Algorithms – handout 6

Matroids

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Let $S = \{e_1, \ldots, e_n\}$ be a finite set, $c : S \to R^+$ a cost function on the elements of S, and $F \subset 2^S$ a collection of subsets of S. We say that F is *hereditary* if it is closed under taking subsets.

We wish to find $X \in F$ that maximizes $\sum_{e \in X} c(e)$. F is a *matroid* if it satisfies the following property:

- For all $X, Y \in F$, if $|X| > |Y|$ then there is some $x \in X \setminus Y$ such that $Y \bigcup \{x\} \in F$.

Theorem: The naive greedy algorithm can be used to optimize over F for every cost function c iff F is a matroid.

Examples of matroids include forests in graphs and linearly independent columns in matrices.

The maximal elements of F are called *bases* of the matroid.

Lemma: A hereditary family F is a matroid iff for every $S' \subset S$,the maximal elements of F contained in S' all have the same cardinality.

Given a matroid (S, F), its *dual* is (S, F^*), where

$$F^* = \{X \subset S | S \setminus X \text{ contains a basis of } F\}$$

Lemma: The dual of a matroid is a matroid. Furthermore, $F^{**} = F$.

For planar graphs the dual has a natural interpretation. It can be used so as to prove Euler's formula $(v - 1) + (f - 1) = e$ where v, f, e are the number of vertices, faces, edges respectively. Via triangluation, this implies that a planar graph has a vertex of degree at most 5, a fact that we used in the inductive coloring algorithm for planar graphs.

The intersection of two matroids in not necessarily a matroid. Still, we can optimize over it via more sophisticated algorithms. Optimizing over the intersection of three matroids is NP-hard (in the general case).

Homework:

Call a planar graph *pd-regular* if both the graph and its dual are regular of the same degree. That is, for some value d, every vertex has degree d and every face touches d edges. For example, K_4, the clique on four vertices, is pd-regular. Show that there is no other planar pd-regular graph. (Hint: this is a consequence of Euler's formula. As an aside, Euler used his formula so as to find all the so called *platonic solids*, K_4 being one of them.)