The Cascade Auction – A Mechanism For Deterring Collusion In Auctions

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Abstract

We introduce a sealed bid auction of a single item in which the winner is chosen at random among the highest k bidders according to a fixed probability distribution, and the price for the chosen winner is the Vickrey-Clarke-Groves price. We call such an auction a *cascade auction*. Our analysis suggests that this type of auction may give higher revenues compared to second price auction in cases of collusion.

Introduction

Consider a sealed bid auction of a single item. *Cascade auctions* are auctions described by the following rule: the winner is chosen at random among the highest k bidders according to a fixed probability distribution and the price for the chosen winner is the Vickrey-Clarke-Groves (VCG) price. We will mainly consider the case $k = 2$, namely the highest bidder wins with probability p (where $p \geq 1/2$) and the second highest bidder wins with probability $1 - p$.

A well known problem with second price auctions is *collusion*. If two or more bidders collude they have a *dominant* strategy of dropping all their bids except for the highest bid. This might substantially lower the revenue for the seller. We demonstrate that in realistic scenarios of collusion by rational bidders, the revenue in cascade auctions is higher than that in second price auctions.

The problem of collusion in auctions is classic. The formal literature on collusion in second-price auctions goes back to Graham and Marshall (Graham and Marshall 1987), while the literature of collusion in first-price auctions goes back to McAfree and McMillan (McAfee and McMillan 1987). There have been extensive literature extending upon the above.

A main motivation to our work is the use of second-price auctions for selling impressions through mediators (aka agents) in electronic exchanges (e.g., the Google double-click ad-exchange,Yahoo RightMedia ad-exchange, or ADECN/Appnexus ad-exchanges used by Microsoft). A typical situation is of an agent (or mediator) who represents several bidders in a second price auction, a situation in which collusion is inherent. Indeed, recently, the computer science literature has put much attention to collusion in second-price auctions (and in Vickrey-Clarke-Groves mechanisms (Conitzer and Sandholm 2006; Bachrach 2010; Mansour, Muthukrishnan, and Nisan 2012)). This is mainly due to the popularity of variants of second price auctions in electronic markets. The problem considered in this paper, tackling the issue of second price auctions through mediators, is central to the above-mentioned billions of dollars ad-exchange market – see Problem 1 in the survey by Muthukrishnan (Muthukrishnan 2009). In this problem an agent who represents the two highest bidders can lead to very small revenue (even 0, assuming no floor price, and only two bidders in the system, which is not a rare event), by submitting a bid of 0 on behalf of the bidder with the lower valuation; the question is how to allow and even advocate such mediators but remove the possibility of such low revenues. We introduce cascade auctions and show they mitigate the above central issue.

The use of cascade auctions may have consequences beyond those considered in the current work. For example, the prospect of winning the auction even with a bid that is not highest among all bids may encourage participation of bidders that might otherwise have chosen to avoid the auction. Here we just remark that such consequences may affect the desirability of using the cascade auction, but the analysis of the effect is left for future work.

This work is concerned with the revenue of the seller (specifically, in face of collusion). An aspect of auctions that partly addresses revenue concerns is the use of a floor price (a.k.a. reserve price), and in certain situations (that do not involve collusion) it is known how to set a floor price in a way that maximizes expected revenue (Myerson 1981). Though cascade auctions may involve a floor price as well, their resistance to collusion is not a consequence of the floor price, and we shall not be concerned with the value of floor price (which we often simply assume to be 0).

The model

Notation and conventions

A *seller* has one *item* for sale. There are multiple *buyers*. The value of the item to buyer i is v_i. Buyers may submit bids (i, b_i), specifying which buyer made the bid and a positive bid value, also referred to as the *offer*. A buyer may submit more than one bid. The collection of bids is called the *bid profile*.

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A (*pure*) auction mechanism is a function that maps the tuple of bids to a winner (the buyer who gets the item) and a price c. For the winner i, the *payoff* of the outcome is $v_i - c$. For all other buyers, the payoff is 0. A *mixed* auction mechanism is a probability distribution over pure auction mechanisms.

