Invitation to Complexity Theory

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Introduction

The strive for efficiency is ancient and universal, as time and other resources are always in shortage. Thus, the question of which tasks can be performed efficiently is central to the human experience.

A key step towards the systematic study of the aforementioned question is a rigorous definition of the notion of a task and of procedures for solving tasks. These definitions were provided by computability theory, which emerged in the 1930's with the work of Turing (and others). This theory focuses on computational tasks, considers automated procedures (i.e., computing devices and algorithms) that may solve such tasks, and studies the class of solvable tasks.

In focusing attention on computational tasks and algorithms, computability theory has set the stage for the study of the computational resources (like time) that are required by such algorithms. When this study focuses on the resources that are necessary for *any* algorithm that solves a particular task (or a task of a particular type), it is viewed as belonging to the theory of Computational Complexity (also known as Complexity Theory). In contrast, when the focus is on the design and analysis of specific algorithms (rather than on the intrinsic complexity of the task), the study is viewed as belonging to a related area that may be called Algorithmic Design and Analysis. Furthermore, Algorithmic Design and Analysis tends to be sub-divided according to the domain of mathematics, science and engineering in which the computational tasks arise. In contrast, Complexity Theory typically maintains a unity of the study of computational tasks that are solvable within certain resources (regardless of the origins of these tasks).

Complexity Theory is a central field of the theoretical foundations of Computer Science. It is concerned with the study of the *intrinsic complexity of computational tasks*. That is, a typical Complexity theoretic study refers to the computational resources required to solve a computational task (or a class of such tasks), rather than referring to a specific algorithm or an algorithmic schema. Actually, research in Complexity Theory tends to *start with and focus on the computational resources themselves*, and addresses the effect of limiting these resources on the class of tasks that can be solved. Thus, Computational Complexity is the general study of the what can be achieved within limited time (and/or other limitations on natural computational resources).

Absolute Goals and Relative Results

Saying that Complexity Theory is concerned with the study of the *intrinsic complexity* of computational tasks means that its "final" goals include the determination of the complexity of any well-defined task. Additional goals include obtaining an understanding of the relations between various computational phenomena (e.g., relating one fact regarding computational complexity to another). Indeed, we may say that the former type of goals is concerned with *absolute* answers

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regarding specific computational phenomena, whereas the latter type is concerned with questions regarding the *relation* between computational phenomena.

Interestingly, so far Complexity Theory has been more successful in coping with goals of the latter ("relative") type. In fact, the failure to resolve questions of the "absolute" type, led to the flourishing of methods for coping with questions of the "relative" type. Musing for a moment, let us say that, in general, the difficulty of obtaining absolute answers may naturally lead to seeking conditional answers, which may in turn reveal interesting relations between phenomena. Furthermore, the lack of absolute understanding of individual phenomena seems to facilitate the development of methods for relating different phenomena. Anyhow, this is what happened in Complexity Theory.

Putting aside for a moment the frustration caused by the failure of obtaining absolute answers, we must admit that there is something fascinating in the success to relate different phenomena: in some sense, relations between phenomena are more revealing than absolute statements about individual phenomena. Indeed, the first example that comes to mind is the theory of NP-completeness. Let us consider this theory, for a moment, from the perspective of these two types of goals.

P, NP, and NP-completeness

Complexity theory has failed to determine the intrinsic complexity of tasks such as finding a satisfying assignment to a given (satisfiable) propositional formula or finding a 3-coloring of a given (3-colorable) graph. But it has succeeded in establishing that these two seemingly different computational tasks are in some sense the same (or, more precisely, are computationally equivalent). We find this success amazing and exciting, and hope that the reader shares these feelings. The same feeling of wonder and excitement is generated by many of the other discoveries of Complexity theory. Indeed, the reader is invited to join a fast tour of some of the other questions and answers that make up the field of Complexity theory.

We will indeed start with the *P versus NP Question*. Our daily experience is that it is harder to solve a problem than it is to check the correctness of a solution (e.g., think of either a puzzle or a research problem). Is this experience merely a coincidence or does it represent a fundamental fact of life (i.e., a property of the world)? Could you imagine a world in which solving any problem is not significantly harder than checking a solution to it? Would the term "solving a problem" not lose its meaning in such a hypothetical (and impossible in our opinion) world? The denial of the plausibility of such a hypothetical world (in which "solving" is not harder than "checking") is what "P different from NP" actually means, where P represents tasks that are efficiently solvable and NP represents tasks for which solutions can be efficiently checked.

The mathematically (or theoretically) inclined reader may also consider the task of proving theorems versus the task of verifying the validity of proofs. Indeed, finding proofs is a special type of the aforementioned task of "solving a problem" (and verifying the validity of proofs is a corresponding case of checking correctness). Again, "P different from NP" means that there are theorems that are harder to prove than to be convinced of their correctness when presented with a proof. This means that the notion of a "proof" is meaningful; that is, proofs do help when seeking to be convinced of the correctness of assertions. Here NP represents sets of assertions that can be efficiently verified with the help of adequate proofs, and P represents sets of assertions that can be efficiently verified from scratch (i.e., without proofs).

