1-\varepsilon)/s}$. Then, using this we get

$$\begin{aligned}
Q\left(M_k \geq C \sum_{k\neq j\leq n} E_\omega^{\nu_{j-1}} \bar{T}_{\nu_j}^{(n)}\right) &\geq Q\left(M_k \geq C\left(E_\omega \bar{T}_{\nu_{k-1}}^{(n)} + 5b_n^4 n^{(1-\varepsilon)/s} + E_\omega^{\nu_{k+b_n}} \bar{T}_{\nu_n}^{(n)}\right), F_n, G_{k,n,\varepsilon}\right) \\
&\geq Q\left(M_k \geq Cn^{1/s}, \quad \nu_k - \nu_{k-1} \leq b_n\right) \\
&\quad \times Q\left(E_\omega \bar{T}_{\nu_{k-1}}^{(n)} + 5b_n^4 n^{(1-\varepsilon)/s} + E_\omega^{\nu_{k+b_n}} \bar{T}_{\nu_n}^{(n)} \leq n^{1/s}, \tilde{F}_n, G_{k,n,\varepsilon}\right),
\end{aligned}$$

where $\tilde{F}_n := F_n \backslash \{\nu_k - \nu_{k-1} \leq b_n\}$. In the last inequality we used the fact that $E_\omega^{\nu_{j-1}} \bar{T}_{\nu_j}^{(n)}$ is independent of $M_k$ for $j < k$ or $j > k + b_n$. Note that we can replace $\tilde{F}_n$ by $F_n$ in the last line above becuase it will only make the probability smaller. Then, using the above and the fact that $E_\omega \bar{T}_{\nu_{k-1}}^{(n)} + E_\omega^{\nu_{k+b_n}} \bar{T}_{\nu_n}^{(n)} \leq E_\omega T_{\nu_n}$ we have

$$\begin{aligned}
&Q\left(M_k \geq C \sum_{k\neq j\leq n} E_\omega^{\nu_{j-1}} \bar{T}_{\nu_j}^{(n)}\right) \\
&\quad \geq Q\left(M_k \geq Cn^{1/s}, \quad \nu_k - \nu_{k-1} \leq b_n\right) Q\left(E_\omega T_{\nu_n} \leq n^{1/s} - 5b_n n^{(1-\varepsilon)/s}, F_n, G_{k,n,\varepsilon}\right) \\
&\quad \geq \left(Q(M_1 \geq Cn^{1/s}) - Q(\nu > b_n)\right)\left(Q(E_\omega T_{\nu_n} \leq n^{1/s}(1 - 5b_n n^{-\varepsilon/s})) - Q(F_n^c) - Q(G_{k,n,\varepsilon}^c)\right) \\
&\quad \sim C_5 C^{-s} L_s(1) \frac{1}{n}\,,
\end{aligned}$$

where the asymptotics in the last line are from (13) and Theorem 1.1. Combining the last display and (31) proves the lemma. $\square$

In Section 3, we showed that the proper scaling for $E_\omega T_{\nu_n}$ (or $E_\omega \bar{T}^{(n)}_{\nu_n}$) was $n^{-1/s}$. The following lemma gives a bound on the moderate deviations, under the measure $P$.

**Lemma 4.2.** *Assume $s \leq 1$. Then for any $\delta > 0$,*

$$P\left(E_\omega T_{\nu_n} \geq n^{1/s+\delta}\right) = o(n^{-\delta s/2}) .$$

*Proof.* First, note that

$$P(E_\omega T_{\nu_n} \geq n^{1/s+\delta}) \leq P(E_\omega T_{2\bar{\nu}n} \geq n^{1/s+\delta}) + P(\nu_n \geq 2\bar{\nu}n) , \tag{33}$$

where $\bar{\nu} := E_P \nu$. To handle the second term on the right hand side of (33) we note that since $\nu_n$ is the sum of $n$ i.i.d. copies of $\nu_1$ and since $\nu$ has exponential tails we have that from Cramér's theorem [3, Theorem 2.2.3] that $P(\nu_n/n \geq 2\bar{\nu}) = \mathcal{O}(e^{-\delta' n})$ for some $\delta' > 0$.
To handle the first term on the right hand side of (33) we note that for any $\gamma < s$ we have $E_P(E_\omega T_1)^\gamma < \infty$ This follows from the fact that $P(E_\omega T_1 > x) = P(1 + 2W_0 > x) \sim K2^s x^{-s}$ by (3) and (9). Then, by Chebychev's inequality and the fact that $\gamma < s \leq 1$ we have

$$P\left(E_\omega T_{2\bar{\nu}n} \geq n^{1/s+\delta}\right) \leq \frac{E_P\left(\sum_{k=1}^{2\bar{\nu}n} E_\omega^{k-1} T_k\right)^\gamma}{n^{\gamma(1/s+\delta)}} \leq \frac{2\bar{\nu}n E_P(E_\omega T_1)^\gamma}{n^{\gamma(1/s+\delta)}} . \tag{34}$$

Then, choosing $\gamma$ arbititrarily close to $s$ we can have that this last term is $o(n^{-\delta s/2})$. $\square$

Throughout the remainder of the paper we will use the following subsequences of integers:

$$n_k := 2^{2^k}, \qquad d_k := n_k - n_{k-1} \tag{35}$$

Note that $n_{k-1} = \sqrt{n_k}$ and so $d_k \sim n_k$ as $k \to \infty$.

**Corollary 4.2.1.** *For any $k$ define*

$$\mu_k := \max\left\{E_\omega^{\nu_{j-1}} \bar{T}^{(d_k)}_{\nu_j} : n_{k-1} < j \leq n_k\right\} .$$

*If $s < 1$, then*

$$\lim_{k\to\infty} \frac{E_\omega^{\nu_{n_{k-1}}} \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k}{E_\omega \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k} = 1, \quad P - a.s.$$

*Proof.* Let $\varepsilon > 0$. Then,

$$\begin{aligned} P\left(\frac{E_\omega^{\nu_{n_{k-1}}} \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k}{E_\omega \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k} \leq 1 - \varepsilon\right) &= P\left(\frac{E_\omega \bar{T}^{(d_k)}_{\nu_{n_{k-1}}}}{E_\omega \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k} \geq \varepsilon\right) \\ &\leq P\left(E_\omega \bar{T}^{(d_k)}_{\nu_{n_{k-1}}} \geq n_{k-1}^{1/s+\delta}\right) + P\left(E_\omega \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \leq \varepsilon^{-1} n_{k-1}^{1/s+\delta}\right) . \end{aligned} \tag{36}$$

Lemma 4.2 gives that $P\left(E_\omega \bar{T}^{(d_k)}_{\nu_{n_{k-1}}} \geq n_{k-1}^{1/s+\delta}\right) \geq P\left(E_\omega T_{\nu_{n_{k-1}}} \geq n_{k-1}^{1/s+\delta}\right) = o(n_{k-1}^{-\delta s/2})$. To handle the second term in the right side of (36), note that if $\delta < \frac{1}{3s}$, then the subsequence $n_k$ grows fast enough such that for all $k$ large enough $n_k^{1/s-\delta} \geq \varepsilon^{-1} n_{k-1}^{1/s+\delta}$. Therefore, for $k$ sufficiently large and $\delta < \frac{1}{3s}$ we have

$$P\left(E_\omega \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \leq \varepsilon^{-1} n_{k-1}^{1/s+\delta}\right) \leq P\left(E_\omega \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \leq n_k^{1/s-\delta}\right) .$$

However, $E_\omega \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \le n_k^{1/s-\delta}$ implies that $M_j < E_\omega^{\nu_{j-1}} \bar{T}^{(d_k)}_{\nu_j} \le n_k^{1/s-\delta}$ for at least $n_k - 1$ of the $j \le n_k$. Thus, since $P(M_1 > n_k^{1/s-\delta}) \sim C_5 n_k^{-1+\delta s}$, we have that

$$P\left(E_\omega \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k \le \varepsilon^{-1} n_{k-1}^{1/s+\delta}\right) \le n_k \left(1 - P\left(M_1 > n_k^{1/s-\delta}\right)\right)^{n_k-1} = o(e^{-n_k^{\delta s/2}}) . \qquad (37)$$

Therefore, for any $\varepsilon > 0$ and $\delta < \frac{1}{3s}$ we have that

$$P\left(\frac{E_\omega^{\nu_{n_{k-1}}} \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k}{E_\omega \bar{T}^{(d_k)}_{\nu_{n_k}} - \mu_k} \le 1 - \varepsilon\right) = o\left(n_{k-1}^{-\delta s/2}\right).$$

By our choice of $n_k$, the sequence $n_{k-1}^{-\delta s/2}$ is summable in $k$. Applying the Borel-Cantelli lemma completes the proof. ☐

**Corollary 4.2.2.** *Assume $s < 1$. Then $P-a.s.$ there exists a random subsequence $j_m = j_m(\omega)$ such that*

$$M_{j_m} \ge m^2 E_\omega \bar{T}^{(j_m)}_{\nu_{j_m-1}} .$$

*Proof.* Recall the definitions of $n_k$ and $d_k$ in (4). Then for any $C > 1$, define the event

$$D_{k,C} := \left\{\exists j \in (n_{k-1}, n_{k-1} + d_k/2] : M_j \ge C\left(E_\omega^{\nu_{n_{k-1}}} \bar{T}^{(d_k)}_{\nu_{j-1}} + E_\omega^{\nu_j} \bar{T}^{(d_k)}_{\nu_{n_k}}\right)\right\}.$$

Note that due to the reflections, the event $D_{k,C}$ depends only on the environment from $\nu_{n_{k-1}-b_{n_k}}$ to $\nu_{n_k-1}$. Then, since $n_{k-1} - b_{d_k} > n_{k-2}$ for all $k \ge 4$, we have that the events $\{D_{2k,C}\}_{k=2}^\infty$ are all independent. Also, since the events do not involve the environment to the left of 0 they have the same probability under $Q$ as under $P$. Then since $Q$ is stationary under shifts of $\nu_i$ we have that for $k \ge 4$,

$$P(D_{k,C}) = Q(D_{k,C}) = Q\left(\exists j \in [1, d_k/2] : M_j \ge C\left(E_\omega \bar{T}^{(d_k)}_{\nu_{j-1}} + E_\omega^{\nu_j} \bar{T}^{(d_k)}_{\nu_{d_k}}\right)\right).$$

Thus for any $C > 1$, we have by Lemma 4.1 that $\liminf_{k\to\infty} P(D_{k,C}) > 0$. This combined with the fact that the events $\{D_{2k,C}\}_{k=2}^\infty$ are independent gives that for any $C > 1$ infinitely many of the events $D_{2k,C}$ occur $P - a.s.$ Therefore, there exists a subsequence $k_m$ of integers such that for each $m$, there exists $j_m \in (n_{k_m-1}, n_{k_m-1} + d_{k_m}/2]$ such that

$$M_{j_m} \ge 2m^2 \left(E_\omega^{\nu_{n_{k_m-1}}} \bar{T}^{(d_{k_m})}_{\nu_{j_m-1}} + E_\omega^{\nu_{j_m}} \bar{T}^{(d_{k_m})}_{\nu_{n_{k_m}}}\right) = 2m^2\left(E_\omega^{\nu_{n_{k_m-1}}} \bar{T}^{(d_{k_m})}_{\nu_{n_{k_m}}} - \mu_{k_m}\right),$$

where the second equality holds due to our choice of $j_m$, which implies that $\mu_{k_m} = E_\omega^{\nu_{j_m-1}} \bar{T}^{(n_{k_m})}_{\nu_{j_m}}$. Then, by Lemma 4.2.1 we have that for all $m$ large enough,

$$M_{j_m} \ge 2m^2\left(E_\omega^{\nu_{k_m-1}} \bar{T}^{(d_{k_m})}_{\nu_{n_{k_m}}} - \mu_{k_m}\right) \ge m^2\left(E_\omega \bar{T}^{(d_{k_m})}_{\nu_{n_{k_m}}} - \mu_{k_m}\right) \ge m^2 E_\omega^{\nu_{j_m-1}} \bar{T}^{(d_{k_m})}_{\nu_{j_m}},$$

where the last inequality is because $\mu_{k_m} = E_\omega^{\nu_{j_m-1}} \bar{T}^{(n_{k_m})}_{\nu_{j_m}}$. Now, for all $k$ large enough we have $n_{k-1} + d_k/2 < d_k$. Thus, we may assume (by possibly choosing a further subsequence) that $j_m < d_{k_m}$ as well, and since allowing less backtracking only decreases the crossing time we have

$$M_{j_m} \ge m^2 E_\omega \bar{T}^{(d_{k_m})}_{\nu_{j_m-1}} \ge m^2 E_\omega \bar{T}^{(j_m)}_{\nu_{j_m-1}} .$$

☐

The following lemma shows that the reflections that we have been using this whole time really do not affect the random walk. We prove a slightly more general version than we need for this section because we will use this lemma again in Section 5.

**Lemma 4.3.** *Let $m_n$ be a sequence of integers such that $n^\eta = o(m_n)$ for some $\eta > 0$. Then*

$$\lim_{n\to\infty} P_\omega\left(T_{\nu_n} \neq \bar{T}^{(m_n)}_{\nu_n}\right) = 0, \quad P-a.s.$$

*Proof.* Let $\varepsilon > 0$. By Chebychev's inequality, $P\left(P_\omega\left(T_{\nu_n} \neq \bar{T}^{(m_n)}_{\nu_n}\right) > \varepsilon\right) \le \varepsilon^{-1}\mathbb{P}\left(T_{\nu_n} \neq \bar{T}^{(m_n)}_{\nu_n}\right)$. Thus by the Borel-Cantelli lemma it is enough to prove that $\mathbb{P}\left(T_{\nu_n} \neq \bar{T}^{(m_n)}_{\nu_n}\right)$ is summable. Now, the event $\{T_{\nu_n} \neq \bar{T}^{(m_n)}_{\nu_n}\}$ implies that there is an $i < \nu_n$ such that after reaching $i$ for the first time, the random walk then backtracts a distance of $b_{m_n}$. Thus, again letting $\bar{\nu} = E_P\nu$ we have

$$\mathbb{P}\left(T_{\nu_n} \neq \bar{T}^{(m_n)}_{\nu_n}\right) \le P(\nu_n \ge 2\bar{\nu}n) + \sum_{i=0}^{2\bar{\nu}n} \mathbb{P}^i(T_{i-b_{m_n}} < \infty) = P(\nu_n \ge 2\bar{\nu}n) + 2\bar{\nu}n\mathbb{P}(T_{-b_{m_n}} < \infty)$$

As noted in Lemma 4.2, $P(\nu_n \ge 2\bar{\nu}n) = \mathcal{O}(e^{-\delta' n})$, so we need only to show that $n\mathbb{P}(T_{-b_{m_n}} < \infty)$ is summable. However, [6, Lemma 3.3] gives that there exists $C_9$ such that for any $k \ge 1$,

$$\mathbb{P}(T_{-k} < \infty) \le e^{-C_9 k}. \tag{38}$$

Thus $n\mathbb{P}(T_{-b_{m_n}} < \infty) \le ne^{-C_9(b_{m_n})}$ which is summable by our assumptions on $m_n$. ∎

We define the random variable $N_t := \max\{k : \exists n \le t, X_n = \nu_k\}$ to be the maximum number of ladder locations crossed by the random walk by time $t$.