The cascade auction

We suggest here a new mixed auction mechanism that we shall call a *cascade single item auction with fixed probabilities*, or *cascade auction* for brevity. It involves a parameter $k \geq 1$ and a vector of non-increasing probabilities $p_1, \ldots, p_k$ (satisfying $p_j \geq 0$, $\sum p_j = 1$, and $p_j \geq p_{j+1}$ for all $j < k$). It also involves a floor price (a.k.a. reserve price) c_0 – bids not higher than the floor price are rejected. Given the bids, their values are sorted in nonincreasing order in $k+1$ *slots* (with ties resolved as in the following subsection). Let $a_1 \geq \ldots \geq a_{k+1}$ be the values in these slots. If there are insufficiently many bids above c_0, the remaining values are c_0. For $1 \leq j \leq k$, the bidder of bid a_j wins the auction with probability p_j. (If a_j corresponds to a floor price rather than an actual bid, the item is not allocated.)

Convention: throughout this manuscript the parameter k will be used only with meaning as in the above paragraph, namely, denoting the number of slots from which the winner in a cascade auction is chosen.

When the winner is the jth highest bidder the payment made by the winner is the Vickrey-Clarke-Groves (VCG) payment $p_j^{-1} \sum_{l>j}^{k+1} a_l(p_{l-1} - p_l)$. In other words, the expected payment of the jth highest bidder is $\sum_{l>j}^{k+1} a_l(p_{l-1} - p_l)$.

A special case of cascade auctions is when the probabilities for the winner are the same for the r highest bidders. For $j > 1$, let a *mixed jth price auction* be an auction that picks a winner uniformly at random among the $j-1$ highest bids, and charges the jth bid from the winner. (Floor prices are handled as above, and ties as in the next sub-section.) Note that the jth price is the VCG price in this case. Observe also that a mixed second price auction is the same as the well known second price auction.

Handling ties

The sorted order of bid values referred to previously is not well defined when there are ties in values of bids. We now explain how the order among tied bids is chosen. Take a permutation over all tied bids, chosen uniformly at random among all permutations that satisfy the following "consecutive slots" requirement: if several of the tied bids come from the same buyer, then in the permutation these bids are required to occupy consecutive slots. Hence if for example we have for some bid value b and two buyers 1 and 2 the bids $(1, b)$, $(1, b)$ and $(2, b)$ then we choose uniformly at random among the following two permutations: $(1, b); (1, b); (2, b)$ and $(2, b); (1, b); (1, b)$. The permutation $(1, b); (2, b); (1, b)$ is not allowed. The reason for introducing the consecutive slots requirement is that under this rule a buyer has nothing to gain by submitting more than k bids (including multiplicities). In contrast, without the consecutive slots requirement, the probability of being first in a random permutation would be strictly increasing in the number of (tied) bids that a buyer supplies, even if this number exceeds k.

Three equivalent views of the cascade auction

We have presented one view of the cascade auction, using a vector of non-increasing probabilities $p_1, \ldots, p_k$. We now present two other equivalent ways of viewing the cascade auction.

Combinations of mixed jth price auctions. In a mixed jth price auction, one of the $j-1$ highest bidders is chosen uniformly at random as the winner, and it pays the jth highest bid. The cascade auction of general form is equivalent to a distribution over mixed jth price auctions for $2 \leq j \leq k+1$, where a jth price auction is chosen with probability $(j-1)(p_{j-1} - p_j)$. The equivalence is in the sense that for every buyer, its probability of receiving the item is identical in both types of auctions, and in case of winning, its *expected* payment in the distribution over mixed jth price auctions is the same as the (deterministic) payment in the cascade auction.

Second price auctions with random masks. We define a *mask* in an auction to be a binary vector that indicates a pattern by which some of the bids are masked (dropped). For example, the mask 1 drops the highest bid, the mask 01 drops the second highest bid, and the mask 101 drops the first and third highest bids. In a masked second price auction, the highest bid that remains after applying the mask wins, and pays the second highest remaining bid. Observe that for every j, a mask with j ones can be assumed to have all the one locations among the $j+1$ highest bids (as otherwise the tail of the ones is irrelevant). Such masks are called plausible for j. A second price auction with random masks is a distribution over second price auctions, where the distribution is taken over choice of masks.

Second price auctions with random masks are more general than the cascade auction (e.g., one may completely mask out the highest bidder). We shall say that a second price auction with random masks is *uniform* if for every j, the probability of choosing any particular mask that is plausible for j is identical (say, to q_j). Uniform second price auctions with masks are equivalent to cascade auctions. This is easily seen by an obvious bijection with combinations of mixed jth price auctions: q_{j-1} is $1/(j-1)$ times the probability of taking the jth price auction.

