TITLES AND ABSTRACTS FOR THE WINTER MASTER CLASS

20 JANUARY – 10 FEBRUARY 2008

1. COURSES:

Anton Alekseev (Geneve)

Title. *The Kashiwara-Vergne conjecture and Drinfeld's associators.*

Abstract: The Kashiwara-Vergne conjecture and the Drinfeld's equations for associators are two important questions stated in terms of free Lie algebras. Recently, an interesting link between these two problems has been discovered using the Kontsevich deformation quantization machinery.

The purpose of this mini-course is to present the background material on the Kashiwara-Vergne conjecture and on Drinfeld's associators, and to end up with a series of results relating the two fields. Plan of the mini-course:

I. Introduction: Free Lie algebras. The Campbell-Hausdorff series. Duflo theorem.

II. The Kashiwara-Vergne (KV) conjecture. Derivations of free Lie algebras. The KV conjecture. KV $\Rightarrow$ Duflo. Symmetries of the KV problem. The group KV_2.

III. Drinfeld's associators. Pentagon and hexagon equations. Drinfeld's twists. The Grothendieck-Teichmueller (GRT) group. Importance and applications of Drinfeld's associators.

IV. New results. The group homomorphism GRT $\to \mathrm{KV}_2$. The group KV_3, associators with values in KV_3. Correspondence between associators and solutions of the KV conjecture.

The mini-course is based on joint works with E. Meinrenken and C.Torossian

Michel Duflo (Paris)

Title. *Transverse Poisson Structures for Coadjoint Orbits and Representation Theory.*

Shlomo Gelaki (Technion)

Title. *Tensor categories.*

Abstract: I will give an introduction to the modern theory of tensor categories, with an emphasis on finite tensor categories (particular fusion categories). The topics include realizability of fusion rings, Ocneanu rigidity, module categories, weak Hopf algebras, Morita theory for tensor categories, lifting theory, categorical dimensions, Frobenius- Perron dimensions, classification of tensor categories.

Suggested reading for the students:

1. Lectures on tensor categories, by Damien Calaque and Pavel Etingof arXiv:math/0401246v3
2. On fusion categories, by P.Etingof, D.Nikshych, V. Ostrik, arXiv:math/0203060
3. Finite tensor categories, by Pavel Etingof, Viktor Ostrik, arXiv:math/0301027

Thierry Levasseur (Brest)

Title. *Invariant Differential Operators, Rational Cherednik Algebras, and Prehomogenous Vector Spaces.*

Abstract: Let G be a complex semisimple algebraic group and $(G : V)$ be a finite dimensional representation. Denote by $\mathcal{D}(X)$ the ring of differential operators on an affine variety X. The group G acts on $\mathcal{D}(V)$ and to each $D \in \mathcal{D}(V)^G$ one can associate a "radial component" $\mathrm{rad}(D) \in \mathcal{D}(V/\!\!/G)$ where $V/\!\!/G$ is the categorical quotient. This defines a morphism $\mathrm{rad} : \mathcal{D}(V)^G \to \mathcal{D}(V/\!\!/G)$.

The aim of the course is to give some information of the image (and the kernel) of rad, more particularly in the case where $V/\!\!/G$ is one dimensional. The Cherednik algebras associated to complex reflections groups will be used in that purpose.

Content of the course. Parts of the following subjects will be discussed.
I) Invariant differential operators and the radial component map.
II) Cherednik algebras and algebras similar to $U(\mathfrak{sl}(2))$.
III) Polar groups, prehomogeneous vector spaces, multiplicity free representations.
IV) Applications and generalizations: holonomic D-modules, symmetric spaces...

Bibliography

[Borel] A. Borel et al., *Algebraic D-modules*, Academic Press, Boston, 1987.

[Björk] J.-E Björk, *Rings of Differential Operators*, North Holland, Amsterdam, 1979.

[Kimura] T. Kimura, *Introduction to Prehomogeneous Vector Spaces*, Translations of Mathematical Monographs, vol. 215, AMS, Providence, 2003.