In light of the foregoing discussion it is clear that the P-versus-NP Question is a fundamental scientific question of far-reaching consequences. The fact that this question seems beyond our current reach led to the development of the theory of *NP-completeness*. Loosely speaking, this

theory identifies a set of computational problems that are as hard as NP. That is, the fate of the P-versus-NP Question lies with each of these problems: if any of these problems is easy to solve then so are all problems in NP. Thus, showing that a problem is NP-complete provides evidence to its intractability (assuming, of course, "P different than NP"). Indeed, demonstrating the NP-completeness of computational tasks is a central tool in indicating hardness of natural computational problems, and it has been used extensively both in computer science and in other disciplines. We note that NP-completeness indicates not only the conjectured intractability of a problem but rather also its "richness" in the sense that the problem is rich enough to "encode" any other problem in NP. The use of the term "encoding" is justified by the exact meaning of NP-completeness, which in turn establishes relations between different computational problems (without referring to their "absolute" complexity).

Some Advanced Topics

The foregoing discussion of NP-completeness hints to *the importance of representation*, since it referred to different problems that encode one another. Indeed, the importance of representation is a central aspect of complexity theory. In general, complexity theory is concerned with problems for which the solutions are implicit in the problem's statement (or rather in the instance). That is, the problem (or rather its instance) contains all necessary information, and one merely needs to process this information in order to supply the answer.[1] Thus, complexity theory is concerned with manipulation of information, and its transformation from one representation (in which the information is given) to another representation (which is the one desired). Indeed, a solution to a computational problem is merely a different representation of the information given; that is, a representation in which the answer is explicit rather than implicit. For example, the answer to the question of whether or not a given Boolean formula is satisfiable is implicit in the formula itself (but the task is to make the answer explicit). Thus, complexity theory clarifies a central issue regarding representation; that is, the distinction between what is explicit and what is implicit in a representation. Furthermore, it even suggests a quantification of the level of non-explicitness.

In general, complexity theory provides new viewpoints on various phenomena that were considered also by past thinkers. Examples include the aforementioned concepts of solutions, proofs, and representation as well as concepts like randomness, knowledge, interaction, secrecy and learning. We next discuss the latter concepts and the perspective offered by complexity theory.

The concept of *randomness* has puzzled thinkers for ages. Their perspective can be described as ontological: they asked "what is randomness" and wondered whether it exists at all (or is the world deterministic). The perspective of complexity theory is behavioristic: it is based on defining objects as equivalent if they cannot be told apart by any efficient procedure. That is, a coin toss is (defined to be) "random" (even if one believes that the universe is deterministic) if it is infeasible to predict the coin's outcome. Likewise, a string (or a distribution on strings) is "random" if it is infeasible to distinguish it from the uniform distribution (regardless of whether or not one can generate the latter). Interestingly, randomness (or rather pseudorandomness) defined this way is efficiently expandable; that is, under a reasonable complexity assumption (to be discussed next), short pseudorandom strings can be deterministically expanded into long pseudorandom strings. Indeed, it turns out that randomness is intimately related to intractability. Firstly, note that the very definition of pseudorandomness refers to intractability (i.e., the infeasibility of distinguishing

[1] In contrast, in other disciplines, solving a problem may require gathering information that is not available in the problem's statement. This information may either be available from auxiliary (past) records or be obtained by conducting new experiments.

a pseudorandomness object from a uniformly distributed object). Secondly, as stated, a complexity assumption, which refers to the existence of functions that are easy to evaluate but hard to invert (called *one-way functions*), implies the existence of deterministic programs (called *pseudorandom generators*) that stretch short random seeds into long pseudorandom sequences. In fact, it turns out that the existence of pseudorandom generators is equivalent to the existence of one-way functions.

Complexity theory offers its own perspective on the concept of *knowledge* (and distinguishes it from information). Specifically, complexity theory views knowledge as the result of a hard computation. Thus, whatever can be efficiently done by anyone is not considered knowledge. In particular, the result of an easy computation applied to publicly available information is not considered knowledge. In contrast, the value of a hard-to-compute function applied to publicly available information is knowledge, and if somebody provides you with such a value then it has provided you with knowledge. This discussion is related to the notion of *zero-knowledge* interactions, which are interactions in which no knowledge is gained. Such interactions may still be useful, because they may convince a party of the *correctness* of specific data that was provided beforehand. For example, a zero-knowledge interactive proof may convince a party that a given graph is 3-colorable without yielding any 3-coloring.

The foregoing paragraph has explicitly referred to *interaction*, viewing it as a vehicle for gaining knowledge and/or gaining confidence. Let us highlight the latter application by noting that it may be easier to verify an assertion when allowed to interact with a prover rather than when reading a proof. Put differently, interaction with a good teacher may be more beneficial than reading any book. We comment that the added power of such *interactive proofs* is rooted in their being randomized (i.e., the verification procedure is randomized), because if the verifier's questions can be determined beforehand then the prover may just provide the transcript of the interaction as a traditional written proof.