Multiple bids

In a cascade auction it is sometimes advantageous for a buyer to submit multiple bids, as then it can occupy multiple slots. In this paper the auction mechanism allows multiple bids by the same buyer, an aspect that is used in order to increase revenue of the seller. If we allow multiple bids, in the mixed jth price auction the highest bidder can guarantee winning the item with probability 1 and paying the second highest bid by making $j-1$ identical bids. In fact, it is easy to see that for the mixed j-price auction, and for every prior that buyers might have on the values of the item to the other buyers, it is a Nash equilibrium point for each buyer to make

$j-1$ identical bids. For uniformity of notation, we may assume that each buyer submits exactly k non-negative bids, where a bid of the form $(i, 0)$ is interpreted as no bid.

Collusion and agents

A *collusion* is a set of buyers that coordinate their bids. We model collusion by a notion of an *agent*, which we also call *mediator*. An agent is a bidding algorithm, and the algorithm of the agent is effectively a contract that the agent offers to buyers. The contract in essence says that the buyers may provide their inputs to the bidding algorithm, and the agent will bid on their behalf the output of the algorithm. Formally, following the literature of mediators (see e.g. (Monderer and Tennenholtz 2006; Rozenfeld and Tennenholtz 2007)) we have m agents, in addition to the n buyers. For uniformity of notation, we may assume that each buyer submits k bids to each of the mediators, in addition to its own bids. Each mediator is a function for its incoming bids to k non-negative bids on behalf of each of the buyers. Again, a zero bid (as an input to the agent, or as its output) will represent no bid.

Notice that buyers may submit bids either directly to the seller, or through one or more mediators. We refer to a buyer submitting bids only directly to the seller as being *independent*. An agent may represent several buyers. In this case these buyers are *colluding*, and are referred to as *siblings* of each other. Recall that bids are associated with particular buyers. In particular, if an agent submits a bid (i, b_i) and the bid wins, the item goes to buyer i. The agent is not allowed to instead give the item to a sibling of i.

The cascade auction as a multi-player game

Given the set of buyers and their valuations, and the set of agents, we get a game in strategic form, where the players are the buyers $N = \{1, 2, \ldots, n\}$, and the set of possible actions of each player is the set of possible pairs $A = (A_I, A_C)$ where A_I is a k-tuple of bids it submits directly to the seller, and $A_C = (A_{C_1}, \ldots, A_{C_m})$ is the corresponding k-tuples of bids it submits to the agents $M = \{1, 2, \ldots, m\}$, respectively. Notice that for each action profile of the buyers, the payoff of each player is well defined, as described above. We can therefore appeal to standard game-theoretic concepts, such as the Nash equilibrium and un-dominated actions.

Solution concepts

Dominant actions. An action A of a buyer B is *dominant* if for every other action A' of buyer B and for every set of actions of the other buyers, the expected payoff for B under action A is at least as large as his expected payoff under action A'. The action A is *strictly dominant* if it is the unique dominant action. (Equivalently, in addition to being dominant, for every A' there is a set of actions for the other buyers under which the expected payoff of A is strictly higher than the expected payoff of A'.)

In second price auctions, bidding one's value is a strictly dominant action. However, in the cascade auction, buyers who submit multiple bids do not have a dominant action. Hence we consider a relaxed concept.

Undominated actions. An action A of a buyer B is *undominated* if for every other action A' of buyer B, either there is a set of actions for the other buyers under which the expected payoff of A is strictly higher than the expected payoff of A', or the payoff of A is always the same as the payoff of A'.

The use of undominated actions in the context of mechanism design is discussed also in (Babaioff, Lavi, and Pavlov 2009). The other two solution concepts that we consider are based on the notion of a *best response* action.

Best response action. Given the set of actions by all other buyers, an action A by buyer B is a *best response* if no other action offers B higher expected payoff.

A standard solution concept based on best responses is the following.

Nash profile. The set of actions performed by the buyers forms a *Nash profile* if the performed action of every buyer is a best response to the performed actions of all other buyers.

While the above concepts are standard, given the notion of agents as mediation devices, we may wish our results (that relate to the expected revenue of the seller) to hold when the output behavior of agents remain fixed. To do so, we refine the notion of a best response.

Semi-best response. For an action A by buyer B, recall we partition it into $A = (A_I, A_C)$, where A_I is the *independent action*, namely, the bids provided by B directly to the seller (if there are any), and A_C is the *colluding action*, namely, the contracts B has with the agents who bid on B's behalf (if there are any). Given the set of actions of all other buyers, A is a *semi-best response* if no independent action $A'_I \neq A_I$ results in an action $A' = (A'_I, A_C)$ that offers B higher expected payoff than A does.