**Lemma 4.4.**

$$\lim_{t\to\infty} \frac{\nu_{N_t} - X_t}{\log^2(t)} = 0, \quad \mathbb{P}-a.s.$$

*Proof.* Let $\delta > 0$. If we can show that $\sum_{t=1}^{\infty} \mathbb{P}(|N_t - X_t| \ge \delta \log^2 t) < \infty$, then by the Borel-Cantelli lemma we will be done. Now, the only way that $N_t$ and $X_t$ can differ by more than $\delta \log^2 t$ is if either one of the gaps between the first $t$ ladder times is larger than $\delta \log^2 t$ or if for some $i < t$ the random walk backtracks $\log^2 t$ steps after first reaching $i$. Thus,

$$\mathbb{P}(|N_t - X_t| \ge \delta \log^2 t) \le P\left(\exists j \in [1, t+1] : \nu_j - \nu_{j-1} > \log^2 t\right) + t\mathbb{P}(T_{-\lceil \delta \log^2 t\rceil} < T_1) \tag{39}$$

So we need only to show that the two terms on the right hand side are summable. For the first term we use Lemma 2.1 we note that

$$P\left(\exists j \in [1, t+1] : \nu_j - \nu_{j-1} > \log^2 t\right) \le (t+1)P(\nu > \log^2 t) \le (t+1)C_1 e^{-C_2 \log^2 t},$$

which is summable in $t$. By (38) the second term on the right side of (39) is also summable. ∎

***Proof of Theorem 1.2:***
By Corollary 4.2.2, $P$-a.s there exists a subsequence $j_m(\omega)$ with the property that $M_{j_m} \ge m^2 E_\omega \bar{T}^{(j_m)}_{\nu_{j_m-1}}$. Define $t_m = t_m(\omega) = \frac{1}{m} M_{j_m}$ and $u_m = u_m(\omega) = \nu_{j_m-1}$. Then,

$$P_\omega\left(\frac{X_{t_m} - u_m}{\log^2 t_m} \notin [-\delta, \delta]\right) \le P_\omega(N_{t_m} \neq j_m - 1) + P_\omega(|\nu_{N_{t_m}} - X_{t_m}| > \delta \log^2 t_m).$$

From Lemma 4.4 the second term goes to zero as $m \to \infty$. Thus, we only need to show that

$$\lim_{m\to\infty} P_\omega(N_{t_m} = j_m - 1) = 1. \tag{40}$$

To see this first note that

$$P_\omega\left(N_{t_m} < j_m - 1\right) = P_\omega\left(T_{\nu_{j_m-1}} > t_m\right) \le P_\omega\left(T_{\nu_{j_m-1}} \neq \bar{T}^{(j_m)}_{\nu_{j_m-1}}\right) + P_\omega\left(\bar{T}^{(j_m)}_{\nu_{j_m-1}} > t_m\right).$$

By Lemma 4.3, $P_\omega\left(T_{\nu_{j_m-1}} \neq \bar{T}^{(j_m)}_{\nu_{j_m-1}}\right) \to 0$ as $m \to \infty$, $P-a.s.$ Also, by our definition of $t_m$ and our choice of the subsequence $j_m$ we have

$$P_\omega\left(\bar{T}^{(j_m)}_{\nu_{j_m-1}} > t_m\right) \le \frac{E_\omega \bar{T}^{(j_m)}_{\nu_{j_m-1}}}{t_m} = \frac{mE_\omega \bar{T}^{(j_m)}_{\nu_{j_m-1}}}{M_{j_m}} \le \frac{1}{m} \underset{m\to\infty}{\longrightarrow} 0.$$

It still remains to show $\lim_{m\to\infty} P_\omega\left(N_{t_m} < j_m\right) = 1$. To prove this, first define the stopping times $T_x^+ := \min\{n > 0 : X_n = x\}$. Then,

$$P_\omega\left(N_{t_m} < j_m\right) = P_\omega(T_{\nu_{j_m}} > t_m) \ge P_\omega^{\nu_{j_m-1}}\left(T_{\nu_{j_m}} > \frac{1}{m}M_{j_m}\right) \ge P_\omega^{\nu_{j_m-1}}\left(T^+_{\nu_{j_m-1}} < T_{\nu_{j_m}}\right)^{\frac{1}{m}M_{j_m}}.$$

Then, using the hitting time calculations given in [15, (2.1.4)], we have that

$$P_\omega^{\nu_{j_m-1}}\left(T^+_{\nu_{j_m-1}} < T_{\nu_{j_m}}\right) = 1 - \frac{1-\omega_{\nu_{j_m-1}}}{R_{\nu_{j_m-1},\nu_{j_m}-1}}.$$

Therefore, since $M_{j_m} \le R_{\nu_{j_m-1},\nu_{j_m}-1}$ we have

$$P_\omega\left(N_{t_m} < j_m\right) \ge \left(1 - \frac{1-\omega_{\nu_{j_m-1}}}{R_{\nu_{j_m-1},\nu_{j_m}-1}}\right)^{\frac{1}{m}M_{j_m}} \ge \left(1 - \frac{1}{M_{j_m}}\right)^{\frac{1}{m}M_{j_m}} \underset{m\to\infty}{\longrightarrow} 1,$$

thus proving (40) and therefore the theorem. □

# 5 Non-local behavior on a Random Subsequence

There are two main goals of this section. The first is to prove the existence of random subsequences $x_m$ where the hitting times $T_{x_m}$ are approximately gaussian random variables. This result is then used to prove the existence of random times $t_m(\omega)$ in which the scaling for the random walk is of the order $t_m^s$ instead of $\log^2 t_m$ as in Theorem 1.2. However, before we can begin proving a quenched CLT for the hitting times $T_n$ (at least along a random subsequence), we first need to understand the tail asymptotics of $Var_\omega T_\nu := E_\omega((T_\nu - E_w T_\nu)^2)$, the quenched variance of $T_\nu$.

## 5.1 Tail Asymptotics of $Q(Var_\omega T_\nu > x)$

The goal of this subsection is to prove the following theorem:

**Theorem 5.1.** *Let Assumptions 1 and 2 hold. Then with $K_\infty > 0$ the same as in Theorem 1.4, we have*

$$Q\left(Var_\omega T_\nu > x\right) \sim Q\left((E_\omega T_\nu)^2 > x\right) \sim K_\infty x^{-s/2} \quad \text{as } x \to \infty, \tag{41}$$

*and for any $\varepsilon > 0$ and $x > 0$,*

$$Q\left(Var_\omega \bar{T}^{(n)}_\nu > xn^{2/s}, \quad M_1 > n^{(1-\varepsilon)/s}\right) \sim K_\infty x^{-s/2}\frac{1}{n} \quad \text{as } n \to \infty. \tag{42}$$

*Consequently,*

$$Q\left(Var_\omega T_\nu > \delta n^{1/s}, M_1 \le n^{(1-\varepsilon)/s}\right) = o(n^{-1}). \tag{43}$$

A formula for the quenched variance of crossing times is given in [7, (2.2)]. Translating to our notation and simplifying we have the formula

$$Var_\omega T_1 := E_\omega(T_1 - E_\omega T_1)^2 = 4(W_0 + W_0^2) + 8\sum_{i<0}\Pi_{i+1,0}(W_i + W_i^2). \tag{44}$$

Now, given the environment the crossing times $T_j - T_{j-1}$ are independent. Thus we get the formula

$$
\begin{aligned}
Var_\omega T_\nu &= 4\sum_{j=0}^{\nu-1}(W_j + W_j^2) + 8\sum_{j=0}^{\nu-1}\sum_{i<j}\Pi_{i+1,j}(W_i + W_i^2) \\
&= 4\sum_{j=0}^{\nu-1}(W_j + W_j^2) + 8R_{0,\nu-1}\left(W_{-1} + W_{-1}^2 + \sum_{i<-1}\Pi_{i+1,-1}(W_i + W_i^2)\right) \\
&\quad + 8\sum_{0\le i<j<\nu}\Pi_{i+1,j}(W_i + W_i^2).
\end{aligned}
$$

In particular, $Var_\omega \bar{T}_\nu^{(n)} \le Var_\omega T_\nu$, because the same expansion for $Var_\omega \bar{T}_\nu^{(n)}$ is obtained by replacing $W_i$ by $W_{\nu_{-b_n}+1,i}$ and restricting the final sum in the second line to $\nu_{-b_n} < i < -1$.

We want to analyze the tails of $Var_\omega T_\nu$ by comparison with $(E_\omega T_\nu)^2$. Using (14) we have

$$
(E_\omega T_\nu)^2 = \left(\nu + 2\sum_{j=0}^{\nu-1}W_j\right)^2 = \nu^2 + 4\nu\sum_{j=0}^{\nu-1}W_j + 4\sum_{j=0}^{\nu-1}W_j^2 + 8\sum_{0\le i<j<\nu}W_iW_j.
$$

Thus, we have

$$
\begin{aligned}
(E_\omega T_\nu)^2 - Var_\omega T_\nu &= \nu^2 + 4(\nu-1)\sum_{j=0}^{\nu-1}W_j + 8\sum_{0\le i<j<\nu}W_i\left(W_j - \Pi_{i+1,j} - \Pi_{i+1,j}W_i\right) && (45)\\
&\quad - 8R_{0,\nu-1}\left(W_{-1} + W_{-1}^2 + \sum_{i<-1}\Pi_{i+1,-1}(W_i + W_i^2)\right) && (46)\\
&=: D^+(\omega) - 8R_{0,\nu-1}D^-(\omega)\,. && (47)
\end{aligned}
$$

The next few lemmas show that the tails of $D^+(\omega)$ and $R_{0,\nu-1}D^-(\omega)$ are much smaller than the tails of $(E_\omega T_\nu)^2$.

**Lemma 5.2.** *For any $\varepsilon > 0$, we have $Q\left(D^+(\omega) > x\right) = o(x^{-s+\varepsilon})$.*

*Proof.* Notice first of all that that from (14) we have $\nu^2 + 4(\nu-1)\sum_{j=0}^{\nu-1}W_j \le 2\nu E_\omega T_\nu$. Also we can re-write $W_j - \Pi_{i+1,j} - \Pi_{i+1,j}W_i = W_{i+2,j}$ when $i < j-1$ (this term is zero when $i = j-1$). Therefore,

$$
Q\left(D^+(\omega) > x\right) \le Q(2\nu E_\omega T_\nu > x/2) + Q\left(\sum_{i=0}^{\nu-3}\sum_{j=i+2}^{\nu-1}W_iW_{i+2,j} > x/2\right).
$$

Lemma 2.1 and Theorem 2.1 give that $Q\left(2\nu E_\omega T_\nu > x\right) \le Q(2\nu > \log^2(x)) + Q\left(E_\omega T_\nu > \frac{x}{\log^2(x)}\right) = o(x^{-s+\varepsilon})$ for any $\varepsilon > 0$. Thus we need only prove that $Q\left(\sum_{i=0}^{\nu-3}\sum_{j=i+2}^{\nu-1}W_iW_{i+2,j} > x\right) = o(x^{-s+\varepsilon})$ for any $\varepsilon > 0$. Note that for $i < \nu$ we have $W_i = W_{0,i} + \Pi_{0,i}W_{-1} \le \Pi_{0,i}(i + W_{-1})$, thus

$$
\begin{aligned}
Q\left(\sum_{i=0}^{\nu-3}\sum_{j=i+2}^{\nu-1}W_iW_{i+2,j} > x\right) &\le Q\left((\nu + W_{-1})\sum_{i=0}^{\nu-3}\sum_{j=i+2}^{\nu-1}\Pi_{0,i}W_{i+2,j} > x\right)\\
&\le Q(\nu > \log^2(x)/2) + Q(W_{-1} > \log^2(x)/2) && (48)\\
&\quad + \sum_{i=0}^{\log^2(x)-3}\sum_{j=i+2}^{\log^2(x)-1}P\left(\Pi_{0,i}W_{i+2,j} > \frac{x}{\log^6(x)}\right), && (49)
\end{aligned}
$$

where we were able to switch to $P$ instead of $Q$ in the last line because the event inside the probability only concerns the environment to the right of 0. Now, Lemmas 2.1 and 2.2 give that (48) is $o(x^{-s+\varepsilon})$ for any $\varepsilon > 0$, so we need only to consider (49). Under the measure $P$ we have that $\Pi_{0,i}$ and $W_{i+2,j}$ are independent, and by (9) we have $P(W_{i+2,j} > x) \le P(W_{i+2} > x) \le K_1 x^{-s}$. Thus,

$$P\left(\Pi_{0,i} W_{i+2,j} > \frac{x}{\log^6(x)}\right) = E_P\left[P\left(W_{i+2,j} > \frac{x}{\log^6(x)\Pi_{0,i}}\middle|\Pi_{0,i}\right)\right] \le K_1 \log^{6s}(x) x^{-s} E_P[\Pi_{0,i}^s] \,.$$

Then because $E_P \Pi_{0,i}^s = (E_P \rho^s)^i = 1$ by Assumption 1, we have

$$\sum_{i=0}^{\log^2(x)-3} \sum_{j=i+2}^{\log^2(x)-1} P\left(\Pi_{0,i} W_{i+2,j} > \frac{x}{\log^6(x)}\right) \le K_1 \log^{4+6s}(x) x^{-s} = o(x^{-s+\varepsilon}) \,.$$