[Benson-Ratcliff] C. Benson and G. Ratcliff, On multiplicity free actions, in: Representations of real and p-adic groups, 221–304, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 2, Singapore Univ. Press, World Scientific, Singapore, 2004.

[Dunkl-Opdam] C.F. Dunkl and E. Opdam, Dunkl operators for complex reflection groups, *Proc. London Math. Soc.*, **86** (2003), 70–108.

[Etingof-Ginzburg] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, *Invent. Math.*, **147** (2002), 243–348.

[Gyoja] A. Gyoja, Theory of Prehomogeneous Vector Spaces without Regularity Condition, *Publ. RIMS*, **27** (1991), 861–922.

[Howe-Umeda] R. Howe and T. Umeda, The Capelli identity, the double commutant theorem and multiplicity free actions, *Math. Annalen*, **290** (1991), 565–619.

[Knop] F. Knop, Some remarks on multiplicity free spaces, in: Proc. NATO Adv. Study Inst. on Representation Theory and Algebraic Geometry (A. Broer, A. Daigneault, G. Sabidussi, eds.), 301-317, Nato ASI Series C, Vol. 514, Dortrecht, Kluwer, 1998.

[Muro] M. Muro, Singular Invariant Tempered Distributions on Regular Prehomogeneous Vector Spaces, *J. Funct. Anal.*, **76** (1988), 317–345.

[Smith] S.P. Smith, A class of algebras similar to the enveloping algebra of $\mathfrak{sl}(2)$, *Trans. Amer. Math.Soc*, **322** (1990), 285–314.

Olivier Mathieu (Lyon)

Title. *On Polyzeta Values.*

Abstract: This course is a Zagier's conjecture introduction about the dimension of the space generated by polyzeta values. For k positive integers $r_1, \dots, r_k$, set:

$$\zeta(r_1, \dots, r_k) = \sum_{0<n_1<\cdots<n_k} n_1^{-r_1} \dots n_k^{-r_k},$$

where the indices $n_1, \dots, n_k$ run over integers. This series is finite iff $r_k \geq 2$. By definition, polyzeta values are the numbers $\zeta(r_1, \dots, r_k)$. Its weight is $\sum_{1\leq i\leq k} r_i$.

These numbers have been introduced by Euler. He already knew some remarkable relations between them. For example, it he proved the famous identity $\zeta(2n) = C_n \pi^{2n}$, where the constant C_n is a well determined rational number. He also proved the relation $\zeta(3) = \zeta(1, 2)$.

Recently, polyzeta values have been investigated again. Zagier, Hoffman and others found beautiful ways to generate identities between polyzeta values. However, the combinatorics is very intricated and is is very difficult to understand the relations connecting basic identies.

Let $E_n \subset \mathbf{C}$ be the $\mathbf{Q}$-vector space generated by all polyzeta values of weight n. Zagier conjectured the following formula: $\dim E_n = b_n$, where the numbers b_n are given by the identity: $\sum_{n=0}^{\infty} b_n\, t^n = \frac{1}{1-t^2-t^3}$.

To prove the inequality $\dim E_n \leq b_n$, one can assume that all identities between polyzeta values are those already described. Then this inequality is reduced to a purely combinatorial problem. This can can be easily solved by a human for small values of n and, a bit further, by a computer. The inequality has been proved in general by Goncarov and Terasoma: it is surprisingOn Polyzeta Values that their proof is based on the cohomology theory of algebraic manifolds (and motivic theory). Up to now, no purely combinatorial proof is known.

The opposite inequality $\dim E_n \geq b_n$ should rely on transcendental number theory. Up to now, it is beyond the standard methods.

Summary:

Ch.1: A brief course on Rieman's zeta function

Ch.2: Polyzeta values

Ch 3: Identities between polyzeta values

Ch 4: A brief outline of Goncarov and Terasoma proof.