The semi-best response concept distinguishes between independent bids and bids that go through an agent. A change from A_I to A'_I leads to a replacing A_I by A'_I in the set of bids received by the seller. In contrast, a change from A_C to A'_C would in general change the set of bids received by the seller in a way that depends not only on A_C and A'_C, but also on actions of agents to which this change applies. These actions may further depend on the set of bids these agents receive from other buyers, and may affect the bids that the agents submit on behalf of other buyers.

Semi-Nash profile. The set of actions performed by the buyers forms a *semi-Nash profile* if the performed action of every buyer is a semi-best response to the performed actions of all other buyers.

Every Nash profile is also a semi-Nash profile, but there might be semi-Nash profiles that are not Nash profiles. This aspect does not weaken our results but rather strengthens them, because it only enlarges the set of behaviors of buyers with respect to which our lower bounds on the expected revenue for the seller hold.

As is well known, every (finite) multi-player game has a mixed Nash equilibrium, but some multi-player games do not have a pure Nash equilibrium. Our Nash profiles relate to pure Nash, rather than mixed Nash. In the absence of agents, the cascade auction game does have pure Nash equilibria (as implied by our analysis). This is also true in the presence of agents, because a pure Nash equilibrium in which no buyer uses the services of agents remains a pure Nash equilibrium even in the presence of agents. Given that pure Nash equilibria always exist in our game, in this work we do not attempt to analyze mixed Nash equilibria for the cascade auction game.

Naive buyers

A buyer is *naive* if he is independent and he submits a single bid. A formal model for a naive buyer is simply as a buyer that does not have the full set of actions available to him, but only the actions that involve submitting a single independent bid (i.e. all other bids he submits are zero).

The following Proposition follows directly from properties of the VCG-price.

Proposition 1 *There is a strictly dominant action for naive buyers, and this action is to simply bid his value.*

Buyers who are not naive do not in general have dominant strategies. The strategy of bidding their value might not be dominant for several reasons. One is that they might benefit from submitting additional bids below their value. Another is that they might benefit from going through an agent instead of submitting an independent bit. Yet another is that when bidding through an agent there are realistic scenarios in which they may benefit from biding *above* their value.

Analysis

In the rest of the paper we focus on the case where $k = 2$, the first instance of cascade auctions that goes beyond 2nd price auction. Our results hold for any number of buyers or agents. Since our results are mainly positive, showing the strength of cascade auctions, the case $k = 2$ is the most interesting (and applicable) one. In this case $p_2 = 1 - p_1$, and we denote p_1 by p. Recall that $p \geq 1/2$. Let $b_1 \geq b_2 \geq b_3$ be the three highest bids (some of which may be the reserve price). Then the highest bidder wins with probability p and his payment when he wins is $\frac{1}{p}((2p-1)b_2 + (1-p)b_3)$, and the second highest bidder will win with probability $1-p$ and his payment when he wins is b_3. The expected payment for the highest bidder is $(2p-1)b_2+(1-p)b_3$, and for the second highest bidder it is $(1-p)b_3$. The expected revenue for the seller is $(2p-1)b_2 + (2-2p)b_3 = b_2 - (2-2p)(b_2 - b_3)$. This revenue approaches the one from the standard second price auction (under the same bidding) as p tends to 1, or alternatively, as b_3 tends to b_2.

As we have previously discussed, this cascade auction is equivalent to having a second price auction with probability $q = 2p - 1$, and mixed third price auction with probability $1 - q = 2 - 2p$. We shall use these two representations of the same auction interchangeably.

Multiple bids as a best response

As noted in Proposition 1, a dominant strategy for a naive buyer who submits only one bid is to bid his value. Here we consider a buyer who submits multiple bids. We assume that the buyer is informed in the sense that he sees all remaining bids. Let v be his value, and let $b_1 \geq b_2$ be the two top bids by other buyers (other remaining bids will not matter). We assume for simplicity that there are no ties ($v \neq b_1$, $v \neq b_2$, $b_1 \neq b_2$). What should the buyer do?

If $v < b_2$ the informed buyer has no reason to bid at all, hence he might either not participate, or he may simply bid v (his dominant naive action) out of precaution (e.g., just in case it turns out that he was mistaken about the values of the other bids).

