Graph coloring – Handout5

June 21, 2015

Homework: Hand in by July 5 (to be checked and returned by July 9).

Present a polynomial time algorithm for coloring n-vertex 4-colorable graphs of maximum degree d (where d can be a function of n) by $\tilde{O}(\sqrt{d})$ colors, where the $\tilde{O}$ notation hides factors that may be polylogarithmic in n. Then show for some $\delta < \frac{1}{2}$ a polynomial time algorithm that colors 4-colorable graphs while using at most $\tilde{O}(n^{\delta})$ colors. In more detail:

1. Formulate an SDP relaxation for 4-coloring, and explain why the SDP is feasible when the graph is 4-colorable.
2. It is known that SDPs can be solved up to arbitrary precision in polynomial time. Fix a level of precision that would suffice for the rounding stage that follows, and for this level of precision, give an upper bound on the number of iterations of the ellipsoid algorithm that guarantee convergence to a solution within this precision.
3. Present a randomized algorithm that rounds the solution that is approximately feasible for the SDP so as to get a coloring of the graph with $\tilde{O}(\sqrt{d})$ colors. Prove that it runs in expected polynomial time.
4. Show how to obtain a coloring with $\tilde{O}(n^{\delta})$ colors for your choice of $\delta < \frac{1}{2}$. This breaks into two cases. In one $d \leq n^{2\delta}$, and then we are already done. In the other, there are vertices of degree larger than $n^{2\delta}$, and there you need to explain how to modify the above algorithm.

Recall that C_5, the cycle on 5 vertices, is the smallest non-perfect graph. Its maximum clique has size 2 and its chromatic number is 3. Hence the value of the theta function satisfies $2 \leq \vartheta(C_5) \leq 3$.

5. Show that $\vartheta(C_5) \leq \sqrt{5}$. (Hint: suppose you want to derive the exact value of $\cos(\frac{4\pi}{5})$. One way of doing so is by using trigonometric identities such as $\cos(\theta) = \cos(2\pi - \theta)$, $\cos(2\theta) = 2(\cos(\theta))^2 - 1$, and $\sum_{i=0}^{k-1} \cos(\frac{i}{k} 2\pi) = 0$.)

We remark that in fact $\vartheta(C_5) = \sqrt{5}$. You may try to think how this can be proved, though do not need to prove it in this homework.