Handout 2: multiway cut and multicut

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In the multiway cut problem (Chapters 8.1 and 8.2 in [5]), there is an input graph $G(V, E)$, a cost function $c : E \rightarrow R^+$ on the edges, and k special vertices (terminals) $s_1, \ldots, s_k$. The goal is to remove a minimum cost set of edges F so that no two terminals are in the same connected component. This problem is NP-hard for $k \geq 3$ [3]. We first saw a factor $2 - \frac{2}{k}$ approximation based on algorithms for minimum s-t-cut. We then saw a factor $\frac{3}{2} - \frac{1}{k}$ approximation based on rounding of an LP relaxation (due to [2]). There are better rounding techniques for the LP. For $k = 3$ the optimal approximation ratio is $\frac{12}{11}$, and for $k > 3$ it is not known what the optimal ratio is. See [1] and references therein.

In the multicut problem, there is an input graph $G(V, E)$, a cost function $c : E \rightarrow R^+$ on the edges, and k source-sink pairs $(s_1, t_1), \ldots, (s_k, t_k)$. The goal is to remove a minimum cost set of edges F so that for every i there is no path between s_i and t_i. This problem is NP-hard for $k \geq 3$ [3]. We will see a factor 2 approximation on trees (which is best possible, unless the approximation ratio for vertex cover can be improved). For general graphs we will see a factor $O(\log k)$ approximation based on rounding of an LP relaxation. The rounding that we present is based on [2]. Chapter 8.3 in [5] presents a different rounding technique, based on work of [4]. The integrality gap of the LP is $\Omega(\log k)$.

Hand in the homework by June 12, 2019.

1. Consider the following variation on the multiway cut problem. In this variation there is a parameter k, the number of terminals is at least k (but could be more than k), and the goal is to remove a minimum cost set of edges F so that the graph decomposes into k connected components, and every connected component contains at least one of the terminals. Consider the following greedy-like approximation algorithm. As long as the number of components is smaller than k, for every pair of terminals x and y that are in the same component, compute a minimum weight x-y-cut (in the subgraph that is the connected component that contains x and y), and add to F the cut of minimum weight among the cuts that were found. (More details appear in Question 8.3 in [5]). Show that the approximation ratio of this algorithm (or a different algorithm of your choice for this problem) is no worse than $2 - \frac{2}{k}$.

2. Explain the *max multicommodity flow* problem (Section 2 in [4]), its LP, the dual of its LP, and its relation to the multicut problem. (Note that

there is another version of the multicommodity flow problem, referred at the top of page 236 in [4] as the *throughput version*, that is not the topic of this question, but relates to the topics of future lectures.)

3. Explain the region growing rounding technique (explained in [4], and in Chapter 8.3 of [5]) and show that it gives an approximation ratio of $O(\log k)$ for the multicut problem.

References

[1] Haris Angelidakis, Yury Makarychev, Pasin Manurangsi: An Improved Integrality Gap for the Clinescu-Karloff-Rabani Relaxation for Multiway Cut. IPCO 2017: 39–50.

[2] Gruia Calinescu, Howard J. Karloff, Yuval Rabani: An Improved Approximation Algorithm for MULTIWAY CUT. J. Comput. Syst. Sci. 60(3): 564–574 (2000).

[3] Elias Dahlhaus, David S. Johnson, Christos H. Papadimitriou, Paul D. Seymour, Mihalis Yannakakis: The Complexity of Multiway Cuts. STOC 1992: 241–251.

[4] Naveen Garg, Vijay V. Vazirani, Mihalis Yannakakis: Approximate Max-Flow Min-(Multi)Cut Theorems and Their Applications. SIAM J. Comput. 25(2): 235–251 (1996).

[5] David P. Williamson, David B. Shmoys: The Design of Approximation Algorithms. Cambridge University Press 2011.