If $b_2 < v < b_1$ the informed buyer has incentive to submit one bid of value between b_1 and b_2, and it makes sense to submit a bid of v. Submitting multiple bids might actually hurt the buyer (if he wins with his highest bid and needs to pay his own second highest bid).

The interesting case is when $v > b_1$. The buyer can certainly submit one bid of v. However, it may be desirable to submit a second bid as well. If at all submitted, the value of the second bid should be marginally higher than b_1, and we denote its value by b_1^+. To see when the buyer gains from the second bid, it is convenient to consider the combination of mixed auctions view. We have a combination of a second price auction and a mixed third price auction. The second price auction is only marginally affected by the additional bid (the payment only changes from b_1 to b_1^+). However, the mixed third price auction does change. Now the buyer occupies both top slots and hence wins for sure rather than only with probability $1/2$, gaining $v/2$ in expectation. On the other hand, upon winning he pays b_1 rather than only b_2. Hence the payment of the buyer increased by $b_1 - b_2/2$. It follows that a second bid can increase the expected payoff of a buyer if and only if $(v + b_2)/2 > b_1$. Thus we established:

Proposition 2 *In a cascade auction with $k = 2$, if a buyer has value v and the two top bids by other buyers are $b_1 \geq b_2$, then submitting two bids is a best response for the buyer if and only if $(v + b_2)/2 > b_1$.*

Two remarks on Proposition 2. One is that when the buyer does submit a second bid b_1^+, and assuming that b_1 was the true value for the buyer who bid b_1 (his dominant strategy), the revenue of the cascade auction becomes equal to the second highest value held by the buyers, which is precisely the revenue in a second price auction (when bidders are truthful). The other is that a buyer may possibly choose to submit a second bid even if $(v + b_2)/2 < b_1$. This may happen for example if the buyer is not concerned only with the expected payoff, but also with the variance. By submitting a second bid the buyer achieves certainty about winning the auction.

Proposition 2 plays an important role in our study of various scenarios of collusion.

Perfect Collusion

We continue to address the case $k = 2$ (highest bid wins with probability $1/2 \leq p < 1$, second highest bid wins with probability $1 - p$). The worst possible collusion from the point of view of the revenue of the seller appears to be when all buyers collude, for example, by disclosing their value to one common agent that bids on behalf of all of them. We call such a situation a *perfect collusion*. With perfect collusion, the revenue for the seller in a second price auction is the floor price. We show that the cascade auction offers higher revenue, assuming buyers are rational (in a game theoretic sense), and furthermore, assuming nontransferable utility among buyers (even if represented by the same agent). Our results are apply in the general setting defined in the previous section, as well as in the restricted setting of perfect collusion.

Since we consider the case $k = 2$ it will suffice to consider in our analysis the three highest values that buyers have, denoted here by $v_1 > v_2 > v_3$ (for simplicity we assume that there are no ties). Without loss of generality, these values are held by buyers 1, 2 and 3 respectively. To simplify notation, we assume that the floor price is 0.

Nash profiles

Recall that a *Nash profile* is a profile of bids in which no buyer has an incentive to unilaterally deviate (replace his bids in the profile by other bids). This is a reasonable solution concept for a situation in which buyers have full information regarding the bids of other buyers, and it has nice fit to the ad exchanges setting discussed in the introduction where bidding is repetitive and bidding logs are accessible (in a delay) to the participants.

Observe that there may be multiple different Nash profiles. For example, in a second price auction, there is a Nash profile in which the buyer with highest value bids his value, and all other buyers bid 0. The revenue for the seller in this Nash profile is 0.

The fact that there may be multiple Nash profiles serves as a connection between Nash profiles and collusion. The nature of collusion might be that among all Nash profiles, the buyers choose the profile in which the revenue of the seller is the smallest. This outcome may come about if one agent represents all buyers, and buyers are truthful (report their true values to the agent).

Here we analyze the revenue of the seller in Nash profiles of the cascade auction. Proposition 3 presents a lower bound that always holds, and then Theorem 4 characterizes cases in which the revenue of the seller exceeds the lower bound of Proposition 3.

Proposition 3 *In every Nash profile, the expected revenue of the seller is at least* $(2p - 1)v_3$.

Proof. In a cascade auction with $k = 2$ and $1/2 \leq p < 1$, the combined event that the highest bid wins and pays the value of the second highest bid has probability $(2p - 1)$. To prove the proposition it suffices to show that the second highest bid is at least v_3.