Ivan Penkov (Bremen)

Title. *Generalized Harish-Chandra modules.*

Abstract: A famous problem in infinite-dimensional representation theory is the classification of unitary representation of a real semisimple Lie group G. A closely related celebrated result is the classification of certain simple modules over the complexified Lie algebra $\mathfrak{g}$, called Harish-Chandra modules. The latter are simple $\mathfrak{g}$-modules on which the Lie algebra $\mathfrak{k}$ of a complexified maximal compact subgroup of G acts locally finitely, for short $(\mathfrak{g}, \mathfrak{k})$-modules. From the point of view of algebraic representation theory it is interesting to consider $(\mathfrak{g}, \mathfrak{k})$-modules for essentially arbitrary subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. For instance one may take $\mathfrak{k} = \mathfrak{h}$ to be a Cartan subalgebra of $\mathfrak{g}$. Then simple $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over $\mathfrak{k}$ are just weight modules. A classification of simple weight modules. was obtained in 1998 by O. Mathieu. In the first part of the course I will recall the necessary preliminaries on general $(\mathfrak{g}, \mathfrak{k})$-modules and will explain Mathieu's classification.

In the central part of the course I will discuss recent results by V. Serganova, G. Zuckerman and myself in the general theory of $(\mathfrak{g}, \mathfrak{k})$-modules. I will state a necessary condition on an arbitrary subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ to admit an irreducible $(\mathfrak{g}, \mathfrak{k})$-module M with finite $\mathfrak{k}$-multiplicities, with the condition that $\mathfrak{k}$ is a maximal subalgebra acting locally finitely on M. Moreover, I will characterize the reductive part of any such subalgebra $\mathfrak{k}$. These results rely on the Beilinson-Bernstein localization theorem.

I will also discuss a classification of $(\mathfrak{g}, \mathfrak{k})$-modules with sufficiently large minimal $\mathfrak{k}$-type given recently by G. Zuckerman and myself. This classification uses the Zuckerman functor and extends some of Vogan's methods from the case of a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ to an arbitrary reductive pair $(\mathfrak{g}, \mathfrak{k})$.

A $(\mathfrak{g}, \mathfrak{k})$-module M with finite $\mathfrak{k}$-multiplicities is called bounded if the $\mathfrak{k}$-multiplicities of M are uniformly bounded. In the final part of the course I will talk about recent results of V. Serganova and myself concerning bounded $(\mathfrak{g}, \mathfrak{k})$-modules. We prove a necessary condition for a subalgebra $\mathfrak{k}$ to admit an infinite-dimensional simple bounded $(\mathfrak{g}, \mathfrak{k})$-module. This condition is based on the study of primitive ideals and nilpotent orbits in the adjoint representation of g. We then provide a construction of bounded $(\mathfrak{g}, \mathfrak{k})$-modules using localization and the theory of spherical varieties.

If time permits, in the last lecture I will explain in detail the classification of simple bounded $(\mathfrak{sp}(4), \mathfrak{sl}(2))$-modules, $\mathfrak{sl}(2)$ being a principal subalgebra, and will write down their characters.

Markus Reineke (Wuppertal)

Title. *Moduli of quiver representations.*

Abstract: Moduli spaces parameterizing isomorphism classes of representations are constructed and studied; they arise as GIT quotients of certain representations of products of general linear groups. Background material on the representation theory of quivers will be recalled. Global geometric properties of the moduli spaces will be studied, focussing on the calculation of cohomology groups. Relations to quantum groups and Hall algebras will be explained. As a particular class of quiver moduli, Hilbert schemes of path algebras of quivers will be analyzed.

Andras Szenes (Geneve)

Title. *Topology and singularities.*

Abstract: We start with an introduction to singularity theory, and define the Thom polynomials, which play the role of characteristic classes for singularities. We describe various classical approaches to the computation of these invariants, and then we turn to a new construction, which produces formulas remarkably similar to those appearing in Conformal Field Theory.

