PDE and Special Functions

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1 A brief history of linear PDE

I am most pleased and honored to start off this conference celebrating my friend and co-worker Yakar Kannai.

It is a prerogative of age to look back and see how we got to where we are now. When Yakar and I were students, one of the fashions in linear PDE was operators of arbitrary order – for example: operators of any type having constant coefficients, and elliptic operators of any even order with variable coefficients:

$$A = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha.$$

This turns out to have been something of a detour in the subject generally, and for us personally. Having started out separately on that detour, we eventually came together on another, much more classical, path. I would like to start with a very brief history of that path.

The history begins in the middle of 18th century, with the (linearized) equation of a vibrating string:

$$u_{tt}(x,t) = u_{xx}(x,t).$$

This was the subject of study and debate by D'Alembert, Euler, Daniel Bernoulli, and Lagrange. The debate was largely settled in the early 19th century after Fourier's work on the heat equation in one variable:

$$u_t = u_{xx}(x,t).$$

See [4] for an extended discussion of the history, the controversy, and the role it played in the development of mathematical analysis.

In the 19th century there was much study of the Laplace and Poisson equations, especially in three space variables:

$$\begin{aligned} \Delta u(\mathbf{x}) &= 0; \\ \Delta u(\mathbf{x}) + u(\mathbf{x}) &= f(\mathbf{x}), \end{aligned}$$

the multidimensional versions of the wave and heat equations:

$$\begin{aligned} u_{tt}(\mathbf{x}, t) &= \Delta u(\mathbf{x}, t); \\ u_t(\mathbf{x}, t) &= \Delta u(\mathbf{x}, t), \end{aligned}$$

and the Helmholtz equation

$$\Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) = f(\mathbf{x}).$$

A general procedure was adopted in studying these equations (especially their homogeneous forms) in various geometries: separate variables and reduce to the study of second order ODEs. This resulted in a zoo of special functions: the Euler–Gauss hypergeometric functions as well as functions associated with the names of Airy, Bessel, Chebyshev, Fresnel, Gegenbauer, Hankel, Jacobi, Kelvin, Kummer, Laguerre, Legendre, Macdonald, Weber, and Whittaker.

You will note that every equation mentioned so far is of second order and has constant coefficients. In the 20th century much effort was devoted to developing the tools – functional analysis, generalized functions and distribution theory, pseudodifferential operators, Fourier integral operators – for dealing with certain kinds of generalization to variable coefficients and to equations of higher order. In particular we have considerably generalized the classical notions of elliptic, parabolic, and hyperbolic equations. Meanwhile, special functions and the quest for explicit solutions nearly dropped from sight.

This brief sketch has brought us to the early 1970s, and has concentrated entirely on the classical elliptic, hyperbolic, and parabolic equations. Nevertheless there were other things going on in PDE – also connected, as we shall see, with some mathematicians of note. An example from the 18th and 19th centuries is:

$$(x + y)u_{xy}(x, y) + u_x(x, y) + u_y(x, y) = 0,$$

the *Euler–Poisson–Darboux equation* [5]. Two examples from the middle third of the 20th century are the kinetic equation

$$x\, u_y(x,y) = u_{xx}(x,y) + f(x,y)$$

considered by Kolmogorov in 1934 [7], and the *Tricomi–Clairaut* equation

$$x\, u_{xx}(x,y) = u_{yy}(x,y)$$

studied by Delache and Leray in 1971 [6]. Both the Euler–Poisson–Darboux equation and the Tricomi–Clairaut equation have fundamental solutions that involve hypergeometric functions.

While Yakar and I were in graduate school in the 1960s, attention in linear PDE began to shift more generally toward other equations that did not fit the classical mold: subelliptic, degenerate elliptic, singular hyperbolic, degenerate hyperbolic, and so on. To some extent we retreated to techniques of earlier eras: study very specific examples in detail in order to understand the range of possibilities. One result was a new appreciation for the role of lower order terms – and for the role of the classical special functions. Here is an example from our joint work, the singular hyperbolic operator

$$x^2 \frac{\partial^2}{\partial t^2} + \lambda \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}.$$

Treves [15] showed that there is uniqueness in the Cauchy problem for this operator if and only if λ is not a negative odd integer. We examined in detail the general form

$$x^{2k-2} \frac{\partial^2}{\partial t^2} + \lambda(k-1) x^{k-2} \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}, \quad k = 2, 3, 4, \ldots.$$

Consider the forward problem for this operator with data at $x_0 > 0$. For most values of λ the propagator (wave operator) is supported on the closure of the union of the following two regions:

$$\begin{aligned} &\Omega_1 : \ |x^k - x_0^k| < kt < |x^k + x_0^k|, \quad x > 0; \\ &\Omega_2 : \ \max\{|x^k - x_0^k|, |x^k + x_0^k| < kt\}. \end{aligned}$$

