Handout 5: Randomized rounding

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We showed a randomized 3/4-approximation algorithm for max-SAT, based on taking the best of two solutions: one is a uniformly random solution, and the other is obtained by randomized rounding of a linear programming relaxation.

We discussed the method of conditional expectations for derandomization. Given a randomized algorithm, the method allows us (if we can compute expectations efficiently) to obtain a deterministic algorithm whose approximaton ratio is at least as good at the expected ratio offered by the randomized algorithm.

We revisited the metric facility location problem, and showed a randomized rounding approach that gives an approximation ratio of 3.

For minimizing congestion in the integer multicommodity flow problem we will show an approximation ratio of $O(\log n)$, which improves to nearly 1 as the optimal congestion grows. This will make use of the Chernoff bounds. One convenient variation of these bounds is stated here. (Note: stronger bounds are known, which may be helpful in cases in which one is interested in tighter results).

Theorem 1 *Let $X_1, \ldots, X_n$ be independent random variables, where X_i takes either value 0 or a_i for some $0 \leq a_i \leq 1$. There for $X = \sum X_i$ and $\mu = E[X]$, $L \leq \mu \leq U$, and $\delta > 0$,*

$$Pr[X \geq (1+\delta)U] < e^{-U\delta^2/3}$$

and

$$Pr[X \leq (1-\delta)L] < e^{-L\delta^2/2}$$

Moreover,

$$Pr[X = 0] < e^{-\mu}$$

Homework – hand in by January 23. (Grader: Yael Hitron. Recall conventions for homework from Handout 1.)

1. Recall the LP relaxation for metric facility location, and the randomized rounding approach that was shown to give a solution of expected cost no worse that three times the optimal cost. Show an example that illustrates

that the analysis of the randomized rounding technique is tight. Namely, for arbitrarily small $\epsilon > 0$, design an instance of metric facility location, present for it an optimal solutions to the LP relaxation and an optimal solution for the dual, such that the expected cost of the rounded solution is no better that $(3 - \epsilon)$ times the optimal cost.

2. (Extension of Problem 5.1 in [WS11].) In the max k-cut problem the input is an undirected graph $G(V, E)$. The goal is to partition V into k parts $V_1, \ldots, V_k$ so as to maximize the number of edges that are cut (an edge is cut if its two endpoints are in different parts).

   (a) Give a randomized algorithm that in expectation cuts at least $\frac{k-1}{k}|E|$ edges.

   (b) Give a deterministic algorithm that cuts at least $\frac{k-1}{k}|E|$ edges.

   (c) Suppose that $|V|$ is divisible by k and it is required that all parts are of equal size $\frac{|V|}{k}$. Give a randomized algorithm that in expectation cuts at least $\frac{k-1}{k}|E|$ edges.

   (d) Suppose that $|V|$ is divisible by k and it is required that all parts are of equal size $\frac{|V|}{k}$. Give a deterministic algorithm that cuts at least $\frac{k-1}{k}|E|$ edges.

3. (Based on questions 5.3 and 5.6 in [WS11].) In the maximum directed cut problem (sometimes called MAX DICUT), we are given as input a directed graph $G = (V, A)$. Each directed arc $(i, j) \in A$ has nonnegative weight $w_{i,j} \geq 0$. The goal is to partition V into two sets U and $W = V - U$ so as to maximize the total weight of the arcs going from U to W (that is, arcs (i, j) with $i \in U$ and $j \in W$).

   (a) Give a randomized $\frac{1}{4}$-approximation algorithm for this problem.

   (b) Show that the following integer program models the maximum directed cut problem:
   maximize $\sum_{(i,j)\in A} w_{ij} z_{ij}$ subject to
   $z_{ij} \leq x_i$, for every arc $(i, j) \in A$,
   $z_{ij} \leq 1 - x_j$, for every arc $(i, j) \in A$,
   $x_i \in \{0, 1\}$, for every vertex $i \in V$,
   $z_{ij} \in \{0, 1\}$, for every arc $(i, j) \in A$.

   (c) Consider a randomized rounding algorithm for MAX DICUT that solves a linear programming relaxation of the integer program and puts vertex i in U with probability $\frac{1}{4} + \frac{x_i}{2}$. Show that this gives a randomized $\frac{1}{2}$-approximation algorithm for the maximum directed cut problem.