Handout 6: randomized rounding and semidefinite programming

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We showed how to 3-color 3-colorable graphs in polynomial time, if they have linear minimum degree.

We showed for the every graph of minimum degree d, its domatic number is at least $(1 - o(1))\frac{d}{\ln n}$, where the $o(1)$ term tends to 0 as d grows. Moreover, we presented a random polynomial time algorithm that finds such a domatic partition.

We described the Ellipsoid algorithm and basic facts about positive definite matrices. We showed how semidefinite programming (SDP) can be used in order to find an embedding of an n point metric space into R^n while (almost) minimizing the maximum distortion of any distance. (Lecture notes about these topics from a previous course are posted on the home page of the current course.)

It is convenient to think of SDPs as *vector programming*. We will show how to use SDP and the random hyperplane rounding technique in order to approximate max-cut within a ratio of roughly 0.878. [Goemans and Williamson: Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming. J. ACM 42(6): 1115–1145 (1995)]

It time permits, we will discuss the unique games conjecture and its relation to max-cut.

Homework – hand in by February 13. (Grader: Yael Hitron. Recall conventions for homework from Handout 1.)

1. (Problem 6.1 in [WS11].) Given a graph $G(V, E)$ with positive edge weights w_{ij} for every $(i, j) \in E$, the dual of the max-cut SDP is:

   minimize $\frac{1}{2}\sum_{(i,j)\in E} w_{ij} + \frac{1}{4}\sum_{i\in V} \gamma_i$

   subject to $W + diag(\gamma)$ is positive semidefinite

   In the above, W is the symmetric matrix of the edge weights w_{ij}, and $diag(\gamma)$ is the diagonal matrix with γ_i as the ith entry on the diagonal. Show that the value of every feasible solution for the dual is an upper bound on the weight of the maximum cut.

2. For the unweighted triangle graph (three vertices, three edges), the max-cut value is 2. Show that the maximum value of the SDP relaxation is exactly $\frac{9}{4}$. (Use the primal SDP to give a lower bound and the dual SDP to give an upper bound.)

3. (Based on Problem 6.2 in [WS11].) Let $\alpha_{GW} \simeq 0.878$ be the approximation ratio for max-cut.

   (a) Consider the following variation of the max-cut problem that we refer to as *max restricted cut*. The graph $G(V,E)$ has $2n$ vertices, and the vertices are arranged in pairs: for every $1 \leq i \leq n$, vertex i is paired with vertex $n+i$. Edges have positive weights w_{ij}. A *legal cut* is one in which for every pair, the two vertices in the pair are in different sides of the cut. The objective is to find a legal cut that maximizes the total weight of edges that are cut.

   Present an SDP relaxation for max restricted cut so that the random hyperplane rounding technique gives an expected approximation ratio of at least α_{GW}.

   (b) Show that every max 2SAT instance ϕ with n variables, m_1 unit clauses and m_2 2-clauses, can be reduced in polynomial time to an instance G of max restricted cut with $2n+2$ vertices and $m_1 + 3m_2$ edges, such that every solution for ϕ gives a solution for G of the same value, and vice versa.

   Hint: every variable of ϕ will give rise to a pair of vertices in G. Two more vertices will correspond to TRUE and FALSE. Every 2-clause of ϕ will correspond to three edges of G, each of weight half that of the clause. These edges involve (a subset of the) vertices that correspond to TRUE and FALSE, and vertices that correspond to the variables of the clause.

   (c) Design an SDP based approximation algorithm for max 2SAT with an approximation ratio of α_{GW}. (The two introductory questions above answer this question implicitly. However, provide now a more direct and simplified SDP formulation, without explicitly going through max restricted cut.)

   (d) Recall that we have seen an LP based algorithm that approximates max SAT within a ratio of $\frac{3}{4}$. Given the improved approximation ratio for max 2-SAT, show that there is some $\epsilon > 0$ such that max SAT can be approximated within a ratio of at least $\frac{3}{4} + \epsilon$.

   We remark that there are other SDP-based approximation algorithms for max 2SAT that provide an approximation ratio (significantly) better than α_{GW}.