One each region the propagator has an explicit representation

$$V(x,t;x_0) = |z_+|^{-a} |z_-|^{-b} F(v),$$

where

$$z_{\pm} = \frac{x^k + x_0^k \pm kt}{2}, \qquad v = \frac{(xx_0)^k}{z_+ z_-},$$

and F is a solution of the hypergeometric equation

$$v(1-v)F''(v) + [c - (a+b+1)v\, F'(v) - ab\, F(v) = 0,$$

with

$$c = 1 - \frac{1}{k}, \qquad a = \left(\frac{1+\lambda}{2}\right)c, \qquad b = \left(\frac{1-\lambda}{2}\right)c.$$

For example, suppose $k = 3$, so that the operator is

$$L_\lambda = x^4 \frac{\partial}{\partial t} + 2\lambda x \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}.$$

Theorem. *If $2\lambda/3$ is not an odd integer, then L_λ has a global forward propagator. The singular support of the forward propagator contains the boundary of $\Omega_1 \cup \Omega_2$. It contains the curve common to the boundaries of Ω_1 and Ω_2 if and only if neither $(2-\lambda)/3$ nor $(1-\lambda)/3$ is a positive integer.*

If $2\lambda/3$ is an odd integer, there is non–uniqueness: L_λ has a solution with support $\Omega_1 \cup \Omega_2$.

Analogous results are true for the degenerate hyperbolic operator obtained by interchanging the "space" and "time" variables here: [14],[2]. To prove them we constructed wave operators with the help of results on Whittaker functions and hypergeometric functions.

2 Special functions

As I have tried to indicate, special functions play an important role in PDE. As a student I absorbed functional analysis, *a priori estimates* and other then–fashionable techniques for studying PDE. Somewhat later (after considerable initial resistance) I learned about pseudodifferential and Fourier integral operators. It was only after falling in with a new crowd – Peter Greiner, Bernard Gaveau, Jacek Szmigielski, and Yakar – that I was dragged into the 19th century and the world of special functions. This had an effect that I would like to describe.

Often, learning a subject as a student, one simply takes one's teachers' advice about what is important. When confronting a new subject later on, one is likely to be harder to convince. And if the subject seems to come in bits and pieces, one might be particularly concerned about getting an overview.

Getting an overview of special functions is particularly difficult. Most textbooks and handbooks of special functions are written by people who have come to know and love the functions as individuals, and do not feel it necessary to provide a convincing, unifying perspective.

One thing that does tie together all the functions mentioned earlier has already been mentioned. Each arises out of separating variables for the Laplacian. To a physicist or applied mathematician, this is probably sufficient: the Laplacian and the associated wave and heat operators are the canonical operators of classical physics, and the coordinate systems in which variables separate reflect natural symmetries. A pure mathematician might ask for a more fundamental view. Here is a result I had to discover on my own, though versions of it are well–known to the specialists.

Question: *Given a Sturm–Liouville problem on an interval $I \subset \mathbb{R}$, self-adjoint with respect to $w(x)\,dx$, when are the eigenfunctions polynomials?*

Answer: *Up to normalizations there are three cases:*

- $I = (-1, 1)$, $w(x) = (1-x)^\alpha (1+x)^\beta$, eigenfunctions are Jacobi polynomials.
- $I = (0, \infty)$, $w(x) = x^\alpha e^{-x}$, eigenfunctions are Laguerre polynomials.
- $I = (-\infty, \infty)$, $w(x) = e^{-x^2}$, eigenfunctions are Hermite polynomials.

Thus this accounts for all the "classical orthogonal polynomials," and nothing else. (Legende polynomials, Chebyshev polynomials, and so on are special cases of Jacobi polynomials.)

The attempt to find unifying principles led to a set of notes. Under prodding by Roderick Wong, the set of notes led to a joint book on the subject. This allowed me to get the whole business out of my system once and for all. Or so I thought.

A few weeks ago I had a visit from Szmigielski, one of the dubious characters mentioned above. In some recent work of his, he had found himself confronted with yet another class of special functions, the Meijer G–functions. My book with Wong carries on a venerable tradition in books about special functions: it contain no mention whatever of the G–functions. This is an exact reflection of my state of knowledge as we began to think about them.

As we thought about them, we became convinced that they deserve a wider audience. The rest of this lecture is my attempt to convince you of this. There are two basic reasons:

- The G–functions play a crucial role in a certain mathematical enterprise.
- When looked at conceptually, they are both natural and attractive.

We all know that there exist table of integrals. They have been around for much longer than G–functions, which date to 1936. What I did not know is that the most extensive such tables, in print [13] and online [17], are constructed almost entirely on the basis of G–functions and their properties; see [16]. I will say a little more about this later, but first I want to argue that they are indeed natural and attractive.