Consider an arbitrary Nash profile, and let b_2 be the second highest bid. For the sake of contradiction, suppose that $b_2 < v_3$. Of the three buyers who hold the highest values, consider a buyer not giving any of the two highest bids (breaking ties among buyers arbitrarily). This buyer strictly gains by replacing his current bid by $v_3 > b_2$, contradicting the assumption that we had a Nash profile. ■

In some sense, Proposition 3 is best possible. Consider the case that $v_3 \geq v_1/2$, buyer 1 (with value v_1) bids v_1 and buyer 2 (with value v_2) bids v_3. This is a Nash profile as no other buyer has any incentive to bid, and the two top buyers do not have any incentive to change their bids.

As v_3 becomes smaller the lower bound provided in Proposition 3 deteriorates. Luckily, as soon as v_3 drops below $v_1/2$, a new lower bound kicks in. Consider first the case that $v_2 \geq v_1/2 > v_3$. If buyer 2 drops his bid b_2 to a value lower than $v_1/2$, then Proposition 2 implies that buyer 1 would provide two bids, the lowest of which is just above b_2. But then this is not a Nash profile, because buyer 2 never wins, despite having a value larger than the second bid. Hence b_2 will not drop below $v_1/2$, ensuring an expected revenue of $(2p - 1)v_1/2$ for the seller (when $v_2 \geq v_1/2$).

It remains to consider the case that $v_1/2 > v_2 \geq v_3$. In this case Proposition 2 implies that buyer 1 will provide the two highest bids. But how large would be the second bid. Surely at least v_2, because otherwise buyer 2 has an incentive to overbid this second bid, and this cannot be a Nash profile. But note further that the buyer 1 would like his second highest bid to be as low as possible, as this lowers his payment when his first bid wins. In a Nash profile, what can prevent his second bid to go below v_2? The only thing that may prevent it is a bid of v_2 by a different buyer. Hence the top three bids in the Nash profile are no worse for the seller than v_1, v_2, v_2, and the revenue for the seller is at least v_2. Combining the above we get:

Theorem 4 *In any Nash profile, if* $v_2 < v_1/2$ *the revenue of the seller is at least* v_2*, and if* $v_2 \geq v_1/2$ *the expected revenue of the seller is at least* $(2p - 1)\max[v_1/2, v_3]]$.

Naive agent

The naive agent asks each buyer for his bids (a buyer may submit any number of bids from 0 to k). The naive agent sorts all bids that he received in order of decreasing value, determines the value of the kth highest bid, and passes to the seller those bids having at least this value. Hence the number of bids that the naive agent passes to the seller may be lower than k (if the agent received less than k bids), exactly k (if the agent received at least k bids and the $(k+1)$th highest bid was smaller than the kth highest bid), or more than k bids (if the $(k + 1)$th highest bid was equal to the kth highest bid). The advantage that the naive agent offers for the buyers is the prospect of dropping all bids beyond the kth highest bid, and hence potentially lowering the payment for the winner of the cascade auction.

The above contract that the naive agent offers may be attractive to buyers, as Proposition 5 shows.

Proposition 5 *If in the cascade auction all agents are naive, then for every buyer, in every undominated action,*

1. *The buyer does not submit an independent bid.*
2. *If the buyer submits multiple bids, all these bids are submitted through the same naive agent.*
3. *At least one bid that the buyer submits to the naive agent is the true value for the buyer, and the other bids (if any) are not higher.*

Proof. Suppose that all agents are naive and the buyer is playing an undominated action. We prove a sequence of claims about his action A.

The buyer submits at most k bids in total. Assume for the sake of contradiction that action A includes at least $k +$

1 bids, and consider the action A' that contains only the k highest bids in A, breaking ties arbitrarily. Regardless of actions of other bidders, action A' wins the item whenever action A does, and does not pay more. Strict domination of A' over A follows from the case in which no other buyer bids, and then the VCG prices under A are strictly larger than those under A', because the $(k+1)$th highest bid in A is strictly positive and hence strictly higher than that in A'.