Another concept related to knowledge is that of *secrecy*: knowledge is something that one party may have while another party does not have (and cannot feasibly obtain by itself) – thus, in some sense knowledge is a secret. In general, complexity theory is related to *Cryptography*, where the latter is broadly defined as the study of systems that are easy to use but hard to abuse. Typically, such systems involve secrets, randomness and interaction as well as a complexity gap between the ease of proper usage and the infeasibility of causing the system to deviate from its prescribed behavior. Thus, much of Cryptography is based on complexity theoretic assumptions and its results are typically transformations of relatively simple computational primitives (e.g., one-way functions) into more complex cryptographic applications (e.g., secure encryption schemes).

We have already mentioned the concept of *learning* when referring to learning from a teacher versus learning from a book. Recall that complexity theory provides evidence to the advantage of the former. This is in the context of gaining knowledge about publicly available information. In contrast, computational learning theory is concerned with learning objects that are only partially available to the learner (i.e., reconstructing a function based on its value at a few random locations or even at locations chosen by the learner). Still, Complexity theory sheds light on the intrinsic limitations of learning (in this sense).

Complexity theory deals with a variety of computational tasks. We have already mentioned two fundamental types of tasks: *searching for solutions* (or rather "finding solutions") and *making decisions* (e.g., regarding the validity of assertions). We have also hinted that in some cases these two types of tasks can be related. Now we consider two additional types of tasks: *counting the number of solutions* and *generating random solutions*. Clearly, both the latter tasks are at least as hard as finding arbitrary solutions to the corresponding problem, but it turns out that for some natural problems they are not significantly harder. Specifically, under some natural conditions on

the problem, approximately counting the number of solutions and generating an approximately random solution is not significantly harder than finding an arbitrary solution.

Having mentioned the notion of *approximation*, we note that the study of the complexity of finding "approximate solutions" is also of natural importance. One type of approximation problems refers to an objective function defined on the set of potential solutions: Rather than finding a solution that attains the optimal value, the approximation task consists of finding a solution that attains an "almost optimal" value, where the notion of "almost optimal" may be understood in different ways giving rise to different levels of approximation. Interestingly, in many cases, even a very relaxed level of approximation is as difficult to obtain as solving the original (exact) search problem (i.e., finding an approximate solution is as hard as finding an optimal solution). Surprisingly, these hardness of approximation results are related to the study of *probabilistically checkable proofs*, which are proofs that allow for ultra-fast probabilistic verification. Amazingly, every proof can be efficiently transformed into one that allows for probabilistic verification based on probing a *constant* number of bits (in the alleged proof). Turning back to approximation problems, we mention that in other cases a reasonable level of approximation is easier to achieve than solving the original (exact) search problem.

Approximation is a natural relaxation of various computational problems. Another natural relaxation is the study of *average-case complexity*, where the "average" is taken over some "simple" distributions (representing a model of the problem's instances that may occur in practice). We stress that, although it was not stated explicitly, the entire discussion so far has referred to "worst-case" analysis of algorithms. We mention that worst-case complexity is a more robust notion than average-case complexity. For starters, one avoids the controversial question of what are the instances that are "important in practice" and correspondingly the selection of the class of distributions for which average-case analysis is to be conducted. Nevertheless, a relatively robust theory of average-case complexity has been suggested, albeit it is less developed than the theory of worst-case complexity.

In view of the central role of randomness in complexity theory (as evident, say, in the study of pseudorandomness, probabilistic proof systems, and cryptography), one may wonder as to whether the randomness needed for the various applications can be obtained in real-life. One specific question, which received a lot of attention, is the possibility of "purifying" randomness (or "extracting good randomness from bad sources"). That is, can we use "defected" sources of randomness in order to implement almost perfect sources of randomness. The answer depends, of course, on the model of such defected sources. This study turned out to be related to complexity theory, where the most tight connection is between some type of *randomness extractors* and some type of pseudorandom generators.

So far we have focused on the time complexity of computational tasks, while relying on the natural association of efficiency with time. However, time is not the only resource one should care about. Another important resource is *space*: the amount of (temporary) memory consumed by the computation. The study of space-complexity has uncovered several fascinating phenomena, which seem to indicate a fundamental difference between space-complexity and time-complexity. For example, in the context of space-complexity, verifying proofs of validity of assertions (of any specific type) has the same complexity as verifying proofs of invalidity for the same type of assertions.

In case the reader feels dizzy, it is no wonder. We took an ultra-fast air-tour of some mountain tops, and dizziness is to be expected. For a totally different touring experience, we refer the interested reader to our book "Computational Complexity: A Conceptual Perspective" (Cambridge University Press, 2008), which offers climbing the aforementioned mountains by foot, while stopping often for appreciation of the view and reflection.

Absolute Results (a.k.a. Lower-Bounds). As stated in the beginning of this essay, absolute results are not known for many of the "big questions" of complexity theory (most notably the P-versus-NP Question). However, several highly non-trivial absolute results have been proved. For example, it was shown that using negation can speed-up the computation of monotone functions (which do not require negation for their mere computation). In addition, many promising techniques were introduced and employed with the aim of providing a low-level analysis of the progress of computation. However, as stated up-front, the focus of this article was elsewhere.