The literature on the G–functions has the same general structure as most writing on special functions by experts: a definition, a list of fairly elementary identities, then a longer list of less elementary identities, with no motivation and little explanation to interrupt the flow. As an afterthought it is mentioned that the function in question is the solution of some homogeneous linear ODE.

As some of us could guess, the ODE is actually the key – both to why the functions are of interest and how they are best understood. This is certainly the case with the G–functions. So let me begin with the ODE, which in this case is essentially the generalized hypergeometric equation.

3 What is special about the generalized hypergeometric equation?

Consider the general linear homogeneous ODE with coefficients that are analytic near $x = 0$. In principle we can construct a formal solution by the power series method. If you try this with a simple example, say

$$u''(x) - e^x u(x) = 0,$$

and ask for the coefficient of x^{10}, or even x^5, it is not a pleasant thing to do by hand. In fact, about the only way such a procedure *would* be pleasant is if it reduced to a simple two–term recursion for the coefficients.

Question: *When does this happen? When does the power series expansion reduce to a two-term recursion?*

With trivial exceptions, and up to a simple change of scale, the answer is: *precisely when the equation is a generalized hypergeometric equation.* Such an equation is most conveniently written using the Euler derivative

$$D = x\frac{d}{dx}.$$

Then the equation has the form

$$\left[D\prod_{j=1}^{q-1}(D+b_j-1) - x\prod_{j=1}^{q}(D+a_j)\right]u(x) = 0.$$

Since $D[x^n] = nx^n$, the associated recursion is

$$n\prod_{j=1}^{q-1}(n+b_j-1)\cdot u_n = \prod_{j=1}^{q}(n+a_j)\cdot u_{n-1}$$

and the formal power series solution with constant term 1 is

$$u(x) = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q-1}(b_j)_n\, n\,!}\, x^n,$$

where

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n > 0.$$

(Here and below we make assumptions like: neither $a-1$ nor $b-1$ is a negative integer, and no two indices differ by an integer.) The series has radius of convergence 0 if $p > q$, 1 if $p = q$, $+\infty$ if $p < q$. If $p \le q$ the function so defined is the generalized hypergeometric function

$${}_pF_{q-1}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{q-1}\end{matrix}\middle|\, x\right).$$

4 The Meijer G–functions

Let us start with a more symmetric version of the generalized hypergeometric equation, obtained by replacing the lone factor D by $D + b_q - 1$:

$$\left[\prod_{j=1}^{q}(D+b_j-1) - x\prod_{j=1}^{q}(D+a_j)\right]u(x) = 0, \quad p \le q. \tag{1}$$

Writing $u(x) = x^{1-b_k}v_k(x)$ leads to a generalized hypergeometric equation for v_k (with indices $\{b_j + 1 - b_k\}, \{a_j + 1 - b_k\}$), so there is a basis of solutions of the form

$$x^{1-b_k}F_k(x), \quad F_k(x) = {}_pF_{q-1}(\dots|x).$$

We look for a more symmetric and conceptual approach. Most of the issues that arise can be seen already in the case $p = q = 1$:

$$(D + b - 1)u(x) - x(D + a)u(x) = 0. \tag{2}$$

Since $D[x^n] = nx^n$ and $x \cdot x^n = x^{n+1}$, it is reasonable to try to write a solution as a (continuous) sum of powers of x, say

$$u(x) = \frac{1}{2\pi i}\int_L \Phi(s)\, x^s\, ds,$$

where L is some complex contour. The equation leads us to

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_L \left[(s + b - 1)\Phi(s)\, x^s - \frac{1}{2\pi i}\int_L (s + a)\Phi(s)\, x^{s+1}\right] ds \\ &= \frac{1}{2\pi i}\int_L (s + b - 1)\Phi(s)\, x^s\, ds - \frac{1}{2\pi i}\int_{L-1} (s + a - 1)\Phi(s - 1)\, x^s\, ds. \end{aligned}$$

Assuming that the translated contour $L - 1$ can be deformed to L without crossing any singularities, we obtain the continuous recursion

$$\frac{\Phi(s)}{\Phi(s-1)} = \frac{a + s - 1}{b + s - 1} = \frac{\Gamma(a + s)}{\Gamma(a + s - 1)} \cdot \frac{\Gamma(b + s - 1)}{\Gamma(b + s)}, \tag{3}$$

so one solution is $\Phi(s) = \Gamma(a + s)/\Gamma(b + s)$. We have been led to

$$u(x) = \frac{1}{2\pi i}\int_L \frac{\Gamma(a + s)}{\Gamma(b + s)}\, x^s\, ds.$$

With appropriate choices of L, two residue calculations show that $u(x)$ vanishes for $|x| < 1$ and is a non–trivial solution of (2) for $|x| > 1$.

