Spectral graph theory (homework)

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The adjacency matrix of a connected undirected graph is nonnegative, symmetric and irreducible (namely, it cannot be decomposed into two diagonal blocks and two off-diagonal blocks, one of which is all-0). As such, the Perron-Frobenius theorem implies that:

1. All its eigenvalues are real. Let us denote them by $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$.
2. Eigenvectors that correspond to different eigenvalues are orthogonal to each other.
3. The eigenvector that corresponds to λ_1 is all positive.
4. $\lambda_1 > \lambda_2$ and $\lambda_1 \geq |\lambda_n|$.

There is a useful characterization of eigenvalues by Raleigh quotients. Let $v_1, \ldots, v_n$ be an orthonormal basis of eigenvalues. For a nonzero vector x, let $a_i =< x, v_i >$ and hence $x = \sum a_i v_i$. Observe that:

$$\frac{x^t A x}{x^t x} = \frac{\sum \lambda_i (a_i)^2}{\sum (a_i)^2}$$

This implies that $\lambda_n \leq \frac{x^t A x}{x^t x} \leq \lambda_1$. Moreover, if x is orthogonal to v_1 then $\frac{x^t A x}{x^t x} \leq \lambda_2$.

The following theorem due to Hoffman is a useful example of how spectral graph theory connects to combinatorial properties of graphs. Let G be a d-regular n-vertex graph and let λ_n be the most negative eigenvalue of its adjacency matrix A. Then the size of its largest independent is at most

$$\alpha(G) \leq -\frac{n\lambda_n}{d - \lambda_n}$$

Homework.

1. Let G be an arbitrary undirected graph, and let $\lambda_1(G)$ be the largest eigenvalue of G's adjacency matrix.
   (a) For every subgraph H of G, prove that $\lambda_1(G)$ is at least as large as the average degree of H.
   (b) Prove that G can be legally colored by $\lfloor \lambda_1(G) \rfloor + 1$ colors.
   (c) Give an example of a graph G that cannot be legally colored with fewer than $\lambda_1(G) + 1$ colors.

2. Let G be a bipartite graph with sides U and V. Let x be an arbitrary eigenvector of its adjacency matrix whose corresponding eigenvalue is nonzero, and let x_i denote the value of x on coordinate i corresponding to vertex i. Prove that $\sum_{i \in U}(x_i)^2 = \sum_{i \in V}(x_i)^2$. (Hint: this has a short proof.)

3. Consider a "star" graph that is a tree with one central vertex and all other vertices are leaves (adjacent only to the central vertex). List *all* eigenvalues of the corresponding adjacency matrix and prove the correctness of your result.

4. Consider an arbitrary graph and let d be its maximum degree. Give a lower bound (as a function of d) on the largest eigenvalue of its adjacency matrix, and prove that your lower bound is best possible (namely, that it holds for every graph, and there are graphs of maximum degree d in which the largest eigenvalue meets this lower bound).

   Remark. d cannot be used in order to provide an interesting upper bound (equivalently, lower bound on absolute value) for the most negative eigenvalue. For example: the complete graph on $d+1$ vertices has largest eigenvalue d and all other eigenvalues are -1. (Please verify that you know how to prove this last statement.)

5. Prove that for the adjacency matrix A of d-regular graphs, $\max[\lambda_2(A), |\lambda_n(A)|] \geq \sqrt{d} - o(1)$, where the $o(1)$ term tends to 0 as the number n of vertices grows (keeping d fixed). Furthermore, prove that $\max[\lambda_2(A), |\lambda_n(A)|] \geq 2^{1/4}\sqrt{d-1}$ for sufficiently large n. (Hint: consider the trace of A^4.)

   Remark. It is not difficult to improve the leading constant beyond $2^{1/4}$ by considering the trace of higher powers of A.