□

**Lemma 5.3.** *For any $\varepsilon > 0$,*

$$Q\left(D^-(\omega) > x\right) = o(x^{-s+\varepsilon}), \tag{50}$$

*and thus for any $\gamma < s$,*

$$E_Q D^-(\omega)^\gamma < \infty. \tag{51}$$

*Proof.* It is obvious that (50) implies (51) and so we will only prove the former. Write

$$D^-(\omega) = W_{-1} + W_{-1}^2 + \sum_{i<-1} \Pi_{i+1,-1}(W_i + W_i^2) = \sum_{i\le -1} \sum_{k \le i} \Pi_{k,-1}\left(1 + \Pi_{k,i} + \sum_{l<k} \Pi_{l,i}\right). \tag{52}$$

Next, for any $c > 0$ and $n \in \mathbb{N}$ consider the event

$$E_{c,n} := \left\{\Pi_{j,i} < e^{-c(i-j+1)}, \quad \forall -n \le i \le -1, \forall j < i-n\right\} = \bigcap_{-n\le i\le -1} \bigcap_{j<i-n} \{\Pi_{j,i} < e^{-c(i-j+1)}\}.$$

Now, under the measure $Q$ we have that $\Pi_{k,-1} < 1$ for all $k \le -1$, and thus on the event $E_{c,n}$ we have

$$\sum_{i\le -1} \sum_{k\le i} \Pi_{k,-1}\left(1 + \Pi_{k,i} + \sum_{l<k} \Pi_{l,i}\right) \le n^2 + \frac{e^{2c} - e^c + 1}{(e^c - 1)^3} + (1+n) \sum_{-n\le i\le -1} W_i + \sum_{i<-n} e^{ci} W_i. \tag{53}$$

Applying Lemma 2.1 with $c < -E_P \log(\rho)$, we have that for all $i \le j$,

$$Q(\Pi_{i,j} > e^{-c(j-i+1)}) \le \frac{1}{P(\mathcal{R})} P(\Pi_{i,j} > e^{-c(j-i+1)}) \le \frac{A_c}{P(\mathcal{R})} e^{-\delta_c (j-i+1)}.$$

Therefore,

$$Q(E_{c,n}^c) \le \sum_{-n\le i\le -1} \sum_{j<i-n} Q(\Pi_{i,j} > e^{-c(i-j+1)}) \le \frac{n A_c e^{-\delta_c(n+2)}}{P(\mathcal{R})(1 - e^{-\delta_c})} = o(e^{-\delta_c n/2}). \tag{54}$$

Then, using (53) with $n = \lfloor \log^2 x \rfloor =: b_x$ we have

$$Q\left(\sum_{i\le -1} \sum_{k\le i} \Pi_{k,-1}\left(1 + \Pi_{k,i} + \sum_{l<k} \Pi_{l,i}\right) > x\right) \tag{55}$$

$$\le Q\left(E_{c,b_x}^c\right) + \mathbf{1}_{\{b_x^2 + \frac{e^{2c}-e^c+1}{(e^c-1)^3} > x/3\}} + Q\left(\sum_{-b_x\le i\le -1} W_i > \frac{x}{3(1+b_x)}\right) + Q\left(\sum_{i<-1} e^{ci} W_i > \frac{x}{3}\right).$$

If we choose $0 < c < -E_P \log\rho$, then applying (54) we have that the first two terms are decreasing in $x$ of order $o(e^{-\delta_c b_x/2}) = o(x^{-s+\varepsilon})$. To handle last two terms in the right side of (55), note first that from (9), $Q\left(W_i > x\right) \le \frac{1}{P(\mathcal{R})} P(W_i > x) = \frac{K_1}{P(\mathcal{R})} x^{-s}$ for any $x > 0$ and any $i$. Thus,

$$Q\left(\sum_{-b_x \le i \le -1} W_i > \frac{x}{3(1+b_x)}\right) \le \sum_{-b_x \le i \le -1} Q\left(W_i > \frac{x}{3(1+b_x)b_x}\right) = o(x^{-s+\varepsilon}),$$

and since $\sum_{i=1}^{\infty} e^{-ci/2} = (e^{c/2}-1)^{-1}$, we have

$$\begin{aligned} Q\left(\sum_{i<-1} e^{ci} W_i > \frac{x}{3}\right) &= Q\left(\sum_{i=1}^{\infty} e^{-ci} W_{-i} > \frac{x}{3}(e^{c/2}-1)\sum_{i=1}^{\infty} e^{-ci/2}\right) \\ &\le \sum_{i=1}^{\infty} Q\left(W_{-i} > \frac{x}{3}(e^{c/2}-1)e^{ci/2}\right) \\ &\le \frac{K_1 3^s}{P(\mathcal{R})(e^{c/2}-1)^s} x^{-s} \sum_{i=1}^{\infty} e^{-csi/2} = \mathcal{O}(x^{-s}). \end{aligned}$$

□

**Corollary 5.3.1.** *For any $\varepsilon > 0$, $Q\left(R_{0,\nu-1} D^-(\omega) > x\right) = o(x^{-s+\varepsilon})$.*

*Proof.* From (11) it is easy to see that for any $\gamma < s$ there exists a $K_\gamma > 0$ such that $P(R_{0,\nu-1} > x) \le P(R_0 > x) \le K_\gamma x^{-\gamma}$. Then, letting $\mathcal{F}_{-1} = \sigma(\ldots, \omega_{-2}, \omega_{-1})$ we have that

$$Q\left(R_{0,\nu-1} D^-(\omega) > x\right) = E_Q\left[Q\left(R_{0,\nu-1} > \frac{x}{D^-(\omega)}\middle| \mathcal{F}_{-1}\right)\right] \le K_\gamma x^{-\gamma} E_Q\left(D^-(\omega)\right)^\gamma .$$

Since $\gamma < s$, the expectation in the last expression is finite by (51). Choosing $\gamma = s - \frac{\varepsilon}{2}$ finishes the proof. □

***Proof of Theorem 5.1:***
Recall from (47) that

$$(E_\omega T_\nu)^2 - D^+(\omega) \le Var_\omega T_\nu \le (E_\omega T_\nu)^2 + 8R_{0,\nu-1} D^-(\omega). \tag{56}$$

The lower bound in (56) gives that for any $\delta > 0$,

$$Q(Var_\omega T_\nu > x) \ge Q\left((E_\omega T_\nu)^2 > (1+\delta)x\right) - Q\left(D^+(\omega) > \delta x\right).$$

Thus, from Lemma 5.2 and Theorem 1.4 we have that

$$\lim_{x\to\infty} x^{s/2} Q(Var_\omega T_\nu > x) \ge K_\infty (1+\delta)^{-s/2}. \tag{57}$$

Similarly, the upper bound in (56) and Corollary 5.3.1 give that for any $\delta > 0$,

$$Q(Var_\omega T_\nu > x) \le Q\left((E_\omega T_\nu)^2 > (1-\delta)x\right) + Q\left(8R_{0,\nu-1} D^-(\omega) > \delta x\right),$$

and then Corollary 5.3.1 and Theorem 1.4 give

$$\lim_{x\to\infty} x^{s/2} Q(Var_\omega T_\nu > x) \le K_\infty (1-\delta)^{-s/2} x^{-s/2}. \tag{58}$$

Letting $\delta \to 0$ in (57) and (58) finishes the proof of (41).
Essentially the same proof works for (42). The difference is that when evaluating the difference $(E_\omega \bar{T}_\nu^{(n)})^2 - Var_\omega \bar{T}_\nu^{(n)}$ the upper and lower bounds in (45) and (46) are smaller in absolute

value. This is because every instance of $W_i$ is replaced by $W_{\nu_{-b_n}+1,i} \le W_i$ and the sum in (46) is taken only over $\nu_{-b_n} < i < -1$. Therefore, the following bounds still hold:

$$\left(E_\omega \bar{T}_\nu^{(n)}\right)^2 - D^+(\omega) \le Var_\omega \bar{T}_\nu^{(n)} \le \left(E_\omega \bar{T}_\nu^{(n)}\right)^2 + 8R_{0,\nu-1}D^-(\omega)\,. \tag{59}$$

The rest of the proof then follows in the same manner, noting that from Lemma 3.3 we have $Q\left(\left(E_\omega \bar{T}_\nu^{(n)}\right)^2 > xn^{2/s}, \quad M_1 > n^{(1-\varepsilon)/s}\right) \sim K_\infty x^{-s/2}\frac{1}{n}$, as $n \to \infty$. ☐

## 5.2 Existence of Random Subsequence of Non-localized Behavior

Introduce the notation:

$$\mu_{i,n,\omega} := E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}, \quad \sigma_{i,n,\omega}^2 := E_\omega^{\nu_{i-1}}\left(\bar{T}_{\nu_i}^{(n)} - \mu_{i,n,\omega}\right)^2 = Var_\omega\left(\bar{T}_{\nu_i}^{(n)} - \bar{T}_{\nu_{i-1}}^{(n)}\right). \tag{60}$$

The first goal of this subsection is to prove a CLT (along random subsequences) for the hitting times $T_n$. We begin by showing that for any $\varepsilon > 0$ only the crossing times of ladder times with $M_k > n^{(1-\varepsilon)/s}$ are relevant in the limiting distribution, at least along a sparse enough subsequence.

**Lemma 5.4.** *Assume $s < 2$. Then for any $\varepsilon, \delta > 0$ there exists an $\eta > 0$ such that for any integer $m$*

$$Q\left(\sum_{i=1}^n \sigma_{i,m,\omega}^2 \mathbf{1}_{M_i \le n^{(1-\varepsilon)/s}} > \delta n^{2/s}\right) = o(n^{-\eta})\,.$$

*Proof.* First, we need an bound on the probability of $Var_\omega \bar{T}_\nu^{(m)}$ being much larger than $M_1$. Note that from (56) we have $Var_\omega T_\nu \le (E_\omega T_\nu)^2 + 8R_{0,\nu-1}D^-(\omega)$. Then, since $R_{0,\nu-1} \le \nu M_1$ we have

$$Q\left(Var_\omega T_\nu > n^{2\beta}, M_1 \le n^\alpha\right) \le Q\left(E_\omega T_\nu > \frac{n^\beta}{\sqrt{2}}, M_1 \le n^\alpha\right) + Q\left(8\nu D^-(\omega) > \frac{n^{\beta-\alpha}}{2}\right).$$

By (15), the first term on the right is $o(e^{-n^{(\beta-\alpha)/5}})$. To bound the second term on the right we use Lemmas 2.1 and 5.3.1 to get that for any $\alpha < \beta$

$$Q\left(8\nu D^-(\omega) > \frac{n^{\beta-\alpha}}{2}\right) \le Q(\nu > \log^2 n) + Q\left(D^-(\omega) > \frac{n^{2\beta-\alpha}}{16\log^2 n}\right) = o(n^{-\frac{s}{2}(3\beta-\alpha)})\,.$$

Therefore, similarly to (15) we have the bound

$$Q\left(Var_\omega T_\nu > n^{2\beta}, M_1 \le n^\alpha\right) = o(n^{-\frac{s}{2}(3\beta-\alpha)})\,. \tag{61}$$

The rest of the proof is similar to the proof of Lemma 3.1. First, from (61),

$$\begin{aligned}
Q\left(\sum_{i=1}^n \sigma_{i,m,\omega}^2 \mathbf{1}_{M_i \le n^{(1-\varepsilon)/s}} > \delta n^{2/s}\right) &\le Q\left(\sum_{i=1}^n \sigma_{i,m,\omega}^2 \mathbf{1}_{\sigma_{i,m,\omega} \le n^{(1-\frac{\varepsilon}{4})/s}} > \delta n^{2/s}\right) \\
&\quad + nQ\left(Var_\omega \bar{T}_\nu^{(m)} > n^{2(1-\frac{\varepsilon}{4})/s}, M_1 \le n^{(1-\varepsilon)/s}\right) \\
&= Q\left(\sum_{i=1}^n \sigma_{i,m,\omega}^2 \mathbf{1}_{\sigma_{i,m,\omega} \le n^{(1-\frac{\varepsilon}{4})/s}} > \delta n^{2/s}\right) + o(n^{-\varepsilon/8})\,.
\end{aligned}$$

Therefore, it is enough to prove that for any $\delta, \varepsilon > 0$ there exists $\eta > 0$ such that

$$Q\left(\sum_{i=1}^n \sigma_{i,m,\omega}^2 \mathbf{1}_{\sigma_{i,m,\omega} \le n^{(1-\varepsilon)/s}} > \delta n^{2/s}\right) = o(n^{-\eta})\,.$$

We prove the above statement by choosing $C \in (1, \frac{2}{s})$ and then using Theorem 5.1 to get bounds the size of the set $\left\{ i \le n : \sigma^2_{i,m,\omega} \in (n^{2(1-\varepsilon C^k)/s}, n^{2(1-\varepsilon C^{k-1})/s}] \right\}$ for all $k$ small enough so that $\varepsilon C^k < 1$. This portion of the proof is similar to that of Lemma 3.1 and thus will be omitted. □

**Corollary 5.4.1.** *Assume $s < 2$. Then there exists an $\eta' > 0$ such that for any $m \le n$ and any $\delta > 0$,*

$$Q\left( \left| \sum_{i=1}^n \left( \sigma^2_{i,m,\omega} - \mu^2_{i,m,\omega} \right) \right| \ge \delta n^{2/s} \right) = o(n^{-\eta'}) \,.$$

*Proof.* For any $\varepsilon > 0$

$$Q\left( \left| \sum_{i=1}^n \left( \sigma^2_{i,m,\omega} - \mu^2_{i,m,\omega} \right) \right| \ge \delta n^{2/s} \right) \le Q\left( \sum_{i=1}^n \sigma^2_{i,n,\omega} \mathbf{1}_{M_i \le n^{(1-\varepsilon)/s}} \ge \frac{\delta}{3} n^{2/s} \right) \tag{62}$$

$$+ Q\left( \sum_{i=1}^n \mu^2_{i,n,\omega} \mathbf{1}_{M_i \le n^{(1-\varepsilon)/s}} \ge \frac{\delta}{3} n^{2/s} \right) \tag{63}$$

$$+ Q\left( \sum_{i=1}^n \left| \sigma^2_{i,m,\omega} - \mu^2_{i,m,\omega} \right| \mathbf{1}_{M_i > n^{(1-\varepsilon)/s}} \ge \frac{\delta}{3} n^{2/s} \right) . \tag{64}$$