One question we need to address is that of *uniqueness*. It is easy to see that $\varphi(s)\Phi(s)$ is a second solution of the recurrence relation (3) if and only if $\varphi(s-1) = \varphi(s)$. We can change the kernel Φ *while staying within the context of gamma functions* by using Euler's reflection identity

$$\Gamma(s)\,\Gamma(1 - s) = \frac{\pi}{\sin \pi s}. \tag{4}$$

Taking

$$\varphi(s) = \frac{\pi}{\sin \pi(b+s)} = \Gamma(b+s)\,\Gamma(1-b-s),$$

we can convert to

$$\Phi(s)\,\varphi(s) = \Gamma(a+s)\,\Gamma(1-b-s),$$

and consider

$$v(x) = \frac{1}{2\pi i}\int_L \Gamma(a+s)\,\Gamma(1-b-s)\,x^s\,ds$$

Now you would be correct to object that this φ is not periodic with period 1, it is antiperiodic: $\varphi(s-1) = -\varphi(s)$. Tracing the argument backward, one can see that this fits with changing the sign of x in (2). Two different residue calculations produce solutions of the version with change of sign

$$(D+b-1)v(x) + x(D+a)v(x) = 0,$$

for $|x| < 1$ and $|x| > 1$ respectively.

In the general case (1), our starting kernel Φ is

$$\Phi(s) = \frac{\prod_{j=1}^{p}\Gamma(a_j+s)}{\prod_{j=1}^{q}\Gamma(b_j+s)}. \tag{5}$$

The corresponding integral

$$G(x) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{p}\Gamma(a_j+s)}{\prod_{j=1}^{q}\Gamma(b_j+s)}\,x^s\,ds$$

is Meier's G–function $G^{0,p}_{p,q}$. The general version

$$G^{m,n}_{p,q}, \qquad 0 \le m \le q, \quad 0 \le n \le p,$$

is obtained by replacing the *first* m factors $\Gamma(b_j+s)$ in the *denominator* of the quotient (5) by factors $\Gamma(1-b_j+s)$ in the *numerator* and the *last* $p-n$ factors $\Gamma(a_j+s)$ in the numerator by factors $\Gamma(1-a_j-s)$ in the *denominator*.

(We need to note a nasty fact about notation. For historical reasons one should replace b_j and a_j by $1-b_j$ and $1-a_j$ throughout this discussion. Thus the Meijer G–function

$$G^{0,p}_{p,q}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle|\, x\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{p}\Gamma(1-a_j+s)}{\prod_{j=1}^{q}\Gamma(1-b_j+s)}\,x^s\,ds$$

is a solution of the equation

$$\left[\prod_{j=1}^{q}(D - b_j) - x\prod_{j=1}^{q}(D + 1 - a_j)\right] u(x) = 0.)$$

5 The convolution theorem

Almost every standard special function, or elementary function, can be written neatly in terms of the Meijer G–functions. This is one of the principal ingredients in current recipes for constructing tables of integrals. Another principal ingredient is the the *convolution theorem*: the multiplicative convolution

$$G(x) = \int_0^\infty G_1(y)\, G_2\left(\frac{x}{y}\right)\, \frac{dy}{y} \tag{6}$$

of G–functions G_1 and G_2 is itself a G–function.

In different terminology, this is a standard result. Consider the Mellin transform normalized as

$$\mathcal{M}f(s) = \int_0^\infty f(x)\, x^{-s}\, \frac{dx}{x}.$$

It is easy to verify that the Mellin transform of a multiplicative convolution is the product of the Mellin transforms, so for (6) we have

$$\mathcal{M}G(s) = \mathcal{M}G_1(s)\, \mathcal{M}G_2(s).$$

But, as you may have guessed from the formulas for G–functions, the G–function itself is (in many cases) the *inverse* Mellin transform of the function we have been denoting Φ or one of its variants. Thus for the function G in (6), the Mellin transform $\Phi_1\Phi_2$ is again a quotient of products of gamma functions, and G is a G–function. In fact if G_j is of type

$$G_j = G^{m_j,n_j}_{p_j,q_j}, \quad j = 1, 2,$$

then G is of type

$$G = G^{m_1+m_2,n_1+n_2}_{p_1+p_2,q_1+q_2},$$

and some bookkeeping will identify the indices $\{a_j\}$, $\{b_j\}$ of G in terms of the corresponding indices of G_1 and G_2.

Remarks. Although the specific source for the Meijer G–functions is [8], integral representations of this kind go back to Barnes [1] for the classical hypergeometric functions, and to Mellin [9] for the generalized hypergeometric functions. For more information, references, and applications in statistics and science, see [8], [9], [10].

6 References

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