Greedy Algorithms and Matroids (homework)

We shall study several problems that can be solved to optimality in polynomial time using the greedy algorithm. One class of such problems is referred to as *matroids*.

Let $S = \{e_1, \ldots, e_n\}$ be a finite set, $c : S \rightarrow R^+$ a cost function on the elements of S, and $F \subset 2^S$ a collection of subsets of S. We say that F is *hereditary* if it is closed under taking subsets.

We wish to find $X \in F$ that maximizes $\sum_{e \in X} c(e)$. F is a *matroid* if it satisfies the following property:

- For all $X, Y \in F$, if $|X| > |Y|$ then there is some $e \in X \setminus Y$ such that $Y \bigcup \{e\} \in F$.

Examples of matroids include partition matroids, forests in graphs, and linearly independent subsets of vectors.

The maximal elements of F are called *bases* of the matroid.

Theorem: The naive greedy algorithm can be used to optimize over F for every cost function c iff F is a matroid.

Homework.

1. (Exercise 17.1-2 from Cormen, Leiserson, Rivest, Stein.) Give a polynomial time greedy algorithm to legally color interval graphs with the minimum number of colors (and prove its correctness), when the interval representation of the graph is given. (That is, the vertices of the graph correspond to intervals on the real line, and two vertices are connected by an edge, and hence need to receive different colors, if their respective intervals overlap.)

2. Let M be a matrix with m rows and n columns, where $n > m$, and with 0/1 entries. Suppose that the rows of m are linearly independent over R (there are no real coefficients, not all 0, such that the weighted sum of all rows is the 0 vector). One needs to select m columns such that the resulting m by m matrix is invertible, and moreover, as sparse as possible (contains the largest number of 0 entries among all m by m invertible submatrices). Prove that this problem can be solved in polynomial time. (You may need to refresh your knowledge about basic linear algebra.)

3. Let (S, F) be a matroid as defined above. Let F^* be the following collection of subsets of S: $X \in F^*$ iff there is a set $Y \subset S \setminus X$ such that Y is a basis of (S, F). Prove that (S, F^*) is a matroid, and furthermore, that $(F^*)^* = F$. (**Remark.** (S, F^*) as defined above is referred to as the *dual* matroid of (S, F), and (S, F) is the dual matroid of (S, F^*). Interestingly, matroid duality provides a characterization of graph planarity: a graph is planar iff the dual of its graphic matroid is a graphic matroid.)