Lemma 5.4 gives that (62) decreases polynomially in $n$. Also, essentially the same proof as in Lemmas 5.4 and 3.1 can be used to show that (63) also decreases polynomially in $n$. Finally (64) is bounded above by

$$Q\left( \# \left\{ i \le n : M_i > n^{(1-\varepsilon)/s} \right\} > n^{2\varepsilon} \right) + nQ\left( \left| Var_\omega \bar{T}^{(m)}_\nu - (E_\omega \bar{T}^{(m)}_\nu)^2 \right| \ge \frac{\delta}{3} n^{2/s - 2\varepsilon} \right),$$

and since by (13), $Q\left( \# \left\{ i \le n : M_i > n^{(1-\varepsilon)/s} \right\} > n^{2\varepsilon} \right) \le \frac{nQ(M_1 > n^{(1-\varepsilon)/s})}{n^{2\varepsilon}} \sim C_5 n^{-\varepsilon}$ we need only show that the second term above is decreasing faster than a power of $n$. However, from (59) we have $\left| Var_\omega \bar{T}^{(m)}_\nu - (E_\omega \bar{T}^{(m)}_\nu)^2 \right| \le D^+(\omega) + 8R_{0,\nu-1} D^-(\omega)$. Thus, Lemma 5.2 and Corollary 5.3.1 give that $Q\left( \left| Var_\omega \bar{T}^{(m)}_\nu - (E_\omega \bar{T}^{(m)}_\nu)^2 \right| > x \right) = o(x^{-s+\varepsilon'})$ for any $\varepsilon' > 0$. Thus, for $\varepsilon < \frac{1}{4s}$,

$$nQ\left( \left| Var_\omega \bar{T}^{(m)}_\nu - (E_\omega \bar{T}^{(m)}_\nu)^2 \right| \ge \frac{\delta}{3} n^{2/s - 2\varepsilon} \right) = o(n^{-1+4\varepsilon s}) \,,$$

which finishes the proof. □

Since $T_{\nu_n} = \sum_{i=1}^n (T_{\nu_i} - T_{\nu_{i-1}})$ is the sum of independent (quenched) random variables, in order to prove a CLT we cannot have any of the first $n$ crossing times of blocks dominating all the others (note this is exactly what happens in the localization behavior we saw in Section 4). Thus, we look for a random subsequence where none of the crossing times of blocks are dominant. Now, for any $\delta \in (0, 1]$ and any positive integer $a < n/2$ define the event

$$\mathcal{S}_{\delta,n,a} := \left\{ \# \left\{ i \le \delta n : \mu^2_{i,n,\omega} \in [n^{2/s}, 2n^{2/s}) \right\} = 2a, \quad \mu^2_{j,n,\omega} < 2n^{2/s} \quad \forall j \le \delta n \right\} .$$

On the event $\mathcal{S}_{\delta,n,a}$, $2a$ of the first $\delta n$ crossings times from $\nu_{i-1}$ to $\nu_i$ have roughly the same size expected crossing times $\mu_{i,n,\omega}$, and the rest are all smaller (we work with $\mu^2_{i,n,\omega}$ instead of $\mu_{i,n,\omega}$ so that comparisons with $\sigma^2_{i,n,\omega}$ are slightly easier). We want a lower bound on the probability of $\mathcal{S}_{\delta,n,a}$. The difficulty in getting a lower bound is that the $\mu^2_{i,n,\omega}$ are not independent. However, we can force all the large crossing times to be independent by forcing them to be separated by at least $b_n$ ladder locations.

Let $\mathcal{I}_{\delta,n,a}$ be the collection of all subsets $I$ of $[1,\delta n]\cap\mathbb{Z}$ of size $2a$ with the property that any two distinct points in $I$ are separated by at least $2b_n$. Also, define the event

$$A_{i,n} := \left\{ \mu_{i,n,\omega}^2 \in \left[ n^{2/s}, 2n^{2/s} \right) \right\}.$$

Then, we begin with a simple lower bound.

$$\begin{aligned} Q(\mathcal{S}_{\delta,n,a}) &\geq Q\left( \bigcup_{I\in\mathcal{I}_{\delta,n,a}} \left( \bigcap_{i\in I} A_{i,n} \bigcap_{j\in[1,\delta n]\setminus I} \left\{ \mu_{j,n,\omega}^2 < n^{2/s} \right\} \right) \right) \\ &= \sum_{I\in\mathcal{I}_{\delta,n,a}} Q\left( \bigcap_{i\in I} A_{i,n} \bigcap_{j\in[1,\delta n]\setminus I} \left\{ \mu_{j,n,\omega}^2 < n^{2/s} \right\} \right). \end{aligned} \tag{65}$$

Now, recall the definition of the event $G_{i,n,\varepsilon}$ from (32), and define the event

$$H_{i,n,\varepsilon} := \left\{ M_j \leq n^{(1-\varepsilon)/s} \text{ for all } j \in [i-b_n, i) \right\}.$$

Also, for any $I \subset \mathbb{Z}$ let $d(j,I) := \min\{|j-i| : i \in I\}$ be the minimum distance from $j$ to the set $I$. Then, with minimal cost, we can assume that for any $I \in \mathcal{I}_{\delta,n,a}$ and any $\varepsilon > 0$ that all $j \notin I$ such that $d(j,I) \leq b_n$ have $M_j \leq n^{(1-\varepsilon)/s}$. Indeed,

$$\begin{aligned} &Q\left( \bigcap_{i\in I} A_{i,n} \bigcap_{j\in[1,\delta n]\setminus I} \left\{ \mu_{j,n,\omega}^2 < n^{2/s} \right\} \right) \\ &\qquad \geq Q\left( \bigcap_{i\in I} (A_{i,n} \cap G_{i,n,\varepsilon} \cap H_{i,n,\varepsilon}) \bigcap_{j\in[1,\delta n]:d(j,I)>b_n} \left\{ \mu_{j,n,\omega}^2 < n^{2/s} \right\} \right) \\ &\qquad\quad - Q\left( \bigcup_{j\notin I, d(j,I)\leq b_n} \left\{ \mu_{j,n,\omega}^2 > n^{2/s}, M_j \leq n^{(1-\varepsilon)/s} \right\} \right) \\ &\qquad \geq \prod_{i\in I} Q(A_{i,n}\cap H_{i,n,\varepsilon}) Q\left( \bigcap_{i\in I} G_{i,n,\varepsilon} \bigcap_{j\in[1,\delta n]:d(j,I)>b_n} \left\{ \mu_{j,n,\omega}^2 < n^{2/s} \right\} \right) \\ &\qquad\quad - 4ab_n Q\left( E_\omega T_\nu > n^{1/s}, M_1 \leq n^{(1-\varepsilon)/s} \right). \end{aligned} \tag{66}$$

From Theorem 1.4 and Lemma 3.3 we have $Q(A_{i,n}) \sim K_\infty(1-2^{-s/2})n^{-1}$. We wish to show the same asymptotics are true for $Q(A_{i,n}\cap H_{i,n,\varepsilon})$ as well. From (13) we have $Q(H_{i,n,\varepsilon}^c) \leq b_n Q(M_1 > n^{(1-\varepsilon)/s}) = o(n^{-1+2\varepsilon})$. Applying this, along with (13) and (15), gives that for $\varepsilon > 0$,

$$\begin{aligned} Q(A_{i,n}) &\leq Q(A_{i,n}\cap H_{i,n,\varepsilon}) + Q\left( M_1 > n^{(1-\varepsilon)/s} \right) Q(H_{i,n,\varepsilon}^c) + Q\left( E_\omega T_\nu > n^{1/s}, M_1 \leq n^{(1-\varepsilon)/s} \right) \\ &= Q(A_{i,n}\cap H_{i,n,\varepsilon}) + o(n^{-2+3\varepsilon}) + o(e^{-n^{\varepsilon/(5s)}}). \end{aligned}$$

Thus, for any $\varepsilon < \frac{1}{3}$ there exists a $C_\varepsilon > 0$ such that

$$Q(A_{i,n}\cap H_{i,n,\varepsilon}) \geq C_\varepsilon n^{-1}. \tag{67}$$

To handle the next probability in (66), note that

$$\begin{aligned} Q\left( \bigcap_{i\in I} G_{i,n,\varepsilon} \bigcap_{j\in[1,\delta n]:d(j,I)>b_n} \left\{ \mu_{j,n,\omega}^2 < n^{2/s} \right\} \right) &\geq Q\left( \bigcap_{j\in[1,\delta n]} \left\{ \mu_{j,n,\omega}^2 < n^{2/s} \right\} \right) - Q\left( \bigcup_{i\in I} G_{i,n,\varepsilon}^c \right) \\ &\geq Q\left( E_\omega T_{\nu_n} < n^{1/s} \right) - 2aQ(G_{i,n,\varepsilon}^c) \\ &= Q\left( E_\omega T_{\nu_n} < n^{1/s} \right) - ao(n^{-1+2\varepsilon}). \end{aligned} \tag{68}$$

Finally, from (15) we have $4ab_nQ\left(E_\omega T_\nu > n^{1/s}, M_1 \le n^{(1-\varepsilon)/s}\right) = ao\left(e^{-n^{\varepsilon/(6s)}}\right)$. This, along with (67) and (68) applied to (65) gives

$$Q\left(\mathcal{S}_{\delta,n,a}\right) \ge \#(\mathcal{I}_{\delta,n,a})\left[\left(C_\varepsilon n^{-1}\right)^{2a}\left(Q\left(E_\omega T_{\nu_n} < n^{1/s}\right) - ao(n^{-1+2\varepsilon})\right) - ao\left(e^{-n^{\varepsilon/(6s)}}\right)\right].$$

An obvious upper bound for $\#(\mathcal{I}_{\delta,n,a})$ is $\binom{\delta n}{2a} \le \frac{(\delta n)^{2a}}{(2a)!}$. To get a lower bound on $\#(\mathcal{I}_{\delta,n,a})$ we note that any set $I \in \mathcal{I}_{\delta,n,a}$ can be chosen in the following way: first choose an integer $i_1 \in [1,\delta n]$ ($\delta n$ ways to do this). Then, choose an integer $i_2 \in [1,\delta n]\backslash\{j \in \mathbb{Z} : |j - i_1| \le 2b_n\}$ (at least $\delta n - 1 - 4b_n$ ways to do this). Continue this process until $2a$ integers have been chosen. When choosing $i_j$, there will be at least $\delta n - (j-1)(1+4b_n)$ integers available. Then, since there are $(2a)!$ orders in which to choose each set if $2a$ integers we have

$$\frac{(\delta n)^{2a}}{(2a)!} \ge \#(\mathcal{I}_{\delta,n,a}) \ge \frac{1}{(2a)!}\prod_{j=1}^{2a}\left(\delta n - (j-1)(1+4b_n)\right) \ge \frac{(\delta n)^{2a}}{(2a)!}\left(1 - \frac{(2a-1)(1+4b_n)}{\delta n}\right)^{2a}.$$

Therefore, applying the upper and lower bounds on $\#(\mathcal{I}_{\delta,n,a})$ we get

$$\begin{aligned} Q\left(\mathcal{S}_{\delta,n,a}\right) \ge & \frac{(\delta C_\varepsilon)^{2a}}{(2a)!}\left(1 - \frac{(2a-1)(1+4b_n)}{\delta n}\right)^{2a}\left(Q\left(E_\omega T_{\nu_n} < n^{1/s}\right) - ao(n^{-1+2\varepsilon})\right) \\ & - \frac{(\delta n)^{2a}}{(2a)!}ao\left(e^{-n^{\varepsilon/(6s)}}\right). \end{aligned}$$

Recall the definitions of $d_k$ in (4) and define

$$a_k := \lfloor \log\log k \rfloor \vee 1, \quad \text{and} \quad \delta_k := a_k^{-1}. \tag{69}$$

Now, replacing $\delta, n$ and $a$ in the above by $\delta_k, d_k$ and $a_k$ respectively we have

$$\begin{aligned} Q\left(\mathcal{S}_{\delta_k,d_k,a_k}\right) \ge & \frac{(\delta_k C_\varepsilon)^{2a_k}}{(2a_k)!}\left(1 - \frac{(2a_k-1)(1+4b_{d_k})}{\delta_k d_k}\right)^{2a_k}\left(Q\left(E_\omega T_{\nu_{d_k}} < d_k^{1/s}\right) - a_k o(d_k^{-1+2\varepsilon})\right) \\ & - \frac{(\delta_k d_k)^{2a_k}}{(2a_k)!}a_k o\left(e^{-d_k^{\varepsilon/(6s)}}\right) \\ \ge & \frac{(\delta_k C_\varepsilon)^{2a_k}}{(2a_k)!}\left(1+o(1)\right)\left(L_{s,b'}(1) - o(1)\right) - o(1). \end{aligned} \tag{70}$$

The last inequality is a result of the definitions of $\delta_k, a_k$, and $d_k$ (it's enough to recall that $d_k \ge 2^{2^{k-1}}$, $a_k \sim \log\log k$, and $\delta_k \sim \frac{1}{\log\log k}$), as well as Theorem 1.1. Also, since $\delta_k = a_k^{-1}$ we get from Sterling's formula that $\frac{(\delta_k C_\varepsilon)^{2a_k}}{(2a_k)!} \sim \frac{(C_\varepsilon e/2)^{2a_k}}{\sqrt{2\pi a_k}}$. Thus since $a_k \sim \log\log k$, we have that $\frac{1}{k} = o\left(\frac{(\delta_k C_\varepsilon)^{2a_k}}{(2a_k)!}\right)$. This, along with (70), gives that $Q\left(\mathcal{S}_{\delta_k,d_k,a_k}\right) > \frac{1}{k}$ for all $k$ large enough.
We now have a good lower bound on the probability of not having any of the crossing times of the first $\delta_k d_k$ blocks dominating all the others. However for the purpose of proving Theorem 1.3 we need a little bit more. We also need that none of the crossing times of succeeding blocks are too large either. Thus, for any $0 < \delta < c$ and $n \in \mathbb{N}$ define the events

$$U_{\delta,n,c} := \left\{\sum_{i=\delta n+1}^{cn}\mu_{i,n,\omega} \le 2n^{1/s}\right\}, \quad \tilde{U}_{\delta,n,c} := \left\{\sum_{i=\delta n+b_n+1}^{cn}\mu_{i,n,\omega} \le n^{1/s}\right\}.$$

**Lemma 5.5.** *Assume $s < 1$. Then there exists a sequence $c_k \to \infty$, $c_k = o(\log a_k)$ such that*

$$\sum_{k=1}^{\infty} Q\left(\mathcal{S}_{\delta_k,d_k,a_k} \cap U_{\delta_k,d_k,c_k}\right) = \infty.$$

