Handout 5, the Lovasz theta function

January 18, 2023

Let $\omega(G)$, $\alpha(G)$, $\chi(G)$, $\bar{\chi}(G)$ denote the maximum clique size, maximum independent set size, chromatic number, and clique cover number of graph G, respectively. A graph G is *perfect* if $\omega(G') = \chi(G')$ for every vertex induced subgraph G' of G. Lovasz proved (the weak perfect graph conjecture) that the complement of a perfect graph is perfect. Hence for perfect graphs, $\alpha(G') = \bar{\chi}(G')$ for every vertex induced subgraph G' of G. Many classes of graphs (and their complements) are known to be perfect, including bipartite graphs, line graphs of bipartite graphs, interval graphs, comparability graphs, chordal graphs (the above classes are not disjoint). A chordless odd cycle of length at least five is not a perfect graph, and neither is its complement. Chudnovsky, Robertson, Seymour and Thomas proved (the strong perfect graph conjecture) that every nonperfect graph must contain one of the above as a vertex induced subgraph.

Let f be a function on graphs with the following "sandwich" property: $\alpha(G) \leq f(G) \leq \bar{\chi}(G)$. If f is computable in polynomial time, it can be used to compute $\omega(G)$, $\alpha(G)$, $\chi(G)$, $\bar{\chi}(G)$ for perfect graphs. In fact, it can be used iteratively in order to find maximum independent sets and cliques in perfect graphs. The ϑ function of Lovasz has the sandwich property, and can be computed with arbitrary precision in polynomial time (via a semidefinite program). In can also be used to find a minimum coloring and minimum clique cover of perfect graphs in polynomial time.

Homework: Hand in by February 1 (no class on January 25).

1. Show (without using the strong perfect graph theorem) that in every perfect graph G, if some vertex v is duplicated (a vertex v' is added, and for every vertex u with an edge (u, v) we also put in the edge (u, v')), then the resulting graph is also perfect.
2. Show that every interval graph is the complement of some comparability graph.
3. Give an example of a nonperfect graph G for which $\omega(G) = \chi(G)$ (and say how you know the values of $\omega(G)$ and $\chi(G)$ in this graph, and how you know that it is not perfect).
4. Show that the question of whether an input graph G has the property $\omega(G) = \chi(G)$ is NP-complete (namely, both in NP and NP-hard).