All the buyer's bids go through the same agent. Assume for the sake of contradiction that action A includes at most k bids, but not all these bids go through the same agent. Let N be the naive agent used by the buyer for his bid that is highest among all bids that are not independent, and let N be an arbitrary agent if all the bids in A are independent. Consider the action A' which has the same bids as A, but all these bids are given through N. We claim that action A' strictly dominates action A. This follows from the fact that when all agents are naive, the bids that the seller receives necessarily include the k highest bids (no naive agent will drop any such bid), and every bid tied with the kth highest. Hence the probability of winning with A' is identical to the probability of winning with A. The difference between A' and A can only be in the payment, due to the $(k+1)$th highest bid received by the seller. A simple case analysis (that is omitted) shows that with A' the $(k+1)$th highest bid received by the seller is never higher than with A, and it is strictly smaller if all buyers happen to use N as an agent, and there is no tie on the kth highest bid.

The highest bid of a buyer equals his value. Here we may already assume that action A includes at most k bids and they all go through the same agent. Let v be the value of the item for the buyer, and let b be the highest among his bids. Assume for the sake of contradiction that $b \neq v$. There are two cases to consider.

$b < v$. Then A' in which the buyer's highest bid is raised to v dominates A (wins at least as often, and pays more only in cases that A could not have won). This is a strict domination due to the possibility that there was only one bid by other buyers, and this bid was of value between v and b.

$b > v$. Then A' in which all bids of the buyer which were higher than v are lowered to v dominates A. (This is perhaps easiest to see when viewing the cascade auction as a combination of mixed jth price auctions, as discussed before). Details omitted.) This is a strict domination due to the possibility that there was only one bid by other buyers, and this bid was of value between v and b. ∎

The following theorem is a straightforward consequence of Proposition 5.

Theorem 6 *In the cascade auction with $k = 2$, if all agents are naive and if every buyer uses an undominated action, then the expected revenue of the seller is at least $(2p-1)v_2$.*

Arbitrary agents

Unlike in previous sections where we had a fairly accurate model of how the buyers collude (either by selecting a Nash profile of minimum revenue for the seller or by using a Naive agent), in this section we prove results that hold regardless of the nature of the bidding algorithms of the agents. The solution concept that we use is the *semi-Nash profile*.

Theorem 7 *In every semi-Nash profile the expected revenue for the seller is at least $(2p-1)\min[v_2, \max[v_1/2, v_3]]$.*

Proof. The proof is via a simple modification of the proof of Theorem 4 for Nash profiles. The analysis for the cases $v_2 \geq v_1/2$ was based on the option of a buyer to provide an additional independent bid, and hence holds for semi-Nash profiles as well. The only difference is in the case $v_2 < v_1/2$. Having only two bids v_1 and v_2 with no other bid present was not a Nash profile. However, it might be a semi-Nash profile, if an agent is providing these bids on behalf of buyer 1. (Recall that in a semi-Nash we only consider what a buyer can gain by changing his independent bids, and here buyer 1 cannot gain by submitting an additional independent bid.) Hence the revenue for the seller might be only $(2p-1)v_2$ instead of v_2 as in Theorem 4 giving the bound claimed in Theorem 7. ∎

Concluding remarks

Our analysis shows that cascade auctions give an advantage over second price auctions in case of collusion, or when mediators are used. The second price auction, used as a standard tool for ad exchanges, is known to suffer in revenue in such cases (Muthukrishnan 2009), yielding potentially 0 revenue (or more generally, not more than a small floor price). Our analysis suggests that cascade auctions will mitigate this problem.

While our approach is both rigorous and handles a realistic issue, let us end with several disclaimers.

1. *Economic efficiency*. Under dominant strategies for the buyers, a second price auction allocates the good to the buyer who values it most. For the cascade auction there is an inherent loss of economic efficiency due to its randomized allocation rule. We note however that some of the loss is compensated for by the possibility that the buyer with highest value will submit multiple bids (as in Proposition 2).
2. *Randomized strategies for agents.* Our analysis for Nash and semi-Nash profiles relate to deterministic strategies. It could be that there are randomized bidding algorithms for agents that will result in mixed-Nash profiles with lower expected revenue for the seller compared to the bounds given in the paper.
3. *Other auction mechanisms.* We compared the cascade auction with the second price auction, because the second price auction (or variants of it) are the de-facto current standards for selling impressions in electronic exchanges. Our goal was to cope with collusion while maintaining as much as possible the truthfulness aspect of second price auctions – we still offer naive buyers a dominant bidding strategy. We remark that if one is willing to give up truthfulness altogether, there are other auction mechanisms, most notably, first price auctions, for which collusion is not considered to be a source of significant loss of revenue for the seller.

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