*Proof.* For any $\delta < c$ and $a < n/2$ we have

$$Q\left(\mathcal{S}_{\delta,n,a} \cap U_{\delta,n,c}\right) \geq Q\left(\mathcal{S}_{\delta,n,a}\right) Q\left(\tilde{U}_{\delta,n,c}\right) - Q\left(\sum_{i=1}^{b_n} \mu_{i,n,\omega} > n^{1/s}\right)$$
$$\geq Q\left(\mathcal{S}_{\delta,n,a}\right) Q\left(E_\omega T_{\nu_{cn}} \leq n^{1/s}\right) - b_n Q\left(E_\omega T_\nu > \frac{n^{1/s}}{b_n}\right)$$
$$\geq Q\left(\mathcal{S}_{\delta,n,a}\right) Q\left(E_\omega T_{\nu_{cn}} \leq n^{1/s}\right) - o(n^{-1/2}), \tag{71}$$

where the last inequality is from Theorem 1.4. Now, define $c_1 = 1$ and for $k > 1$ let

$$c'_k := \max\left\{c \in \mathbb{N} : Q\left(E_\omega T_{\nu_{cd_k}} \leq d_k^{1/s}\right) \geq \frac{1}{\log k}\right\} \vee 1\,.$$

Note that by Theorem 1.1 we have that $c'_k \to \infty$, and so we can define $c_k = c'_k \wedge \log\log(a_k)$. Then applying (71) with this choice of $c_k$ we have

$$\sum_{k=1}^{\infty} Q\left(\mathcal{S}_{\delta_k,d_k,a_k} \cap U_{\delta_k,d_k,c_k}\right) \geq \sum_{k=1}^{\infty}\left[Q\left(\mathcal{S}_{\delta_k,d_k,a_k}\right) Q\left(E_\omega T_{\nu_{c_k d_k}} \leq d_k^{1/s}\right) - o(d_k^{-1/2})\right] = \infty,$$

and the last sum is infinite because $d_k^{-1/2}$ is summable and for all $k$ large enough we have

$$Q\left(\mathcal{S}_{\delta_k,d_k,a_k}\right) Q\left(E_\omega T_{\nu_{c_k d_k}} \leq d_k^{1/s}\right) \geq \frac{1}{k \log k}.$$

□

**Corollary 5.5.1.** *Assume $s < 1$, and let $c_k$ be as in Lemma 5.5. Then, $P$-a.s. there exists a random subsequence $n_{k_m} = n_{k_m}(\omega)$ of $n_k = 2^{2^k}$ such that for the sequences $\alpha_m, \beta_m$, and $\gamma_m$ defined by*

$$\alpha_m := n_{k_m - 1}, \qquad \beta_m := n_{k_m-1} + \delta_{k_m} d_{k_m}, \qquad \gamma_m := n_{k_m-1} + c_{k_m} d_{k_m}, \tag{72}$$

*we have that for all $m$*

$$\max_{i \in (\alpha_m, \beta_m]} \mu^2_{i,d_{k_m},\omega} \leq 2 d_{k_m}^{2/s} \leq \frac{1}{a_{k_m}} \sum_{i=\alpha_m+1}^{\beta_m} \mu^2_{i,d_{k_m},\omega}, \tag{73}$$

*and*

$$\sum_{\beta_m+1}^{\gamma_m} \mu_{i,d_{k_m},\omega} \leq 2 d_{k_m}^{1/s}\,.$$

*Proof.* Define the events

$$\mathcal{S}'_k := \left\{\#\left\{i \in (n_{k-1}, n_{k-1} + \delta_k d_k] : \mu^2_{i,d_k,\omega} \in [d_k^{2/s}, 2d_k^{2/s})\right\} = 2a_k\right\}$$
$$\cap \left\{\mu^2_{j,d_k,\omega} < 2d_k^{2/s} \quad \forall j \in (n_{k-1}, n_{k-1} + \delta_k d_k]\right\},$$

$$U'_k := \left\{\sum_{n_{k-1}+\delta_k d_k + 1}^{n_{k-1}+c_k d_k} \mu_{i,d_k,\omega} \leq 2 d_k^{1/s}\right\}.$$

Note that due to the reflections of the random walk, the event $\mathcal{S}'_k \cap U'_k$ depends on the environment between ladder locations $n_{k-1} - b_{d_k}$ and $n_{k-1} + c_k d_k$. Thus, for $k_0$ large enough

$\{\mathcal{S}'_{2k} \cap U'_{2k}\}_{k=k_0}^{\infty}$ is an independent sequence of events. Similarly, for $k$ large enough $\mathcal{S}'_k \cap U'_k$ does not depend on the environment to left of the origin. Thus

$$P(\mathcal{S}'_k \cap U'_k) = Q(\mathcal{S}'_k \cap U'_k) = Q\left(\mathcal{S}_{\delta_k,d_k,a_k} \cap U_{\delta_k,d_k,c_k}\right)$$

for all $k$ large enough. Lemma 5.5 then gives that $\sum_{k=1}^{\infty} P(\mathcal{S}'_{2k} \cap U'_{2k}) = \infty$, and the Borel-Cantelli lemma then implies that infinitely many of the events $\mathcal{S}'_{2k} \cap U'_{2k}$ occur $P-a.s.$ Finally, note that $\mathcal{S}'_{k_m}$ implies the event in (73). $\square$

Before proving a quenched CLT (along a subsequence) for the hitting times $T_n$, we need one more lemma that gives us some control on the quenched tails of crossing times of blocks. We can get this from an application of Kac's moment formula. Let $\bar{T}_y$ be the hitting time of $y$ when we add a reflection at the starting point of the random walk. Then Kac's moment formula [5, (6)] and the Markov property give that $E_\omega^x(\bar{T}_y)^j \le j!\left(E_\omega^x \bar{T}_y\right)^j$. Thus,

$$E_\omega^{\nu_{i-1}}(\bar{T}_{\nu_i}^{(n)})^j \le E_\omega^{\nu_{i-1}-b_n}(\bar{T}_{\nu_i})^j \le j!\left(E_\omega^{\nu_{i-1}-b_n}\bar{T}_{\nu_i}\right)^j \le j!\left(E_\omega^{\nu_{i-1}-b_n}\bar{T}_{\nu_{i-1}} + \mu_{i,n,\omega}\right)^j. \tag{74}$$

**Lemma 5.6.** *For any $\varepsilon < \frac{1}{3}$, there exists an $\eta > 0$ such that*

$$Q\left(\exists i \le n, \quad j \in \mathbb{N} : M_i > n^{(1-\varepsilon)/s}, \quad E_\omega^{\nu_{i-1}}(\bar{T}_{\nu_i}^{(n)})^j > j! 2^j \mu_{i,n,\omega}^j\right) = o(n^{-\eta}).$$

*Proof.* We use (74) to get

$$\begin{aligned}
Q\Big(\exists i \le n, \quad j \in \mathbb{N} : M_i > n^{(1-\varepsilon)/s}, \quad & E_\omega^{\nu_{i-1}}(\bar{T}_{\nu_i}^{(n)})^j > j! 2^j \mu_{i,n,\omega}^j\Big) \\
& \le Q\left(\exists i \le n : M_i > n^{(1-\varepsilon)/s}, \quad E_\omega^{\nu_{i-1}-b_n}\bar{T}_{\nu_{i-1}} > \mu_{i,n,\omega}\right) \\
& \le nQ\left(M_1 > n^{(1-\varepsilon)/s}, \quad E_\omega^{\nu-b_n} T_0 > n^{(1-\varepsilon)/s}\right) \\
& = nQ\left(M_1 > n^{(1-\varepsilon)/s}\right) Q\left(E_\omega^{\nu-b_n} T_0 > n^{(1-\varepsilon)/s}\right),
\end{aligned}$$

where the second inequality is due to a union bound and the fact that $\mu_{i,n,\omega} > M_i$. Now, by (13) we have $nQ\left(M_1 > n^{(1-\varepsilon)/s}\right) \sim C_5 n^{\varepsilon}$, and by Theorem 1.4

$$Q\left(E_\omega^{\nu-b_n} T_0 > n^{(1-\varepsilon)/s}\right) \le b_n Q\left(E_\omega T_\nu > \frac{n^{(1-\varepsilon)/s}}{b_n}\right) \sim K_\infty b_n^{1+s} n^{-1+\varepsilon}.$$

Therefore, $Q\left(\exists i \le n, \quad j \in \mathbb{N} : M_i > n^{(1-\varepsilon)/s}, \quad E_\omega^{\nu_{i-1}}(\bar{T}_{\nu_i}^{(n)})^j > j! 2^j \mu_{i,n,\omega}^j\right) = o(n^{-1+3\varepsilon})$. $\square$

**Theorem 5.7.** *Let Assumptions 1 and 2 hold, and let $s < 1$. Then $P-a.s.$ there exists a random subsequence $n_{k_m} = n_{k_m}(\omega)$ of $n_k = 2^{2^k}$ such that for $\alpha_m$, $\beta_m$ and $\gamma_m$ as in (72) and any sequence $x_m \in [\nu_{\beta_m}, \nu_{\gamma_m}]$, we have*

$$\lim_{m\to\infty} P_\omega\left(\frac{T_{x_m} - E_\omega T_{x_m}}{\sqrt{v_{m,\omega}}} \le y\right) = \Phi(y), \tag{75}$$

*where*

$$v_{m,\omega} := \sum_{i=\alpha_m+1}^{\beta_m} \mu_{i,d_{k_m},\omega}^2.$$

*Proof.* Let $n_{k_m}(\omega)$ be the random subsequence specified in Corollary 5.5.1. For ease of notation, set $\tilde{a}_m = a_{k_m}$ and $\tilde{d}_m = d_{k_m}$. We have

$$\max_{i\in(\alpha_m,\beta_m]} \mu_{i,\tilde{d}_m,\omega}^2 \le 2\tilde{d}_m^{2/s} \le \frac{1}{\tilde{a}_m}\sum_{i=\alpha_m+1}^{\beta_m} \mu_{i,\tilde{d}_m,\omega}^2 = \frac{v_{m,\omega}}{\tilde{a}_m}, \quad \text{and} \quad \sum_{i=\beta_m+1}^{\gamma_m} \mu_{i,\tilde{d}_m,\omega} \le 2\tilde{d}_m^{1/s}.$$

Now, let $\{x_m\}_{m=1}^{\infty}$ be any sequence of integers (even depending on $\omega$) such that $x_m \in [\nu_{\beta_m}, \nu_{\gamma_m}]$. Then, since $(T_{x_m} - E_\omega T_{x_m}) = (T_{\nu_{\alpha_m}} - E_\omega T_{\nu_{\alpha_m}}) + (T_{x_m} - T_{\nu_{\alpha_m}} - E_\omega^{\nu_{\alpha_m}} T_{x_m})$, it is enough to prove

$$\frac{T_{\nu_{\alpha_m}} - E_\omega T_{\nu_{\alpha_m}}}{\sqrt{v_{m,\omega}}} \xrightarrow{\mathcal{D}_\omega} 0, \quad \text{and} \quad \frac{T_{x_m} - T_{\nu_{\alpha_m}} - E_\omega^{\nu_{\alpha_m}} T_{x_m}}{\sqrt{v_{m,\omega}}} \xrightarrow{\mathcal{D}_\omega} Z \sim N(0,1) \tag{76}$$

where we use the notation $Z_n \xrightarrow{\mathcal{D}_\omega} Z$ to denote quenched convergence in distribution, that is $\lim_{n\to\infty} P_\omega(Z_n \le z) = P_\omega(Z \le z), P - a.s.$ For the first term in (76) note that for any $\varepsilon > 0$, we have from Chebychev's inequality and $v_{m,\omega} \ge \tilde{d}_m^{2/s}$, that

$$P_\omega\left( \left| \frac{T_{\nu_{\alpha_m}} - E_\omega T_{\nu_{\alpha_m}}}{\sqrt{v_{m,\omega}}} \right| \ge \varepsilon \right) \le \frac{Var_\omega T_{\nu_{\alpha_m}}}{\varepsilon^2 v_{m,\omega}} \le \frac{Var_\omega T_{\nu_{\alpha_m}}}{\varepsilon^2 \tilde{d}_m^{2/s}} .$$

Thus, the first claim in (76) will be proved if we can show that $Var_\omega T_{\nu_{\alpha_m}} = o(\tilde{d}_m^{2/s})$. For this we need the following lemma:

**Lemma 5.8.** *Assume $s \le 2$. Then for any $\delta > 0$,*

$$P\left( Var_\omega T_{\nu_n} \ge n^{2/s+\delta} \right) = o(n^{-\delta s/4}) .$$

*Proof.* First, we claim that

$$E_P (Var_\omega T_1)^\gamma < \infty \text{ for any } \gamma < \frac{s}{2} . \tag{77}$$

Indeed, from (44), we have that for any $\gamma < \frac{s}{2} \le 1$

$$\begin{aligned} E_P(Var_\omega T_1)^\gamma &\le 4^\gamma E_P(W_0 + W_0^2)^\gamma + 8^\gamma \sum_{i<0} E_P\left( \Pi_{i+1,0}^\gamma (W_i + W_i^2)^\gamma \right) \\ &= 4^\gamma E_P(W_0 + W_0^2)^\gamma + 8^\gamma \sum_{i=1}^{\infty} (E_P \rho_0^\gamma)^i E_P(W_0 + W_0^2)^\gamma, \end{aligned}$$

where we used that $P$ is i.i.d. in the last equality. Since $E_P \rho_0^\gamma < 1$ for any $\gamma \in (0, s)$, we have that (77) follows as soon as $E_P(W_0 + W_0^2)^\gamma < \infty$. However, since $W_0$ has the same distribution as $R_0$, we get the latter from (9) when $\gamma < \frac{s}{2}$.

