Perfect graphs, and the Lovasz theta function

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Let $\omega(G)$, $\alpha(G)$, $\chi(G)$, $\bar{\chi}(G)$ denote the maximum clique size, maximum independent set size, chromatic number, and clique cover number of graph G, respectively. A graph G is *perfect* if $\omega(G') = \chi(G')$ for every vertex induced subgraph G' of G. Lovasz proved that the complement of a perfect graph is perfect. Hence for perfect graphs, $\alpha(G') = \bar{\chi}(G')$ for every vertex induced subgraph G' of G. Many classes of graphs (and their complements) are known to be perfect, including bipartite graphs, interval graphs, comparability graphs, chordal graphs (the above classes are not disjoint). A chordless odd cycle of length at least five is not a perfect graph, and neither is its complement. It is conjectured that every nonperfect graph must contain one of the above as a vertex induced subgraph.

Let f be a function on graphs with the following "sandwich" property: $\alpha(G) \leq f(G) \leq \bar{\chi}(G)$. If f is computable in polynomial time, it can be used to compute $\omega(G)$, $\alpha(G)$, $\chi(G)$, $\bar{\chi}(G)$ for perfect graphs. In fact, it can be used iteratively in order to find maximum independent sets and cliques in perfect graphs. The ϑ function of Lovasz has the sandwich property, and can be computed with arbitrary precision in polynomial time (via a semidefinite program). In can also be used to find a minimum coloring and minimum clique cover of perfect graphs in polynomial time.

Homework:

1. The vertex-arc incidence matrix of a directed graph has the vertices as its rows, the arcs as its columns, and for arc (i, j), its corresponding column has $+1$ at row i, -1 at row j, and 0 elsewhere. Show that this matrix is totally unimodular.

2. (a) Give an example of a nonperfect graph G for which $\omega(G) = \chi(G)$.

   (b) Show that the question of whether an input graph G has the property $\omega(G) = \chi(G)$ is NP-complete. (Hint: it is NP-hard to tell whether a graph is 3-colorable.)