Homework assignment 6
Hand in by July 12.

1. There are n points. For every pair of points we may or may not be given a distance constraint specifying a lower bound on the distance between them, and we may or may not be given a distance constraint specifying an upper bound on the distance between them. We have seen in class that we can check in polynomial time whether the points can be embedded in Euclidean space such that all distance constraints that are given are satisfied up to an arbitrarily small additive term of ϵ. Show that with the extra restriction that the embedding has to be in two dimensional space (in the plane), the problem becomes NP-hard. (Hint: given a graph G, let the points represent the vertices of the graph. It may help to also include an auxiliary point that will serve as the origin. Show how to write distance constraints between the points that can be satisfied in the plane if and only if G is 3-colorable. As warm-up, show how to write distance constraints between the points that can be satisfied in the one dimensional line if and only if G is 2-colorable, namely, bipartite.)

2. Given a graph $G(V, E)$ on n vertices, consider the following optimization problem, in which one is required to find a matrix of order n.

   maximize $\lambda_1(M)$ subject to the constraints:

   (a) M is symmetric.
   (b) $M_{ii} = 0$ for every $1 \leq i \leq n$.
   (c) $-1 \leq M_{ij} \leq 0$ for all $(i, j) \notin E$.
   (d) $0 \leq M_{ij} \leq 1$ for all $(i, j) \in E$.

   Which of the following options is correct?

   (a) The ellipsoid algorithm can be used in order to solve this problem exactly.
   (b) The ellipsoid algorithm can be used in order to solve this problem up to arbitrary precision.
   (c) The ellipsoid algorithm is not applicable to this problem.

   Explain your answer.

3. Consider the following two semi-random models for the planted independent set problem. In both models, n is the number of vertices, k is the size of the planted independent set, and $\epsilon > 0$ is some small parameter that may depend on n (for example, it might be that $\epsilon = \frac{1}{\sqrt{n}}$).

   Monotone adversary. For a value of $p = \epsilon$, one first generates a graph G_1 from the distribution $G_{n,p}$. Then one plants in G_1 uniformly at random an independent set K of size k, thus obtaining G_2. Thereafter, an adversary may add arbitrary edges to G_2, as long as these edges are not inside K. This gives the input graph G_3.

   Combinatorial smoothed analysis. An adversary generates an arbitrary graph H_1 on n vertices that contains an independent set K of size k. (H_1 may contain other independent sets of size k, or even larger, but only one is called K.) Then the graph is *smoothed* by the following process. For every pair of vertices i, j such that at least one of them is not in K, the adversary's decision on whether there is an edge between i and j is independently flipped with probability ϵ (edges change to non-edges and vice versa). This gives the input graph H_2.

For the same given values of n, k, ϵ, is one of the two models easier than the other, in the sense that every algorithm for finding an independent set of size k in one model can be used for the other? Are the models equivalent (each model is easier than the other)? Are they incomparable? Explain your answer.