As in Lemma 4.2 let $\bar{\nu} = E_P \nu$. Then,

$$P\left( Var_\omega T_{\nu_n} \ge n^{2/s+\delta} \right) \le P(Var_\omega T_{2\bar{\nu} n} \ge n^{2/s+\delta}) + P(\nu_n \ge 2\bar{\nu} n) .$$

As in Lemma 4.2, the second term is $\mathcal{O}\left(e^{-\delta' n}\right)$ for some $\delta' > 0$. To handle the first term on the right side, we note that for any $\gamma < \frac{s}{2} \le 1$

$$P(Var_\omega T_{2\bar{\nu} n} \ge n^{2/s+\delta}) \le \frac{E_P\left( \sum_{k=1}^{2\bar{\nu} n} Var_\omega(T_k - T_{k-1}) \right)^\gamma}{n^{\gamma(2/s+\delta)}} \le \frac{2\bar{\nu} n E_P(Var_\omega T_1)^\gamma}{n^{\gamma(2/s+\delta)}}. \tag{78}$$

Then since $E_P(Var_\omega T_1)^\gamma < \infty$ for any $\gamma < \frac{s}{2}$, we can choose $\gamma$ arbitrarily close to $\frac{s}{2}$ so that the last term on the right of (78) is $o(n^{-\delta s/4})$. ◻

As a result of Lemma 5.8 and the Borel-Cantelli lemma, we have that $Var_\omega T_{\nu_{n_k}} = o(n_k^{2/s+\delta})$ for any $\delta > 0$. Therefore, for any $\delta \in (0, \frac{2}{s})$ we have $Var_\omega T_{\nu_{\alpha_m}} = o(\alpha_m^{2/s+\delta}) = o(n_{k_m-1}^{2/s+\delta}) = o(\tilde{d}_m^{2/s})$ (in the last equality we use that $d_k \sim n_k$ to grow much faster than exponentially in $k$).

For the next step in the proof, we show that reflections can be added without changing the limiting distribution. Specifically, we show that it is enough to prove the following lemma, whose proof we postpone:

**Lemma 5.9.** *With notation as in Theorem 5.7, we have*

$$\lim_{m\to\infty} P_\omega^{\nu_{\alpha_m}}\left(\frac{\bar{T}_{x_m}^{(\tilde{d}_m)} - E_\omega \bar{T}_{x_m}^{(\tilde{d}_m)}}{\sqrt{v_{m,\omega}}} \le y\right) = \Phi(y)\,. \tag{79}$$

Assuming Lemma 5.9, we complete the proof of Theorem 5.7. It is enough to show that

$$\lim_{m\to\infty} P_\omega^{\nu_{\alpha_m}}(\bar{T}_{x_{k_m}}^{(\tilde{d}_m)} \neq T_{x_m}) = 0, \quad \text{and} \quad \lim_{m\to\infty} E_\omega^{\nu_{\alpha_m}}(T_{x_m} - \bar{T}_{x_{k_m}}^{(\tilde{d}_m)}) = 0.$$

However, since $P_\omega^{\nu_{\alpha_m}}(\bar{T}_{x_m}^{(\tilde{d}_m)} \neq T_{x_m}) = P_\omega^{\nu_{\alpha_m}}\left(T_{x_m} - \bar{T}_{x_m}^{(\tilde{d}_m)} \ge 1\right) \le E_\omega^{\nu_{\alpha_m}}(T_{x_m} - \bar{T}_{x_m}^{(\tilde{d}_m)})$, and $x_m \le \gamma_m = n_{k_m-1} + c_{k_m}\tilde{d}_m \le n_{k_m+1}$ for all $m$ large enough, it is enough to prove

$$\lim_{k\to\infty} E_\omega^{\nu_{n_{k-1}}}\left(T_{\nu_{n_{k+1}}} - \bar{T}_{\nu_{n_{k+1}}}^{(d_k)}\right) = 0, \quad P-a.s. \tag{80}$$

Now, from Lemma 3.2 we have that for any $\varepsilon > 0$

$$Q\left(E_\omega^{\nu_{n_{k-1}}}\left(T_{\nu_{n_{k+1}}} - \bar{T}_{\nu_{n_{k+1}}}^{(d_k)}\right) > \varepsilon\right) \le n_{k+1} Q\left(E_\omega T_\nu - E_\omega \bar{T}_\nu^{(d_k)} > \frac{\varepsilon}{n_{k+1}}\right) = n_{k+1}\mathcal{O}\left(n_{k+1}^s e^{-\delta' b_{d_k}}\right).$$

Since $n_k \sim d_k$, the last term on the right is summable. Therefore, by the Borel-Cantelli lemma,

$$\lim_{k\to\infty} E_\omega^{\nu_{n_{k-1}}}\left(T_{\nu_{n_{k+1}}} - \bar{T}_{\nu_{n_{k+1}}}^{(d_k)}\right) = 0, \quad Q-a.s. \tag{81}$$

This is almost the same as (80), but with $Q$ instead of $P$. To use this to prove (80) note that for $i > b_n$ using (19) we can write

$$E_w^{\nu_{i-1}} T_{\nu_i} - E_w^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)} = A_{i,n}(\omega) + B_{i,n}(\omega) W_{-1}\,,$$

where $A_{i,n}(\omega)$ and $B_{i,n}(\omega)$ are random variables depending only on the environment to the right of 0. Thus, $E_\omega^{\nu_{n_{k-1}}}\left(T_{\nu_{n_{k+1}}} - \bar{T}_{\nu_{n_{k+1}}}^{(d_k)}\right) = A_{d_k}(\omega) + B_{d_k}(\omega)W_{-1}$ where $A_{d_k}(\omega)$ and $B_{d_k}(\omega)$ only depend on the environment to the right of zero (so $A_{d_k}$ and $B_{d_k}$ have the same distribution under $P$ as under $Q$). Therefore (80) follows from (81), which finishes the proof of the theorem. ☐

*Proof of Lemma 5.9.* Clearly, it suffices to show the following claims:

$$\frac{\bar{T}_{x_m}^{(\tilde{d}_m)} - \bar{T}_{\nu_{\beta_m}}^{(\tilde{d}_m)} - E_\omega^{\nu_{\beta_m}} \bar{T}_{x_m}^{(\tilde{d}_m)}}{\sqrt{v_{m,\omega}}} \xrightarrow{\mathcal{D}_\omega} 0, \tag{82}$$

and

$$\frac{\bar{T}_{\nu_{\beta_m}}^{(\tilde{d}_m)} - \bar{T}_{\nu_{\alpha_m}}^{(\tilde{d}_m)} - E_\omega^{\nu_{\alpha_m}} \bar{T}_{\nu_{\beta_m}}^{(\tilde{d}_m)}}{\sqrt{v_{m,\omega}}} \xrightarrow{\mathcal{D}_\omega} Z \sim N(0,1)\,. \tag{83}$$

To prove (82), we note that

$$P_\omega\left(\left|\frac{\bar{T}_{x_m}^{(\tilde{d}_m)} - \bar{T}_{\nu_{\beta_m}}^{(\tilde{d}_m)} - E_\omega^{\nu_{\beta_m}} \bar{T}_{x_m}^{(\tilde{d}_m)}}{\sqrt{v_{m,\omega}}}\right| \ge \varepsilon\right) \le \frac{Var_\omega(\bar{T}_{x_m}^{(\tilde{d}_m)} - \bar{T}_{\nu_{\beta_m}}^{(\tilde{d}_m)})}{\varepsilon^2 v_{m,\omega}} \le \frac{\sum_{i=\beta_m+1}^{\gamma_m} \sigma_{i,\tilde{d}_m,\omega}^2}{\varepsilon^2 \tilde{a}_m \tilde{d}_m^{2/s}},$$

where the last inequality is because $x_m \le \gamma_m$ and $v_{m,\omega} \ge \tilde{a}_m \tilde{d}_m^{2/s}$. However, by Corollary 5.4.1 and the Borel-Cantelli lemma,

$$\sum_{i=\beta_m+1}^{\gamma_m} \sigma_{i,\tilde{d}_m,\omega}^2 = \sum_{i=\beta_m+1}^{\gamma_m} \mu_{i,\tilde{d}_m,\omega}^2 + o\left((c_{k_m}\tilde{d}_m)^{2/s}\right).$$

The application of Corollary 5.4.1 uses the fact that for $k$ large enough the reflections ensure that the events in question do not involve the environment to the left of zero and thus have the same probability under $P$ or $Q$. (This type of argument will be used a few more times in the remainder of the proof without mention.) By our choice of the subsequence $n_{k_m}$ we have

$$\sum_{i=\beta_m+1}^{\gamma_m} \mu_{i,\tilde{d}_m,\omega}^2 \leq \left( \sum_{i=\beta_m+1}^{\gamma_m} \mu_{i,\tilde{d}_m,\omega} \right)^2 \leq 4\tilde{d}_m^{2/s} .$$

Therefore,

$$\lim_{m\to\infty} P_\omega \left( \left| \frac{\bar{T}_{x_m}^{(\tilde{d}_m)} - \bar{T}_{\nu_{\beta_m}}^{(\tilde{d}_m)} - E_\omega^{\nu_{\beta_m}} \bar{T}_{x_m}^{(\tilde{d}_m)}}{\sqrt{v_{m,\omega}}} \right| \geq \varepsilon \right) \leq \lim_{m\to\infty} \frac{4\tilde{d}_m^{2/s} + o\left( (c_{k_m}\tilde{d}_m)^{2/s} \right)}{\varepsilon^2 \tilde{a}_m \tilde{d}_m^{2/s}} = 0, \; P-a.s.$$

where the last limit equals zero because $c_k = o(\log a_k)$.

It only remains to prove (83). Since re-writing we express

$$\bar{T}_{\nu_{\beta_m}}^{(\tilde{d}_m)} - \bar{T}_{\nu_{\alpha_m}}^{(\tilde{d}_m)} - E_\omega^{\nu_{\alpha_m}} \bar{T}_{\nu_{\beta_m}}^{(\tilde{d}_m)} = \sum_{i=\alpha_m+1}^{\beta_m} \left( (\bar{T}_{\nu_i}^{(\tilde{d}_m)} - \bar{T}_{\nu_{i-1}}^{(\tilde{d}_m)}) - \mu_{i,\tilde{d}_m,\omega} \right)$$

as the sum of independent, zero-mean random variables (quenched), we need only show the Lindberg-Feller condition. That is, we need to show

$$\lim_{m\to\infty} \frac{1}{v_{m,\omega}} \sum_{i=\alpha_m+1}^{\beta_m} \sigma_{i,\tilde{d}_m,\omega}^2 = 1, \quad P-a.s. \tag{84}$$

and for all $\varepsilon > 0$

$$\lim_{m\to\infty} \frac{1}{v_{m,\omega}} \sum_{i=\alpha_m+1}^{\beta_m} E_\omega^{\nu_{i-1}} \left[ \left( \bar{T}_{\nu_i}^{(\tilde{d}_m)} - \mu_{i,\tilde{d}_m,\omega} \right)^2 \mathbf{1}_{|\bar{T}_{\nu_i}^{(\tilde{d}_m)} - \mu_{i,\tilde{d}_m,\omega}| > \varepsilon\sqrt{v_{m,\omega}})} \right] = 0, \quad P-a.s. \tag{85}$$

To prove (84) note that

$$\frac{1}{v_{m,\omega}} \sum_{i=\alpha_m+1}^{\beta_m} \sigma_{i,\tilde{d}_m,\omega}^2 = 1 + \frac{\sum_{i=\alpha_m+1}^{\beta_m} \left( \sigma_{i,\tilde{d}_m,\omega}^2 - \mu_{i,\tilde{d}_m,\omega}^2 \right)}{v_{m,\omega}}$$

However, again by Lemma 5.4.1 and the Borel-Cantelli lemma we have $\sum_{i=\alpha_m+1}^{\beta_m}(\sigma_{i,\tilde{d}_m,\omega}^2 - \mu_{i,\tilde{d}_m,\omega}^2) = o\left( (\delta_{k_m}\tilde{d}_m)^{2/s} \right)$. Recalling that $v_{m,\omega} \geq \tilde{a}_m \tilde{d}_m^{2/s}$ we have that (84) is proved.
To prove (85) we break the sum up into two parts depending on whether $M_i$ is "small" or "large". Specifically, for $\varepsilon' \in (0, \frac{1}{3})$ we decompose the sum as

$$\frac{1}{v_{m,\omega}} \sum_{i=\alpha_m+1}^{\beta_m} E_\omega^{\nu_{i-1}} \left[ \left( \bar{T}_{\nu_i}^{(\tilde{d}_m)} - \mu_{i,\tilde{d}_m,\omega} \right)^2 \mathbf{1}_{|\bar{T}_{\nu_i}^{(\tilde{d}_m)} - \mu_{i,\tilde{d}_m,\omega}| > \varepsilon\sqrt{v_{m,\omega}})} \right] \mathbf{1}_{M_i \leq \tilde{d}_m^{(1-\varepsilon')/s}} \tag{86}$$

$$+ \frac{1}{v_{m,\omega}} \sum_{i=\alpha_m+1}^{\beta_m} E_\omega^{\nu_{i-1}} \left[ \left( \bar{T}_{\nu_i}^{(\tilde{d}_m)} - \mu_{i,\tilde{d}_m,\omega} \right)^2 \mathbf{1}_{|\bar{T}_{\nu_i}^{(\tilde{d}_m)} - \mu_{i,\tilde{d}_m,\omega}| > \varepsilon\sqrt{v_{m,\omega}}} \right] \mathbf{1}_{M_i > \tilde{d}_m^{(1-\varepsilon')/s}}. \tag{87}$$

We get an upper bound for (86) by first omitting the indicator function inside the expectation, and then expanding the sum to be up to $n_{k_m} \geq \beta_m$. Thus (86) is bounded above by

$$\frac{1}{v_{m,\omega}} \sum_{i=\alpha_m+1}^{\beta_m} \sigma_{i,\tilde{d}_m,\omega}^2 \mathbf{1}_{M_i \leq \tilde{d}_m^{(1-\varepsilon')/s}} \leq \frac{1}{v_{m,\omega}} \sum_{i=n_{k_m-1}+1}^{n_{k_m}} \sigma_{i,\tilde{d}_m,\omega}^2 \mathbf{1}_{M_i \leq \tilde{d}_m^{(1-\varepsilon')/s}}.$$

However, since $d_k$ grows exponentially fast, the Borel-Cantelli lemma and Lemma 5.4 give that

$$\sum_{i=n_{k-1}+1}^{n_k} \sigma^2_{i,d_k,\omega} \mathbf{1}_{M_i \leq d_k^{(1-\varepsilon')/s}} = o(d_k^{2/s}). \tag{88}$$

Therefore, since our choice of the subsequence $n_{k_m}$ gives that $v_{m,\omega} \geq \tilde{d}_m^{2/s}$, we have that (86) tends to zero as $m \to \infty$.