Charles Torossian (Paris)

Title. *Kontsevich quantization and application in Lie theory.*

Abstract: We will first recall the geometric construction of Kontsevich (weights, graphs etc...) and we will explain why the Stokes formula is the main ingredient for the associativity. The same argument holds for the homotopy. Then we will focus on the following topic : what's new for Lie theory. We will focus the following application: the description of the algebra $(U/U\mathfrak{h})^{\mathfrak{h}}$ where $U = U(\mathfrak{g})$ is the enveloping algebra of any Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ is any sub algebra (the bi quantization of Cattaneo-Felder will be used). Applications for symmetric spaces will be detailed

Suggested reading – notes of two conferences:
http://www.institut.math.jussieu.fr/ torossian/IHP.pdf
http://www.institut.math.jussieu.fr/
Notes of the course :
http://uk.arxiv.org/PS_cache/math/pdf/0403/0403135v2.pdf
http://trefoil.math.ucdavis.edu/0711.3553

2. **TALKS:**

Yuri Bazlov (Warwick)

Title. *Quantum universal enveloping superalgerbas of type Q.*

Abstract: Lie superalgebras of type Q are known as "strange Lie superalgebras"; their Cartan subalgebras are not abelian and have both even and odd parts. Although the quantum universal enveloping superalgebra of type Q was constructed by Olshanski a decade and a half ago using the Faddeev-Reshetikhin-Takhtajan method, the Drinfeld-Jimbo style presentation was not found. The goal of my talk will be to describe an approach to the Q type QUE superalgebras via braided

doubles, which can be thought of as a vast generalisation of Drinfeld-Jimbo's construction, where the Hopf algebra structure is "lost". Joint work with A. Berenstein.

Alexander Elashvili (Tbilisi)

Title. *Lie algebras and singularity theory.*

Abstract: In my talk I shall present a number of results obtained jointly with G. Khimshiashvili and M. Jibladze about the finite dimensional Lie algebras, which can be naturally associated with germs of isolated hypersurface singularities (IHS). Recall that for any IHS germ $X := X(f) = \{f = 0\}$, one considers the Lie algebra of derivations $L(X) := O_n = (df)$, where O_n is the algebra of convergent power series in n variables, $f \in O_n$ and (df) is the ideal in O_n generated by all partial derivatives $\partial_i f = \frac{\partial f}{\partial x_i}$. According to S.S-T. Yau, $L(X)$ is a finite dimensional solvable Lie algebra called the Lie algebra of the singularity X.

According to Arnolds classification, IHS with modality ≤ 1 are subdivided into simple IHS (mod $= 0$), parabolic and hyperbolic ones and moreover there are 14 exceptional classes. We have investigated Lie algebras of vector field germs for these exceptional singularities, i. e. spaces of derivations of their moduli algebras equipped with the commutator bracket. Namely, explicit bases, and the structure constant presentation in terms of these bases have been constructed for derivation Lie algebras of all exceptional singularities.

In the work of Yau and collaborators the corresponding Lie algebras for parabolic singularities have been already computed.

It turns out that in all these cases, the singularities are determined by their Lie algebras, in the sense that all these Lie algebras are pairwise nonisomorphic. Note that for simple singularities this is not the case: Lie algebras of singularities of types A_6 and D_5 are isomorphic.

Another important property of these Lie algebras is that they are all complete, i. e. they all have trivial centers, and all their derivations are inner.

All these Lie algebras have natural gradings, and Poincaré polynomials with respect to these grading have been also calculated for all exceptional IHS.

Florence Millet(St. Etienne)

Title. *Link between semicentres in the two extremal cases of parabolics.*

Abstract: Let $\mathfrak{g}$ be a semisimple Lie algebra of finite dimension over $\mathbb{C}$ and $\mathfrak{p}$ a parabolic sublagebra of $\mathfrak{g}$. We denote by $S(\mathfrak{p})$ the symmetric algebra of $\mathfrak{p}$ (endowed with the Poisson bracket coming from Lie bracket on $\mathfrak{p}$) and by $U(\mathfrak{p})$ the universal enveloping algebra of $\mathfrak{p}$. The Poisson semicentre of $S(\mathfrak{p})$ is denoted by $Sy(\mathfrak{p})$ and the semicentre of $U(\mathfrak{p})$ is denoted by $Sz(\mathfrak{p})$. These semicentres are well known when $\mathfrak{p}$ is equal to Borel $\mathfrak{b}$ of $\mathfrak{g}$ or to the whole $\mathfrak{g}$ (in the last case $Sy(\mathfrak{g}) := Y(\mathfrak{g})$ is the Poisson centre of $S(\mathfrak{g})$ and $Sz(\mathfrak{g}) := Z(\mathfrak{g})$ is the centre of $U(\mathfrak{g})$).

