Semidefinite programming and max-cut

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Read sections 6.1 and 6.2 (pages 137–143) in the book [2]. (The book should be available on the web. If you do not find it, ask me and I will send you the relevant chapter.) These sections show how to use semidefinite programming (SDP) and the random hyperplane rounding technique in order to approximate max-cut within a ratio of roughly 0.878 (a result of [1]).

Homework.

Maximization linear programs have dual minimization linear programs, where the optimal values of both programs are the same. The same applies to semidefinite programs (under some mild conditions that do hold for the max-cut SDP).

1. (Problem 6.1 in [2].) Given a complete graph $G(V, E)$ with non-negative edge weights w_{ij}, the dual of the max-cut SDP is:

   minimize $\frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\sum_{i\in V} \gamma_i$
   subject to $W + diag(\gamma)$ is positive semidefinite

   In the above, W is the symmetric matrix of the edge weights w_{ij}, and $diag(\gamma)$ is the diagonal matrix with γ_i as the ith entry on the diagonal. Show (directly, without appealing to SDP duality) that the value of every feasible solution for the dual is an upper bound on the weight of the maximum cut.

2. For the unweighted triangle graph (three vertices, three edges), the max-cut value is 2. Show that the maximum value of the SDP relaxation is exactly $\frac{9}{4}$. (Use the primal SDP to give a lower bound and the dual SDP to give an upper bound.)

References

[1] Michel Goemans and David Williamson: Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming. J. ACM 42(6): 1115–1145 (1995).

[2] David Williamson, David Shmoys: The Design of Approximation Algorithms. Cambridge University Press 2011.