Hint for exercise 1 in homework assignment 5

Consider a bipartite graph H with k vertices on each side. It is well known that if H is d-regular then its edge set can be decomposed into d perfect matchings. (Hall's condition can be shown to imply that it has at least one perfect matching. After removing the edges of this perfect matching, the graph becomes $d-1$ regular, and the proof can be continued by induction.)

Suppose now that H is a random k by k bipartite graph, with edge probability $\frac{1}{2}$. Let δ denote the minimum degree in H (which is roughly $\frac{k}{2} - O(\sqrt{k \log k})$, w.o.p.). Then surely H cannot have more than δ disjoint perfect matchings. However, with overwhelming probability, it does have δ disjoint perfect matchings. In exercise 1 in homework assignment 5, you may use this fact without proof (though you may need to be clever in choosing the graph H on which to apply this fact).

For completeness, here is a sketch of proof of the above fact. Give each edge of H capacity 1. Add a vertex s and k edges, connecting s to the vertices of one side of H. Add a vertex t and k edges, connecting t to the vertices of the other side of H. Give each edge incident with s and each edge incident with t capacity δ. Standard probabilistic arguments show that w.o.p., the minimum cut between s and t has capacity δk. Consequently, the maximum flow between s and t is δk. As all capacities are integral, this flow can be assumed to be integral as well. The edges of H that support this integral flow form a δ-regular bipartite graph.