We will show in this talk how to construct a linear morphism ϕ between $Sz(\mathfrak{b})$ and $Z(\mathfrak{g})$ and a linear morphism $gr\,\phi$ between $Sy(\mathfrak{b})$ and $Y(\mathfrak{g})$. These morphisms are isomorphisms only when $\mathfrak{g}$ is a product of simple Lie algebras of type A or C (shortly denoted by $\mathfrak{g}$ of type AC). More precisely ϕ and $gr\,\phi$ are injective if and only if $\mathfrak{g}$ is of type AC. It remains the conjecture that ϕ and $gr\,\phi$ are surjective.

Carlo Rossi(Zürich)

Title. *Fourier transforms on toric varieties and the Joseph Ideal.*

Abstract: We discuss Fourier transforms on toric varieties and the resulting isomorphisms on twisted rings of differential operators thereof. Specializing to the case of complex projective space of dimension n, Fourier transforms yield

i) a surjective homomorphism from the UEA of $\mathfrak{sl}_{n+1}$ and twisted rings of differential operators on the blow-up at the origin of the complex affine n-space;
ii) isomorphisms between twisted rings of differential operators on the n irreducible components of the intersection of the minimal nilpotent orbit of $\mathfrak{sl}_{n+1}$ with a Borel subalgebra.

If time permits we will briefly discuss the case of the Lie algebra $\mathfrak{sp}_{2n}$, where the intersection of the minimal nilpotent orbit of $\mathfrak{sp}_{2n}$ with a Borel subalgebra admits a toric structure and a simple resolution of singularities: Fourier transforms let weighted projective spaces appear in this framework.

Evgeny Smirnov (Moscow)

Title. *Schubert decomposition for double Grassmannians.*

Abstract: Classical Schubert calculus deals with orbits of a Borel subgroup $B \subset \mathrm{GL}(V)$ acting on a Grassmann variety $\mathrm{Gr}(k, V)$ of k-planes in a finite-dimensional space V. These orbits (Schubert cells) and their closures (Schubert varieties) are very well studied both from the combinatorial and the geometric points of view.

One can go one step farther, considering the direct product of two Grassmannians $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$ and the Borel subgroup $B \subset \mathrm{GL}(V)$ acting diagonally on this variety. In this case, the number of orbits still remains finite, but their combinatorics and geometry of their closures become much more involved. It would be challenging to extend the whole body of the Schubert calculus to this situation.

I will explain how to index the B-orbit closures in $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$ combinatorially, describe the inclusion relations between them, and construct their desingularizations, which are analogous to Bott–Samelson desingularizations for ordinary Schubert varieties. If time allows, I will also discuss relations of this situation with the study of spherical nilpotent orbits in the set of strictly upper-triangular matrices, due to Anna Melnikov, and geometry of quiver representations, studied by Grzegorz Bobiński and Grzegorz Zwara.

Geordie Williamson (Freiburg)

Title. *Link homology and equivariant cohomology.*

Abstract: Khovanov-Rozansky link homology is a categorical knot invariant: starting with any knot (or link) one obtains an invariant consisting of a triply graded vector space. Moreover, taking the Euler characteristic yields the HOMFLYPT polynomial. I will describe one way of constructing Khovanov-Rozansky homology (due to Khovanov) in terms of Soergel bimodules and explain joint work with Ben Webster in which we reinterpret the steps in its construction geometrically. Along the way we calculate the cohomology of certain smooth orbit closures in a reductive complex algebraic group.