To get an upper bound for (87), first note that our choice of the subsequence $n_{k_m}$ gives that $\varepsilon\sqrt{v_{m,\omega}} \geq \varepsilon\sqrt{\tilde{a}_m}\mu_{i,\tilde{d}_m,\omega}$ for any $i \in (\alpha_m, \beta_m]$. Thus, for $m$ large enough we can replace the indicators inside the expectations in (87) by the indicators of the events $\{\bar{T}^{(\tilde{d}_m)}_{\nu_i} > (1 + \varepsilon\sqrt{\tilde{a}_m})\mu_{i,\tilde{d}_m,\omega}\}$. Thus, for $m$ large enough and $i \in (\alpha_m, \beta_m]$, we have

$$\begin{aligned}
&E^{\nu_{i-1}}_\omega \left[ \left( \bar{T}^{(\tilde{d}_m)}_{\nu_i} - \mu_{i,\tilde{d}_m,\omega} \right)^2 \mathbf{1}_{|\bar{T}^{(\tilde{d}_m)}_{\nu_i} - \mu_{i,\tilde{d}_m,\omega}| > \varepsilon\sqrt{v_{m,\omega}}} \right] \\
&\quad \leq E^{\nu_{i-1}}_\omega \left[ \left( \bar{T}^{(\tilde{d}_m)}_{\nu_i} - \mu_{i,\tilde{d}_m,\omega} \right)^2 \mathbf{1}_{\bar{T}^{(\tilde{d}_m)}_{\nu_i} > (1+\varepsilon\sqrt{\tilde{a}_m})\mu_{i,\tilde{d}_m,\omega}} \right] \\
&\quad = \int_{1+\varepsilon\sqrt{\tilde{a}_m}}^\infty P^{\nu_{i-1}}_\omega \left( \bar{T}^{(\tilde{d}_m)}_{\nu_i} > x\mu_{i,\tilde{d}_m,\omega} \right) 2(x-1)\mu^2_{i,\tilde{d}_m,\omega} dx \,.
\end{aligned}$$

We want to use Lemma 5.6 get an upper bound on the probability inside the integral on the last line above. Lemma 5.6 and the Borel-Cantelli lemma give that for $k$ large enough, $E^{\nu_{i-1}}_\omega \left( \bar{T}^{(d_k)}_{\nu_i} \right)^j \leq 2^j j! \mu^j_{i,d_k,\omega}$, for all $n_{k-1} < i \leq n_k$ such that $M_i > d_k^{(1-\varepsilon')/s}$. Multiplying by $(4\mu_{i,d_k,\omega})^{-j}$ and summing over $j$ gives that $E^{\nu_{i-1}}_\omega e^{\bar{T}^{(d_k)}_{\nu_i}/(4\mu_{i,d_k,\omega})} \leq 2$. Therefore, Chebychev's inequality gives

$$P^{\nu_{i-1}}_\omega \left( \bar{T}^{(d_k)}_{\nu_i} > x\mu_{i,d_k,\omega} \right) \leq e^{-x/4} E^{\nu_{i-1}}_\omega e^{\bar{T}^{(d_k)}_{\nu_i}/(4\mu_{i,d_k,\omega})} \leq 2e^{-x/4} \,.$$

Thus, for all $m$ large enough we have for all $\alpha_m < i \leq \beta_m \leq n_{k_m}$ with $M_i > \tilde{d}_m^{(1-\varepsilon')/s}$ that

$$\begin{aligned}
\int_{1+\varepsilon\sqrt{\tilde{a}_m}}^\infty P^{\nu_{i-1}}_\omega \left( \bar{T}^{(\tilde{d}_m)}_{\nu_i} > x\mu_{i,\tilde{d}_m,\omega} \right) 2(x-1)\mu^2_{i,\tilde{d}_m,\omega} dx &\leq \int_{1+\varepsilon\sqrt{\tilde{a}_m}}^\infty 2e^{-x/4} 2(x-1)\mu^2_{i,\tilde{d}_m,\omega} dx \\
&= 16(4+\varepsilon\sqrt{\tilde{a}_m}) e^{-(1+\varepsilon\sqrt{\tilde{a}_m})/4} \mu^2_{i,\tilde{d}_m,\omega} \,.
\end{aligned}$$

Recalling the definition of $v_{m,\omega} = \sum_{i=\alpha_m+1}^{\beta_m} \mu^2_{i,\tilde{d}_m,\omega}$, we have that as $m \to \infty$, (87) is bounded above by

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{v_{m,\omega}} \sum_{i=\alpha_m+1}^{\beta_m} 16(4+\varepsilon\sqrt{\tilde{a}_m}) e^{-(1+\varepsilon\sqrt{\tilde{a}_m})/4} \mu^2_{i,\tilde{d}_m,\omega} \mathbf{1}_{M_i > \tilde{d}_m^{(1-\varepsilon')/s}} \\
&\quad \leq \lim_{m\to\infty} 16(4+\varepsilon\sqrt{\tilde{a}_m}) e^{-(1+\varepsilon\sqrt{\tilde{a}_m})/4} = 0 \,.
\end{aligned}$$

This finishes the proof of (85) and thus of Lemma 5.9. □

***Proof of Theorem 1.3:***
Note first that from Lemma 4.2 and the Borel-Cantelli lemma, we have that for any $\varepsilon > 0$, $E_\omega T_{\nu_{n_k}} = o(n_k^{(1+\varepsilon)/s})$, $P-a.s.$ This is equivalent to

$$\limsup_{k\to\infty} \frac{\log E_\omega T_{\nu_{n_k}}}{\log n_k} \leq \frac{1}{s}, \quad P-a.s. \tag{89}$$

We can also get bounds on the probability of $E_\omega T_{\nu_n}$ being small. Since $E^{\nu_{i-1}}_\omega T_{\nu_i} \geq M_i$ we have

$$P\left( E_\omega T_{\nu_n} \leq n^{(1-\varepsilon)/s} \right) \leq P\left( M_i \leq n^{(1-\varepsilon)/s}, \quad \forall i \leq n \right) \leq \left( 1 - P\left( M_1 > n^{(1-\varepsilon)/s} \right) \right)^n,$$

and since $P(M_1 > n^{(1-\varepsilon)/s}) \sim C_5 n^{1-\varepsilon}$, see (9), we have $P\left(E_\omega T_{\nu_n} \le n^{(1-\varepsilon)/s}\right) \le e^{-n^{\varepsilon/2}}$. Thus, by the Borel-Cantelli lemma, for any $\varepsilon > 0$ we have that $E_\omega T_{\nu_{n_k}} \ge n_k^{(1-\varepsilon)/s}$ for all $k$ large enough, $P-a.s.$, or equivalently

$$\liminf_{k\to\infty} \frac{\log E_\omega T_{\nu_{n_k}}}{\log n_k} \ge \frac{1}{s}, \quad P-a.s. \tag{90}$$

Let $n_{k_m}$ be the subsequence specified in Theorem 5.7, and define $t_m := E_\omega T_{n_{k_m}}$. Then, by (89) and (90), $\lim_{m\to\infty} \frac{\log t_m}{\log n_{k_m}} = 1/s$.
For any $t$ define $X_t^* := \max\{X_n : n \le t\}$. Then, for any $x \in (0,\infty)$ we have

$$\begin{aligned} P_\omega\left(\frac{X^*_{t_m}}{n_{k_m}} < x\right) &= P\left(X^*_{t_m} < x n_{k_m}\right) = P_\omega\left(T_{xn_{k_m}} > t_m\right) \\ &= P_\omega\left(\frac{T_{xn_{k_m}} - E_\omega T_{xn_{k_m}}}{\sqrt{v_{m,\omega}}} > \frac{E_\omega T_{n_{k_m}} - E_\omega T_{xn_{k_m}}}{\sqrt{v_{m,\omega}}}\right). \end{aligned}$$

Now, with notation as in Theorem 5.7, we have that for all $m$ large enough $\nu_{\beta_m} < x n_{k_m} < \nu_{\gamma_m}$ (note that this also uses the fact that $\nu_n/n \to E_P\nu$, $P-a.s.$). Thus $\frac{T_{xn_{k_m}} - E_\omega T_{xn_{k_m}}}{\sqrt{v_{m,\omega}}} \xrightarrow{\mathcal{D}_\omega} Z \sim N(0,1)$. Then, we will have proved that $\lim_{m\to\infty} P_\omega\left(\frac{X^*_{t_m}}{n_{k_m}} < x\right) = \frac{1}{2}$ for any $x \in (0,\infty)$ if we can show

$$\lim_{m\to\infty} \frac{E_\omega T_{n_{k_m}} - E_\omega T_{xn_{k_m}}}{\sqrt{v_{m,\omega}}} = 0\,, \quad P-a.s. \tag{91}$$

For $m$ large enough we have $n_{k_m}, xn_{k_m} \in (\nu_{\beta_m}, \nu_{\gamma_m})$. Thus, for $m$ large enough,

$$\left|\frac{E_\omega T_{xn_{k_m}} - E_\omega T_{n_{k_m}}}{\sqrt{v_{m,\omega}}}\right| \le \frac{E_\omega^{\nu_{\beta_m}} T_{\nu_{\gamma_m}}}{\sqrt{v_{m,\omega}}} = \frac{1}{\sqrt{v_{m,\omega}}}\left(E_\omega^{\nu_{\beta_m}}\left(T_{\nu_{\gamma_m}} - \bar{T}^{(\tilde{d}_m)}_{\nu_{\gamma_m}}\right) + \sum_{i=\beta_m+1}^{\gamma_m} \mu_{i,\tilde{d}_m,\omega}\right).$$

Since $\alpha_m \le \beta_m \le \gamma_m \le n_{k_m+1}$ for all $m$ large enough, we can apply (80) to get

$$\lim_{m\to\infty} E_\omega^{\nu_{\beta_m}}\left(T_{\nu_{\gamma_m}} - \bar{T}^{(\tilde{d}_m)}_{\nu_{\gamma_m}}\right) \le \lim_{m\to\infty} E_\omega^{\nu_{\alpha_m}}\left(T_{\nu_{n_{k_m+1}}} - \bar{T}^{(\tilde{d}_m)}_{\nu_{n_{k_m+1}}}\right) = 0.$$

Also, from our choice of $n_{k_m}$ we have that $\sum_{i=\beta_m+1}^{\gamma_m} \mu_{i,\tilde{d}_m,\omega} \le 2\tilde{d}_m^{1/s}$ and $v_{m,\omega} \ge \tilde{a}_m \tilde{d}_m^{2/s}$. Thus (91) is proved. Therefore

$$\lim_{m\to\infty} P_\omega\left(\frac{X^*_{t_m}}{n_{k_m}} \le x\right) = \frac{1}{2}, \quad \forall x \in (0,\infty),$$

and obviously $\lim_{m\to\infty} P_\omega\left(\frac{X^*_{t_m}}{n_{k_m}} < 0\right) = 0$ since $X_n$ is transient to the right $\mathbb{P}-a.s.$ due to Assumption 1. Finally, note that

$$\frac{X_t^* - X_t}{\log^2 t} = \frac{X_t^* - \nu_{N_t}}{\log^2 t} + \frac{\nu_{N_t} - X_t}{\log^2 t} \le \frac{\max_{i\le t}(\nu_i - \nu_{i-1})}{\log^2 t} + \frac{\nu_{N_t} - X_t}{\log^2 t}.$$

However, Lemma 4.4 and an easy application of Lemma 2.1 and the Borel-Cantelli lemma gives that

$$\lim_{t\to\infty} \frac{X_t^* - X_t}{\log^2 t} = 0, \quad P-a.s.$$

This finishes the proof of the theorem. ∎

## 6 Asymptotics of the tail of $E_\omega T_\nu$

Recall that $E_\omega T_\nu = \nu + 2\sum_{j=0}^{\nu-1} W_j = \nu + 2\sum_{i\le j, 0\le j<\nu} \Pi_{i,j}$, and for any $A > 1$ define

$$\sigma = \sigma_A = \inf\{n \ge 1 : \Pi_{0,n-1} \ge A\}\,.$$

Note that $\sigma - 1$ is a stopping time for the sequence $\Pi_{0,k}$. For any $A > 1$, $\{\sigma > \nu\} = \{M_1 < A\}$. Thus we have by (15) that for any $A > 1$,

$$Q(E_\omega T_\nu > x, \sigma > \nu) = Q(E_\omega T_\nu > x, M_1 < A) = o(x^{-s}). \tag{92}$$

Thus, we may focus on the tail estimates $Q(E_\omega T_\nu > x, \sigma < \nu)$ in which case we can use the following expansion of $E_\omega T_\nu$:

$$\begin{aligned} E_\omega T_\nu &= \nu + 2\sum_{i<0\le j<\sigma-1} \Pi_{i,j} + 2\sum_{0\le i\le j<\sigma-1} \Pi_{i,j} + 2\sum_{\sigma\le i\le j<\nu} \Pi_{i,j} + 2\sum_{i\le\sigma-1\le j<\nu} \Pi_{i,j} \\ &= \nu + 2W_{-1}R_{0,\sigma-2} + 2\sum_{j=0}^{\sigma-2} W_{0,j} + 2\sum_{i=\sigma}^{\nu-1} R_{i,\nu-1} + 2W_{\sigma-1}(1 + R_{\sigma,\nu-1})\,. \end{aligned} \tag{93}$$

We will show that the dominant term in (93) is the last term: $2W_{\sigma-1}(1 + R_{\sigma,\nu-1})$. A few easy consequences of Lemmas 2.1 and 2.2 are that the tails of the first three terms in the expansion (93) are negligible. The following statements are true for any $\delta > 0$ and any $A > 1$:

$$Q(\nu > \delta x) = P(\nu > \delta x) = o(x^{-s})\,, \tag{94}$$

$$\begin{aligned} Q(2W_{-1}R_{0,\sigma-2} > \delta x, \sigma < \nu) &\le Q(W_{-1} > \sqrt{\delta x}) + P(2R_{0,\sigma-2} > \sqrt{\delta x}, \sigma < \nu) \\ &\le Q(W_{-1} > \sqrt{\delta x}) + P(2\nu A > \sqrt{\delta x}) = o(x^{-s}), \end{aligned} \tag{95}$$

$$Q\left(2\sum_{j=0}^{\sigma-2} W_{0,j} > \delta x, \sigma < \nu\right) \le P\left(2\sum_{j=1}^{\sigma-1} jA > \delta x, \sigma < \nu\right) \le P(\nu^2 A > \delta x) = o(x^{-s}). \tag{96}$$

The fourth term in (93) is not negligible, but we can make it arbitrarily small by taking $A$ large enough.

**Lemma 6.1.** *For all $\delta > 0$, there exists an $A_0 = A_0(\delta) < \infty$ such that*

$$P\left(2\sum_{\sigma_A\le i<\nu} R_{i,\nu-1} > \delta x\right) < \delta x^{-s}, \quad \forall A \ge A_0(\delta)\,.$$

*Proof.* This proof is essentially a copy of the proof of Lemma 3 in [10].

$$\begin{aligned} P\left(2\sum_{\sigma_A\le i<\nu} R_{i,\nu-1} > \delta x\right) &\le P\left(\sum_{\sigma_A\le i<\nu} R_i > \frac{\delta}{2}x\right) = P\left(\sum_{i=1}^{\infty} \mathbf{1}_{\sigma_A\le i<\nu} R_i > \frac{\delta}{2}x\frac{6}{\pi^2}\sum_{i=1}^{\infty} i^{-2}\right) \\ &\le \sum_{i=1}^{\infty} P\left(\mathbf{1}_{\sigma_A\le i<\nu} R_i > x\frac{3\delta}{\pi^2}i^{-2}\right)\,. \end{aligned}$$

However, since the event $\{\sigma_A \le i < \nu\}$ depends only on $\rho_j$ for $j < i$, and $R_i$ depends only on $\rho_j$ for $j \ge i$, we have that

$$P\left(2\sum_{\sigma_A\le i<\nu} R_{i,\nu-1} > \delta x\right) \le \sum_{i=1}^{\infty} P(\sigma_A \le i < \nu)\, P\left(R_i > x\frac{3\delta}{\pi^2}i^{-2}\right)\,.$$

Now, from (11) we have that there exists a $K_1 > 0$ such that $P(R_0 > x) \le K_1 x^{-s}$ for all $x > 0$. We then conclude that

$$P\left(\sum_{\sigma_A \le i < \nu} R_{i,\nu-1} > \delta x\right) \le K_1 \left(\frac{3\delta}{\pi^2}\right)^{-s} x^{-s} \sum_{i=1}^{\infty} P\left(\sigma_A \le i < \nu\right) i^{2s}$$
$$= K_1 \left(\frac{3\delta}{\pi^2}\right)^{-s} x^{-s} E_P\left[\sum_{i=1}^{\infty} \mathbf{1}_{\sigma_A \le i < \nu} i^{2s}\right]$$
$$\le K_1 \left(\frac{3\delta}{\pi^2}\right)^{-s} x^{-s} E_P[\nu^{2s+1} \mathbf{1}_{\sigma_A < \nu}] . \tag{97}$$

Since $E_P \nu^{2s+1} < \infty$ and $\lim_{A\to\infty} P(\sigma_A < \nu) = 0$, we have that the right side of (97) can be made less than $\delta x^{-s}$ by choosing $A$ large enough. $\square$

We need one more lemma before analyzing the dominant term in (93).

**Lemma 6.2.** *$E_Q\left[W_{\sigma-1}^t \mathbf{1}_{\sigma<\nu}\right] < \infty$ for all $A > 1$ and all $t > 0$.*

*Proof.* Since $W_{\sigma-1} = W_{0,\sigma-1} + \Pi_{0,\sigma-1} W_{-1}$, we need only to show that $E_Q[W_{0,\sigma-1}^t \mathbf{1}_{\sigma<\nu}] < \infty$ and $E_Q[\Pi_{0,\sigma-1}^t W_{-1}^t \mathbf{1}_{\sigma<\nu}] < \infty$.
By Assumption 2 we have $\Pi_{0,\sigma-1} < \rho_{\max} A$, and Lemma 2.1 gives $E_P \nu^t < \infty$. Thus,

$$E_Q[W_{0,\sigma-1}^t \mathbf{1}_{\sigma<\nu}] \le E_P[\sigma^t \Pi_{0,\sigma-1}^t \mathbf{1}_{\sigma<\nu}] \le \rho_{\max}^t A^t E_P[\nu^t] < \infty.$$

Similarly, since Lemma 2.2 gives $E_Q W_{-1}^t < \infty$ we have

$$E_Q[\Pi_{0,\sigma-1}^t W_{-1}^t \mathbf{1}_{\sigma<\nu}] \le \rho_{\max}^t A^t E_Q[W_{-1}^t] < \infty .$$

$\square$

Finally, we turn to the asymptotics of the tail of $2W_{\sigma-1}(1 + R_{\sigma,\nu-1})$, which is the dominant term in (93).

**Lemma 6.3.** *For any $A > 1$, there exists a constant $K_A \in (0, \infty)$ such that*

$$\lim_{x\to\infty} x^s Q\left(W_{\sigma-1}(1 + R_{\sigma,\nu-1}) > x\right) = K_A ,$$

*where we use the convention that $W_{\sigma-1} = R_{\sigma,\nu-1} = 0$ when $\sigma > \nu$.*

*Proof.* The strategy of the proof is as follows. First, note that on the event $\{\sigma < \nu\}$ we have $W_{\sigma-1}(1+R_\sigma) = W_{\sigma-1}(1+R_{\sigma,\nu-1})+W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu$. We will begin by analyzing the asymptotics of the tails of $W_{\sigma-1}(1 + R_\sigma)$ and $W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu$. Next we will show that $W_{\sigma-1}(1 + R_{\sigma,\nu-1})$ and $W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu$ are essentially independent in the sense that they cannot both be large. This will allow us to use the asymptotics of the tails of $W_{\sigma-1}(1 + R_\sigma)$ and $W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu$ to compute the asymptotics of the tails of $W_{\sigma-1}(1 + R_{\sigma,\nu-1})$.
To analyze the asymptotics of the tail of $W_{\sigma-1}(1 + R_\sigma)$, we first recall from (11) that there exists a $K > 0$ such that $P(R_0 > x) \sim Kx^{-s}$. Let $\mathcal{F}_{\sigma-1} = \sigma(\ldots, \omega_{\sigma-2}, \omega_{\sigma-1})$ be the $\sigma-$algebra generated by the environment to the left of $\sigma$. Then on the event $\{\sigma < \infty\}$, $R_\sigma$ has the same distribution as $R_0$ and is independent of $\mathcal{F}_{\sigma-1}$. Thus,

$$\lim_{x\to\infty} x^s Q(W_{\sigma-1}(1 + R_\sigma) > x, \sigma < \nu) = \lim_{x\to\infty} E\left[x^s Q\left(1 + R_\sigma > \frac{x}{W_{\sigma-1}}, \sigma < \nu \middle| \mathcal{F}_{\sigma-1}\right)\right]$$
$$= KW_{\sigma-1}^s \mathbf{1}_{\sigma<\nu}. \tag{98}$$

A similar calculation yields

$$\begin{aligned}\lim_{x\to\infty} x^s Q\left(W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu > x, \sigma<\nu\right) &= \lim_{x\to\infty} E_Q\left[x^s Q\left( R_\nu > \frac{x}{W_{\sigma-1}\Pi_{\sigma,\nu-1}}, \sigma<\nu \middle| \mathcal{F}_{\nu-1}\right)\right] \\ &= E_Q\left[W_{\sigma-1}^s \Pi_{\sigma,\nu-1}^s \mathbf{1}_{\sigma<\nu}\right] K. \end{aligned} \tag{99}$$

Next, since $\Pi_{\sigma,\nu-1} < \frac{1}{A}$ on the event $\{\sigma<\nu\}$ we have for any $\varepsilon>0$ that

$$\begin{aligned} &Q\left(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > \varepsilon x,\, W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu > \varepsilon x,\, \sigma<\nu\right) \\ &\quad \le Q\left(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > \varepsilon x, W_{\sigma-1}R_\nu > A\varepsilon x, \sigma<\nu\right) \\ &\quad = E_Q\left[Q\left(1+R_{\sigma,\nu-1} > \frac{\varepsilon x}{W_{\sigma-1}}, \sigma<\nu|\mathcal{F}_{\sigma-1}\right) Q\left(R_\nu > A\frac{\varepsilon x}{W_{\sigma-1}}, \sigma<\nu|\mathcal{F}_{\sigma-1}\right)\right] \\ &\quad \le E_Q\left[Q\left(1+R_\sigma > \frac{\varepsilon x}{W_{\sigma-1}}, \sigma<\nu|\mathcal{F}_{\sigma-1}\right) Q\left(R_\nu > A\frac{\varepsilon x}{W_{\sigma-1}}|\mathcal{F}_{\sigma-1}\right)\right], \end{aligned} \tag{100}$$

where the inequality inequality on the third line is because $R_{\sigma,\nu-1}$ and $R_\nu$ are independent when $\sigma<\nu$ (note that $\{\sigma<\nu\}\in\mathcal{F}_{\sigma-1}$), and the last inequality is because $R_{\sigma,\nu-1}\le R_\sigma$. Now, conditioned on $\mathcal{F}_{\sigma-1}$, $R_\sigma$ and $R_\nu$ have the same distribution as $R_0$. Then, since by (11) there exists a $\tilde{K}_1>0$ such that $P(1+R_0>x)\le \tilde{K}_1 x^{-s}$, we have that (100) is bounded above by

$$E_Q\left[W_{\sigma-1}^{2s}\mathbf{1}_{\sigma<\nu}\right]\tilde{K}_1^2 A^s \varepsilon^{-2s} x^{-2s}.$$

Since $E_Q\left[W_{\sigma-1}^{2s}\mathbf{1}_{\sigma<\nu}\right]<\infty$ by Lemma 6.2, we have that

$$\lim_{x\to\infty} x^s Q(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > \varepsilon x,\, W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu > \varepsilon x,\, \sigma<\nu) = 0. \tag{101}$$

Therefore, since $R_\sigma = R_{\sigma,\nu-1} + \Pi_{\sigma,\nu-1}R_\nu$, we have that for any $\varepsilon>0$

$$\begin{aligned} Q(W_{\sigma-1}(1+R_\sigma) > (1+\varepsilon)x, \sigma<\nu) &\le Q(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > \varepsilon x,\, W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu > \varepsilon x,\, \sigma<\nu) \\ &\quad + Q(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > x, \sigma<\nu) \\ &\quad + Q(W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu > x, \sigma<\nu). \end{aligned}$$

Applying (98), (99) and (101) we get that for any $\varepsilon>0$

$$\liminf_{x\to\infty} x^s Q(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > x, \sigma<\nu) \ge K E_Q[W_{\sigma-1}^s \mathbf{1}_{\sigma<\nu}](1+\varepsilon)^{-s} - K E_Q[W_{\sigma-1}^s \Pi_{\sigma,\nu-1}^s \mathbf{1}_{\sigma<\nu}]. \tag{102}$$

Similarly, for a bound in the other direction we have

$$\begin{aligned} Q(W_{\sigma-1}(1+R_\sigma) > x, \sigma<\nu) &\ge Q(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > x, \text{ or } W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu > x, \sigma<\nu) \\ &= Q(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > x, \sigma<\nu) \\ &\quad + Q(W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu > x, \sigma<\nu) \\ &\quad - Q(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > x, W_{\sigma-1}\Pi_{\sigma,\nu-1}R_\nu > x, \sigma<\nu). \end{aligned}$$

Thus, again applying (98),(99) and (101) we get

$$\limsup_{x\to\infty} x^s Q(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > x, \sigma<\nu) \le K E_Q[W_{\sigma-1}^s \mathbf{1}_{\sigma<\nu}] - K E_Q[W_{\sigma-1}^s \Pi_{\sigma,\nu-1}^s \mathbf{1}_{\sigma<\nu}]. \tag{103}$$

Finally, applying (102) and (103) and letting $\varepsilon\to 0$, we get that

$$\lim_{x\to\infty} x^s Q(W_{\sigma-1}(1+R_{\sigma,\nu-1}) > x, \sigma<\nu) = K E_Q[W_{\sigma-1}^s(1-\Pi_{\sigma,\nu-1}^s)\mathbf{1}_{\sigma<\nu}] =: K_A,$$

and $K_A\in(0,\infty)$ by Lemma 6.2, and the fact that $1-\Pi_{\sigma,\nu-1}\in(1-\frac{1}{A},1)$. ☐

Finally, we are ready to analyze the tail of $E_\omega T_\nu$ under the measure $Q$.

***Proof of Theorem 1.4:***
Let $\delta > 0$, and choose $A \geq A_0(\delta)$ as in Lemma 6.1. Then using (93) we have

$$\begin{aligned} Q(E_\omega T_\nu > x) &= Q(E_\omega T_\nu > x, \sigma > \nu) + Q(E_\omega T_\nu > x, \sigma < \nu) \\ &\leq Q(E_\omega T_\nu > x, \sigma > \nu) + Q(\nu > \delta t) + Q(2W_{-1}R_{0,\sigma-2} > \delta t, \sigma < \nu) \\ &\quad + Q\left(2\sum_{j=0}^{\sigma-2} W_{0,j} > \delta t, \sigma < \nu\right) + Q\left(2\sum_{\sigma \leq i < \nu} R_{i,\nu-1} > \delta t\right) \\ &\quad + Q(2W_{\sigma-1}(1 + R_{\sigma,\nu-1}) > (1-4\delta)x, \sigma < \nu)\,. \end{aligned}$$

Thus combining equations (92), (94), (95), and (96), and Lemmas 6.1 and 6.3, we get that

$$\limsup_{x\to\infty} x^s Q(E_\omega T_\nu > x) \leq \delta + 2^s K_A (1-4\delta)^{-s}. \tag{104}$$

The lower bound is easier, since $Q(E_\omega T_\nu > x) \geq Q(2W_{\sigma-1}(1+R_{\sigma,\nu-1}) > x, \sigma < \nu)$. Thus

$$\liminf_{x\to\infty} x^s Q(E_\omega T_\nu > x) \geq 2^s K_A\,. \tag{105}$$

From (104) and (105) we get that $\overline{K} := \limsup_{A\to\infty} 2^s K_A < \infty$. Therefore, letting $\underline{K} := \liminf_{A\to\infty} 2^s K_A$ we have from (104) and (105) that

$$\overline{K} \leq \liminf_{x\to\infty} x^s Q(E_\omega T_\nu > x) \leq \limsup_{x\to\infty} x^s Q(E_\omega T_\nu > x) \leq \delta + \underline{K}(1-4\delta)^{-s}$$

Then, letting $\delta \to 0$ completes the proof of the theorem with $K_\infty = \underline{K} = \overline{K}$. □

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